Gegenbauer polynomials

1 Definition

Generating function:

$$\frac{1}{(1-2xt+t^2)^\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x) t^n$$

for $\alpha \neq 0$.

The Gegenbauer polynomials $C_n^{(\alpha)}(x)$ are also known as ultraspherical polynomials (see Arfken-Weber-Harris section 18.4).

The Gegenbauer polynomials include a number of polynomials we have seen previously as special cases: for example, $\alpha = 1/2$ gives the Legendre polynomials, and $\alpha = 1$ gives the type II Chebyshev polynomials. The case $\alpha = 0$ gives the type I Chebyshev polynomials, though this case must be handled differently than $\alpha \neq 0$. Specifically, one defines

$$C_n^{(0)}(x) = \lim_{\alpha \to 0} \frac{C_n^{(\alpha)}}{\alpha}$$

and the relation to type I Chebyshev polynomials is discussed in Arfken-Weber-Harris section 18.4.

The first few Gegenbauer polynomials are

$$\begin{aligned}
C_0^{(\alpha)}(x) &= 1 \\
C_1^{(\alpha)}(x) &= 2\alpha x \\
C_2^{(\alpha)}(x) &= -\alpha + 2\alpha(1+\alpha)x^2 \\
C_3^{(\alpha)}(x) &= -2\alpha(1+\alpha)x + \frac{4}{3}\alpha(1+\alpha)(2+\alpha)x^3
\end{aligned}$$

By comparison, note that the first few Legendre polynomials ($\alpha = 1/2$) are given by (Arfken-Weber-Harris table 15.1)

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right) \\
P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right)
\end{aligned}$$

and the first few type II Chebyshev polynomials ($\alpha = 1$) are given by (Arfken-Weber-Harris table 18.4)

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_2(x) &= 4x^2 - 1 \\ U_3(x) &= 8x^3 - 4x \end{aligned}$$

In terms of hypergeometric functions,

$$\begin{aligned} C_n^{(\alpha)}(x) &= \begin{pmatrix} n+2\alpha-1 \\ n \end{pmatrix} {}_2F_1\left(-n, n+2\alpha, \alpha+1/2; (1/2)(1-x)\right) \\ &= 2^n \begin{pmatrix} n+\alpha-1 \\ n \end{pmatrix} (x-1)^n {}_2F_1\left(-n, -n-\alpha+1/2, -2n-2\alpha+1; \frac{2}{1-x}\right) \\ &= \begin{pmatrix} n+2\alpha+1 \\ n \end{pmatrix} \left(\frac{x+1}{2}\right)^n {}_2F_1\left(-n, -n-\alpha+1/2, \alpha+1/2; \frac{x-1}{x+1}\right) \end{aligned}$$

Normalization:

$$\int_{-1}^{1} (1-x^2)^{\alpha-1/2} [C_n^{(\alpha)}(x)]^2 dx = 2^{1-2\alpha}\pi \frac{\Gamma(n+2\alpha)}{(n+\alpha)\Gamma(\alpha)^2\Gamma(n+1)}$$

for $\alpha > -1/2$.

A few relations include

$$\begin{aligned} \frac{d}{dx} C_n^{(\alpha)}(x) &= 2\alpha C_{n-1}^{(\alpha+1)}(x) \\ n C_n^{(\alpha)}(x) &= 2(n+\alpha-1)x C_{n-1}^{(\alpha)}(x) - (n+2\alpha-2) C_{n-2}^{(\alpha)}(x) \text{ for } n \geq 2 \end{aligned}$$

2 Exercises

1. Check that the normalization condition stated above for Gegenbauer polynomials reduces for $\alpha = 1/2$ to the normalization condition for Legendre polynomials

$$\int_{-1}^{1} P_n(x) P_n(x) dx = \frac{2}{2n+1} \quad \text{(Arfken-Weber-Harris (15.38)}$$

   and for $\alpha = 1$ to the normalization condition for type II Chebyshev polynomials

$$\int_{-1}^{1} (1-x^2)^{1/2} U_n(x) U_n(x) dx = \frac{\pi}{2} \quad \text{(Arfken-Weber-Harris (18.118)}$$

2. Check that the recursion relation

$$nC_n^{(\alpha)}(x) \ = \ 2(n+\alpha-1)xC_{n-1}^{(\alpha)}(x) \ - \ (n+2\alpha-2)C_{n-2}^{(\alpha)}(x) \text{ for } n \geq 2$$

reduces to known statements about Legendre and type II Chebyshev polynomials.

3. Show that

$$\frac{d}{dx}C_n^{(\alpha)}(x) \ = \ 2\alpha C_{n-1}^{(\alpha+1)}(x)$$

References

[1] http://mathworld.wolfram.com/GegenbauerPolynomial.html

[2] P. Morse, F. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, 1953.