1. (a) Show that the Fourier cosine series expansion for $\cos ax$ on $[0, \pi]$ is given by

$$\cos ax = \frac{2a \sin a\pi}{\pi} \left[\frac{1}{2a^2} - \frac{\cos x}{a^2 - 1^2} + \frac{\cos 2x}{a^2 - 2^2} - \cdots \right]$$

$$a_n = (-)^n \frac{2a \sin a\pi}{\pi(a^2 - n^2)}$$

(b) From the preceding result show that

$$a\pi \cot a\pi = 1 - 2 \sum_{p=1}^{\infty} \zeta(2p) a^{2p}$$

Cultural note: this last expression also gives a relationship between the zeta function and the Bernoulli numbers, using the fact that

$$x \cot x = \sum_{n=0}^{\infty} (-)^n B_{2n} \frac{(2x)^{2n}}{(2n)!} \text{ for } -\pi < x < \pi$$

2. The electrostatic potential of a charged conducting disk is known to have the general form

$$\Phi(\rho, z) = \int_0^{\infty} \exp(-k|z|) J_0(k\rho) f(k) dk$$

with $f(k)$ unknown. At large distances ($z \to \infty$) the potential must approach the Coulomb potential $Q/(4\pi\epsilon_0 z)$. Show that

$$\lim_{k \to 0} f(k) = \frac{Q}{4\pi\epsilon_0}$$

(Hint: at large distances, one can approximate $\rho \cong 0$.)

3. A calculation of the magnetic field of a circular current loop leads to the integral

$$\int_0^{\infty} \exp(-kz) k J_1(ka) dk$$

Show that this integral equals

$$\frac{a}{(z^2 + a^2)^{3/2}}$$

This is problem 20.8.12 in the text.

4. Heaviside expansion theorem. If the Laplace transform $f(s)$ may be written as a ratio

$$f(s) = \frac{g(s)}{h(s)}$$

where $g(s)$, $h(s)$ are analytic functions, $h(s)$ having simple isolated zeroes at $s = s_i$, show that

$$F(t) = \mathcal{L}^{-1} \left\{ \frac{g(s)}{h(s)} \right\} = \sum_i \frac{g(s_i)}{h'(s_i)} \exp(s_i t)$$

This is problem 20.10.11 in the text.

5. A Fredholm equation of the first kind has a kernel $\exp(-(x-t)^2)$:

$$f(x) = \int_{-\infty}^{\infty} \exp\left(-(x-t)^2\right) \varphi(t)dt$$

Show that the solution is

$$\varphi(x) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{2^n n!} H_n(x)$$

in which $H_n(x)$ is an nth order Hermite polynomial. This is problem 21.2.8 in the text.

6. The Kronig-Penney model

In this problem we shall study a simple model of electrons in a one-dimensional crystal, interacting only weakly with the ion cores. Begin with Schrödinger's equation for a particle of mass m in one-dimension:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi(x) = E\psi$$

where E is the energy of the particle, a constant.

(a) First, show that

$$-\frac{\hbar^2}{2m}(-i\omega)^2\tilde{\psi}(\omega) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega' \tilde{U}(\omega-\omega')\tilde{\psi}(\omega') = E\tilde{\psi}(\omega)$$

where $\tilde{U}$ and $\tilde{\psi}$ are the Fourier transforms of U and ψ, respectively.

(b) What kind of integral equation is this?

(c) Assume that the potential is periodic with period a, described by delta functions:

$$U(x) = A\sum_{n=-\infty}^{\infty} \delta(x+na)$$

Show that the integral equation above reduces to

$$\left(\frac{\hbar^2}{2m}\omega^2 - E\right)\tilde{\psi}(\omega) + \frac{A}{a}\sum_{n=-\infty}^{\infty} \tilde{\psi}(\omega + 2\pi n/a) = 0$$

(d) Define

$$f(\omega) = \sum_{n=-\infty}^{\infty} \tilde{\psi}(\omega + 2\pi n/a)$$

Show that

$$\tilde{\psi}(\omega + 2\pi n/a) = -\frac{\left(\frac{2mA}{\hbar^2 a}\right) f(\omega)}{(\omega + 2\pi n/a)^2 - 2mE/\hbar^2}$$

(e) Sum over all n to derive the constraint

$$\frac{\hbar^2 a}{2mA} \;=\; -\sum_{n=-\infty}^{\infty} \left[(\omega + 2\pi n/a)^2 \;-\; 2mE/\hbar^2\right]^{-1}$$

(f) Use the identity

$$\cot x \;=\; \sum_{n=-\infty}^{\infty} \frac{1}{x \,+\, n\pi}$$

and trig identities (differences of cotangents, products of pairs of sines and cosines) to show that the constraint above can be rewritten as

$$\cos a\omega \;=\; \cos aK \;+\; \frac{mA}{\hbar^2 K} \sin aK$$

where

$$K \;=\; \sqrt{\frac{2mE}{\hbar^2}}$$

The left-hand side of the equation above can only take values between -1 and $+1$, whereas the right-hand side can take values outside that range. That means there are some values of K, and hence E, for which Schrödinger's equation does not have a solution. In fact, the allowed solutions break up into bands – stretches of allowed energies (known as bands) separated by gaps (the band gaps).

Attached are a pair of figures showing these gaps. The top figure is a plot of the right-hand side of the equation above, as a function of K. In the second figure the relation above has been inverted to compute the energy E as a function of ω. The band gaps are clearly visible in the second plot, as places where the energy periodically jumps at certain ω.

Although this model is very simple (one-dimensional crystal, potential described by delta functions), it has correctly captured a typical qualitative characteristic of metals, the bands and band gaps. Whether a given metal is an insulator or a conductor depends upon the occupancy of the bands: if all bands are either full or empty, the metal is an insulator, whereas if some bands are only partially full, the metal is a conductor.

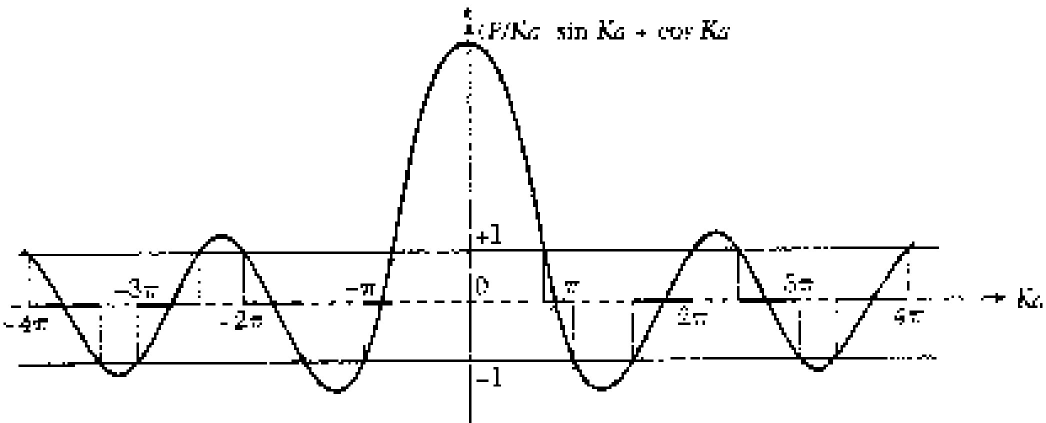

Figure 5 Plot of the function $(P/Ka)\sin Ka + \cos Ka$, for $P = 3\pi/2$. The allowed values of the energy ϵ are given by those ranges of $Ka = (2m\epsilon/\hbar^2)^{1/2}a$ for which the function lies between ± 1. For other values of the energy there are no traveling wave or Bloch-like solutions to the wave equation, so that forbidden gaps in the energy spectrum are formed.

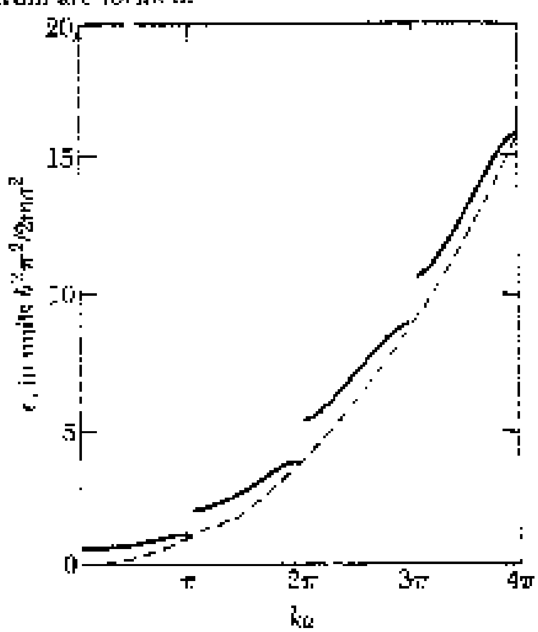

Figure 6 Plot of energy vs. wavenumber for the Kronig-Penney potential, with $P = 3\pi/2$. Notice the energy gaps at $ka = \pi, 2\pi, 3\pi \ldots$.