1. Show that

$$\Gamma(-n+\epsilon) \;=\; \frac{(-)^n}{n!}\left[\frac{1}{\epsilon} \;+\; F(n) \;+\; \mathcal{O}(\epsilon)\right]$$

where F is the digamma function ($F(z) = \psi(z+1)$), n is a positive integer, and ϵ is small. Use the identity

$$F(n) \;=\; 1 \;+\; \frac{1}{2} \;+\; \frac{1}{3} \;+\; \cdots \;+\; \frac{1}{n} \;-\; \gamma$$

(γ the Euler-Mascheroni constant) and think inductively.

2. Derive the "Jacobi-Angier" expansion

$$\exp(ix\cos\theta) \;=\; \sum_{n=-\infty}^{\infty} i^n J_n(x)\exp(in\theta)$$

This is an expansion of a plane wave in a series of cylindrical waves. (This is also problem 14.1.5 in the text.)

3. Use the integral representation of the Hankel function

$$H_0^{(1)}(x) \;=\; \frac{2}{i\pi}\int_0^\infty \exp(ix\cosh s)ds$$

to show that

(a)

$$J_0(x) \;=\; \frac{2}{\pi}\int_1^\infty \frac{\sin xt}{\sqrt{t^2-1}}dt$$

(b)

$$N_0(x) \;=\; -\frac{2}{\pi}\int_1^\infty \frac{\cos xt}{\sqrt{t^2-1}}dt$$

4. Show that a series

$$\sum_{n=1}^{\infty}\frac{A_n}{x^n}$$

is asymptotic to $f(x)$ if and only if it is also asymptotic to $f(x)+\exp(-x)$.

5. The modified Bessel function $K_0(x)$ has the integral representation

$$K_0(x) \;=\; \int_0^\infty \exp(-x\cosh t)dt$$

Use this integral representation and the method of steepest descent to derive the leading term in an asymptotic expansion of $K_0(x)$. For full credit you must use the method of steepest descent.

6. Define the Sharpe polynomials $S_n(x)$ by the generating function

$$\sum_{n=0}^{\infty} S_n(x)t^n \;=\; \frac{\exp(xt)}{1-t}, \;\; |t| < 1$$

   (a) Compute $S_n(0)$ for all n.

   (b) Derive the recurrence relations

$$\begin{aligned} S_n' &= S_{n-1} \\ (n+1)S_{n+1} &= xS_n \;+\; [S_n \;+\; S_{n-1} \;+\; \cdots \;+\; S_0] \end{aligned}$$

   (c) Describe a Schläfli-type (contour integral) representation of $S_n(x)$.

   (d) Find a closed-form expression for $S_n(x)$.

7. Show that

$$\int_0^x L_n(t)dt \;=\; L_n(x) \;-\; L_{n+1}(x)$$

where L_n denotes the nth Laguerre polynomial.

8. Let $P_n(x)$ denote the Legendre polynomials and $U_n(x)$ denote the type II Chebyshev polynomials. Show that

$$\begin{aligned} U_{2n} &= (P_n)^2 \;+\; 2P_{n-1}P_{n+1} \;+\; 2P_{n-2}P_{n+2} \;+\; \cdots \;+\; 2P_0P_{2n} \\ U_{2n+1} &= 2P_nP_{n+1} \;+\; 2P_{n-1}P_{n+2} \;+\; \cdots \;+\; 2P_0P_{2n+1} \end{aligned}$$

for all n. (Hint: use the generating functions.)

9. Define the generalized hypergeometric functions by

$${}_pF_q(\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q; x) \;=\; \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{x^n}{n!}$$

Show that

$$\int_0^z {}_pF_q(\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q; x)dx \;=\; \frac{(\beta_1-1)(\beta_2-1)\cdots(\beta_q-1)}{(\alpha_1-1)(\alpha_2-1)\cdots(\alpha_p-1)} \left[{}_pF_q(\alpha_1-1, \cdots, \alpha_p-1, \beta_1-1, \cdots, \beta_q-1; z) \;-\; 1\right]$$