1. Assuming the validity of the Hankel transform,

$$\begin{aligned} g(\alpha) &= \int_0^\infty f(t) J_n(\alpha t)\, t dt \\ f(t) &= \int_0^\infty g(\alpha) J_n(\alpha t)\, \alpha d\alpha \end{aligned}$$

show that the Dirac delta function has a Bessel representation

$$\delta(t - t') = t \int_0^\infty J_n(\alpha t) J_n(\alpha t')\, \alpha d\alpha$$

2. From the Fourier transform show that the coordinate change

$$t \mapsto \ln x, \quad i\omega \mapsto \alpha - \gamma$$

leads to the Mellin transform

$$\begin{aligned} \tilde{f}(\alpha) &= \int_0^\infty f(x) x^{\alpha-1} dx \\ f(x) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{f}(\alpha)\, x^{-\alpha} d\alpha \end{aligned}$$

3. Verify the following Mellin transforms:

   (a)

$$\int_0^\infty x^{\alpha-1} \sin(kx)\, dx = k^{-\alpha}(\alpha-1)!\, \sin\left(\frac{\pi\alpha}{2}\right) \quad \text{for } -1 < \alpha < 1$$

   (b)

$$\int_0^\infty x^{\alpha-1} \cos(kx)\, dx = k^{-\alpha}(\alpha-1)!\, \cos\left(\frac{\pi\alpha}{2}\right) \quad \text{for } 0 < \alpha < 1$$

4. $F(\rho)$ and $G(\rho)$ are the Hankel transforms of $f(r)$, $g(r)$. Derive the Hankel transform analogue of the Parseval relation:

$$\int_0^\infty F^*(\rho) G(\rho) \rho d\rho = \int_0^\infty f^*(r) g(r) r dr$$