Rethinking Gauge Theory through Connes' Noncommutative Geometry

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Work with Ufuk Aydemir, Djordje Minic, Tatsu Takeuchi:

Phys. Rev. D **91**, 045020 (2015) [arXiv:1409.7574],
Pati-Salam Unification from Non-commutative Geometry and the TeV-scale W_R boson [arXiv:1509.01606],
Review of NCG in preparation.

For background of NCG, c.f. Chamseddine, Connes, et. al.:
Nucl. Phys. Proc. Suppl. **18B**, 29 (1991)
Commun. Math. Phys. **182**, 155 (1996) [hep-th/9603053],
Adv. Theor. Math. Phys. **11**, 991 (2007) [hep-th/0610241],

and for superconnection, c.f. Neeman, Fairlie, et. al.:
Phys. Lett. B **81**, 190 (1979),
J. Phys. G **5**, L55 (1979),
Phys. Lett. B **82**, 97 (1979).

The quickest review of gauge theory

Given

ψ	element in rep' space $\mathcal{H}$, e.g. Dirac spinors,
$\hat{O}$	operator on $\mathcal{H}$, e.g. $\not{\partial}$,

we say the operator is 'covariant' if under the transformation

$$\psi \mapsto u\psi,$$

the operator trasforms as

$$\hat{O} \mapsto u\hat{O}u^{-1},$$

since that gives us

$$\hat{O}\psi \mapsto u\hat{O}\psi.$$

At the end, a theory built with

$$\mathcal{L} \sim \langle \psi | \hat{O}\psi \rangle$$

is invariant under the transformation.

The quickest review of gauge theory -Cont'd

When we localize the transformation u, things sometimes change

$$\hat{O} \mapsto u\hat{O}u^{-1} + \text{local terms.}$$

Therefore, we need to come up with another operator that transforms as

$$\hat{A} \mapsto u\hat{A}u^{-1} - \text{local terms,}$$

so that the combination of the two gives

$$\hat{O} + \hat{A} \mapsto u(\hat{O} + \hat{A})u^{-1}.$$

Then we have made the combo operator $\hat{O} + \hat{A}$ a 'covariant' operator, denoted $\hat{O}_A$.

Example: U(1) from global to local

We have

$$\mathcal{L} = \overline{\psi} i \partial\!\!\!/ \psi.$$

Invariant under global $U(1)$:

$$\psi \mapsto e^{i\theta}\psi,$$
$$\mathcal{L} \mapsto \mathcal{L}' = \mathcal{L}.$$

When we localize the $U(1)$ symmetry, i.e. $\theta = \theta(x)$,

$$\psi \mapsto e^{i\theta(x)}\psi,$$
$$\mathcal{L} \mapsto \mathcal{L}' = \mathcal{L} - \partial\theta\overline{\psi}\psi.$$

Example: U(1) from global to local -Cont'd

Therefore we come up with a $U(1)$ gauge field A, which transforms as

$$A \mapsto uAu^{-1} + \partial\theta.$$

and modify the Lagrangian as

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ + A\!\!\!/)\psi.$$

All together, we acquire an invariant theory.

Description in spectral triple

Suppose we have

$$\begin{aligned}\mathcal{A} =& C^\infty(M),\\ \mathcal{H} =& \Gamma(M, S),\\ D =& i\partial\!\!\!/.\end{aligned}$$

The unitary transformations are

$$\{u \in \mathcal{A} | u^\dagger u = u u^\dagger = 1\}.$$

Under transformations u, we have

$$\begin{aligned}\psi &\mapsto u\psi,\\ D\psi &\mapsto Du\psi = uD\psi + [D, u]\psi.\end{aligned}$$

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The theory built with $\overline{\psi}D\psi$ is invariant

$$\begin{aligned}&\Leftrightarrow [D, u] = 0,\\ &\Leftrightarrow \partial(u) = 0,\\ &\Leftrightarrow u \text{ is a global symmetry.}\end{aligned}$$

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What if $[D, u] \neq 0$?

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$$\begin{aligned} &\Leftrightarrow [D, u] = 0, \\ &\Leftrightarrow \partial(u) = 0, \\ &\Leftrightarrow u \text{ is a global symmetry.} \end{aligned}$$

What if $[D, u] \neq 0$?

– Old trick: use a gauge field to absorb the extra term.

– What should the gauge look like?

$[D, u] \neq 0$

In this case, D transforms as

$$D \mapsto u(D + u^\dagger [D, u])u^\dagger .$$

Apparently it is not covariant. It is ‘perturbed’ during the transformation, with the extra term is of the form

$$u^\dagger [D, u].$$

We want to ‘absorb’ the extra term into D, with the hope the overall operator is recovered covariant. Therefore we define another operator as

$$A = \sum a_i [D, b_i],$$

where a_i, $b_i \in \mathcal{A}$. We can immediately tell the extra term is nothing but of the form of A, thus can be absorbed.

$$D \mapsto u(D + u^\dagger[D, u])u^\dagger = u(D + A_0)u^\dagger.$$

With transformation u:

$$D \mapsto D + A_0.$$

Need

$$A \mapsto A - A_0.$$

Using the language we are familiar with, we have (up to order one condition)

$$\begin{aligned} \psi &\mapsto u\psi, \\ D &\mapsto u(D + u^\dagger[D, u])u^\dagger, \\ A &\mapsto u(A - u^\dagger[D, u])u^\dagger, \\ D + A &\mapsto u(D + A)u^\dagger. \end{aligned}$$

Formally, D works similarly to a differential operator as in $W = W_\mu \mathrm{d}x^\mu$, and A works like the gauge field. In this way, we can define the new differential one forms as elements in

$$\Omega^1 = \left\{ \sum a_i[D, b_i] | a_i, b_i \in \mathcal{A} \right\}.$$

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Define the 'perturbed' D_A to be the combination of the two

$$D_A = D + A.$$

Generalization

As it is shown above, $(\mathcal{A}, \mathcal{H}, D) = (C^\infty(M), \Gamma(M, S), i\partial\!\!\!/)$ gives us a $U(1)$ gauge theory.

But, what for?

With a few modifications, we can build a generalized gauge theory.

$\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$

According to Gelfand-Naimark, if we study all the algebra in $C^\infty(M)$, we can get all the information of the geometry M.

$$\begin{aligned} f : M &\to \mathbb{C}, \\ p &\mapsto f(p), \end{aligned}$$

where $f \in C^\infty(M)$.

By analogy:
Consider changing $\mathcal{A} = C^\infty(M)$ to $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$, $\forall a \in \mathcal{A}$, we denote $a = (\lambda, \lambda')$. This is the map,

$$\begin{aligned} a : \{p_1, p_2\} &\to \mathbb{C}, \\ p_1 &\mapsto a(p_1) = \lambda, \\ p_2 &\mapsto a(p_2) = \lambda'. \end{aligned}$$

Similar to $C^\infty(M) \leftrightarrow M$, roughly, we have $\mathbb{C} \oplus \mathbb{C} \leftrightarrow \{p_1, p_2\}$, a two point space.

- At this point, there is no relation for the two points space.
- In $\mathcal{A} = C^\infty(M)$, the distance is

$$\mathrm{d}(x, y) = \inf \int_\gamma \mathrm{d}s,$$
$$\mathrm{d}^2 s = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu.$$

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- How to extract this information from the algebra, if Gelfand-Naimark is correct?
- $\mathrm{d}(x, y) = \sup\{|f(x) - f(y)| : f \in C^\infty(M), |\partial f(x)| \leq 1\}.$

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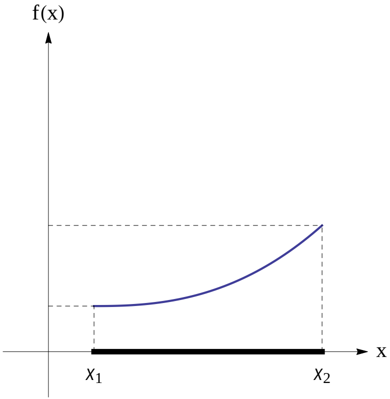
Distance formula:

$$\mathrm{d}(x, y) = \sup\{|f(x) - f(y)| : f \in C^{\infty}(M), |\partial f(x)| \le 1\}.$$

Translate:



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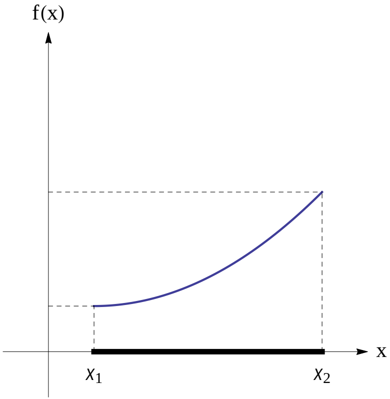

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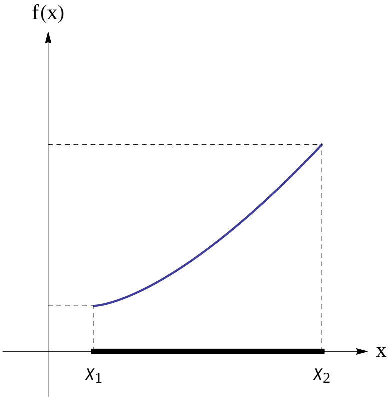

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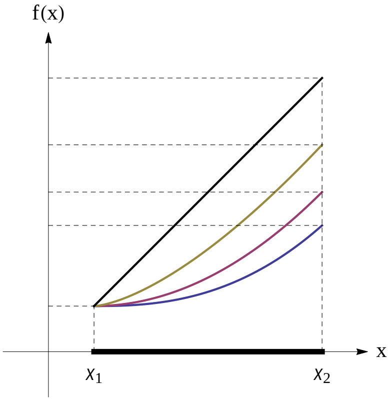

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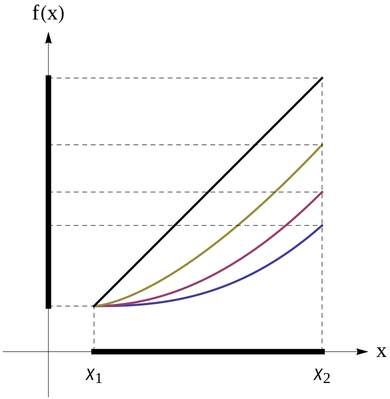

- By analogy, can calculate the 'distance' between the two points in $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$.
- Introduce the third element, the generalization of Dirac operator,

$$D = \begin{bmatrix} 0 & \overline{m} \\ m & 0 \end{bmatrix}.$$

- The distance formula is

$$\mathrm{d}(x, y) = \sup\{|a(x) - a(y)| : a \in \mathcal{A}, \|[D, a]\| \leq 1\}.$$

- Distance between the two points

$$\begin{aligned} d(p_1, p_2) &= \sup_{a \in \mathcal{A}, \|[D,a]\| \leq 1} \{|a(p_1) - a(p_2)|\} \\ &= \sup_{(\lambda,\lambda') \in \mathcal{A}, \|[D,a]\| \leq 1} |\lambda - \lambda'| \\ &= \frac{1}{|m|}. \end{aligned}$$

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The generalized Dirac operator encodes the distance information!

$$\mathcal{A} = \mathbb{C} \oplus \mathbb{C},$$
$$\mathcal{H} = \mathbb{C}^N \oplus \mathbb{C}^N,$$
$$D = \begin{bmatrix} 0 & M^\dagger \\ M & 0 \end{bmatrix}.$$

For $a \in \mathcal{A} = (\lambda, \lambda')$, the 'differential' is $\sim (\lambda - \lambda')$:

$$[D, a] = (\lambda - \lambda') \begin{bmatrix} 0 & -M^\dagger \\ M & 0 \end{bmatrix},$$

By analogy with

$$df = \partial_\mu f \, \mathrm{d}x^\mu = \lim_{\epsilon \to 0} (f(x+\epsilon) - f(x)) \, \frac{\mathrm{d}x^\mu}{\epsilon}.$$

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The 'integral' is $\sim (\lambda + \lambda')$:

$$Tr(a) = \lambda + \lambda'.$$

By analogy with

$$\int f(x) \mathrm{d}x.$$

Grading

- Physically, we are specifically interested in the type of algebra $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. e.g. the model with $U(1)_Y \times SU(2)_L$, or $SU(2)_R \times SU(2)_L$, etc.

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A device that helps us distinguish one part from the other.

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A device that helps us distinguish one part from the other.
D, $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \sim$ two sheets structure.

$\mathcal{A} = \mathbb{C} \oplus \mathbb{H}$ – A toy model

$$
\begin{aligned}
\mathcal{A} &= \mathbb{C} \oplus \mathbb{H}, \\
\mathcal{H} &= \mathbb{C}^2 \oplus \mathbb{C}^2, \\
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How do we fit this with our particle spectrum? 'Flavor' space:

$$\nu_R = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ e_R = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \nu_L = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ e_L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\text{For any } a \in \mathcal{A}, \ a = \begin{bmatrix} \lambda & & & \\ & \bar{\lambda} & & \\ & & \alpha & \beta \\ & & -\bar{\beta} & \bar{\alpha} \end{bmatrix}.$$

To give mass terms out of $\psi^\dagger D \psi$, let $M = \begin{bmatrix} m_\nu & 0 \\ 0 & m_e \end{bmatrix}$.

The unitary transformations are

$$\{u \in \mathcal{A} | u^\dagger u = uu^\dagger = 1\}.$$

This implies

$$u = \begin{bmatrix} \mathrm{e}^{i\theta} & & & \\ & e^{-i\theta} & & \\ & & \alpha & \beta \\ & & -\overline{\beta} & \overline{\alpha} \end{bmatrix}, \quad \text{s.t.} \quad |\alpha|^2 + |\beta|^2 = 1.$$

which automatically fulfills $\det u = 1$. This is the symmetry $U(1)_R \times SU(2)_L$. The $U(1)_R$ charge is

	$\|\uparrow\rangle$	$\|\downarrow\rangle$
$\mathbf{2}_R$	1	−1
$\mathbf{2}_L$	0	0

When we make the $U(1)_R \times SU(2)_L$ transformation,

$$\begin{aligned}\mathcal{L} =& \Psi^\dagger D \Psi \\ \mapsto& \Psi^\dagger u^\dagger D u \Psi = \Psi^\dagger D \Psi + \underbrace{\Psi^\dagger u^\dagger [D, u] \Psi}_{\text{the 'local' twist}} .\end{aligned}$$

In general $[D, u] \neq 0$, therefore, this demands for a 'gauge' field to absorb the local twist, in the discrete direction.

According to our recipe, we do have a gauge field between the two sheets,

$$A = \sum_i a_i [D, b_i].$$

$$\begin{aligned}\mathcal{L} =& \Psi^\dagger (D + A) \Psi \\ \mapsto& \Psi^\dagger D \Psi + \Psi^\dagger u^\dagger [D, u] \Psi + \Psi^\dagger A \Psi - \Psi^\dagger u^\dagger [D, u] \Psi \\ =& \Psi^\dagger (D + A) \Psi\end{aligned}$$

Demanded to be Hermitian, this gauge field is

$$A = \begin{bmatrix} & M^\dagger \Phi^\dagger \\ \Phi M & \end{bmatrix},$$

$$\Phi = [\phi_1 \ \ \phi_2] = \begin{bmatrix} \phi_1 & \phi_2 \\ -\overline{\phi_2} & \overline{\phi_1} \end{bmatrix}$$

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The perturbation of 'D^2' derived from the (spectral) action:

$$\mathrm{Tr}\left((D+A)^2 - D^2\right) \mathrm{Tr}\left((D+A)^2 - D^2\right) \sim \mathrm{Tr}\left((MM^\dagger)^2\right)\left(|\Phi + 1|^2 - 1\right)^2.$$

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SSB now has a reason:

$D + A$ gives a VEV shift.

By analogy,

local 'twist'	$e^{-i\theta}\partial_\mu e^{i\theta} = \partial_\mu \theta$	$u^\dagger [D, u]$
ω	$A_\mu \mathrm{d}^\mu x$	$\sum a_i [D, b_i] = \begin{bmatrix} & M^\dagger \Phi^\dagger \\ \Phi M & \end{bmatrix}$
basis	$d^\mu x$	$\begin{bmatrix} & M^\dagger \\ M & \end{bmatrix}$
comp'	A_μ	Φ
θ $\sim F^{\mu\nu}$	$(d + A) \wedge (d + A)$ $\sim \partial_\mu A_\nu + [A_\mu, A_\nu]$	$\mathrm{Tr}\left((D + A)^2 - D^2\right)$ $\sim DA + A^2$
S	$\int F^{\mu\nu} F_{\mu\nu} \mathrm{d}^4 x$	$\left(\mathrm{Tr}\left((D + A)^2 - D^2\right)\right)^2$

Product geometry

Consider the algebra:

$$\mathcal{A} = C^\infty(M) \oplus C^\infty(M)$$
$$\sim C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{C}).$$

This corresponds to a geometry

$$F = M \oplus M,$$
$$\sim M \times \{p_1, p_2\}.$$

Product geometry

Consider the algebra:

$$\begin{aligned}\mathcal{A} =& C^\infty(M) \oplus C^\infty(M)\\ \sim& C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{C}).\end{aligned}$$

This corresponds to a geometry

$$\begin{aligned}F =& M \oplus M,\\ \sim& M \times \{p_1, p_2\}.\end{aligned}$$

Combining continuous part with $\mathbb{C} \oplus \mathbb{H}$,

$$\mathcal{A} = C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H}).$$

$\sim$ a double-layer structure.

The Dirac operator of the product geometry:

$$D_x = i\partial\!\!\!/ + \gamma^5 \otimes D.$$

The gauge field:

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$$\sim \begin{bmatrix} B & \Phi^* \\ \Phi & W \end{bmatrix}.$$

Highlights:

- A two sheet structure.
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$\sim$ Implies a Higgs as the discrete gauge, generated similarly as the continous gauge fields.

Color sector

In order to reproduce SM, color sector must be involved.

- Introduce the 'color' space. $\mathcal{H} = \mathbb{C} \oplus \mathbb{C}^3$, with basis

$$\ell = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ r = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ g = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

- $\mathcal{A} = \mathbb{C} \oplus M_3(\mathbb{C})$, with $\forall a \in \mathcal{A}$,

$$a = \begin{bmatrix} \lambda & & & \\ & m_{11} & m_{12} & m_{13} \\ & m_{21} & m_{22} & m_{23} \\ & m_{31} & m_{32} & m_{33} \end{bmatrix}.$$

- Symmetry group is

$$\{u \in \mathcal{A} | u^\dagger u = uu^\dagger = 1\},$$

- together with the 'unimodularity' condition, $\det u = 1$.

$$a = \begin{bmatrix} e^{-i\theta} & \\ & e^{i\theta/3} m' \end{bmatrix}, \ m' \in SU(3).$$

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This gives the $U(1)$ charge

$$\begin{array}{cccc} \ell & r & g & b \\ -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}$$

We recognize them as $B - L$ charge, and this gives us the symmetry $U(1)_{B-L} \times SU(3)_C$.

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$$d_R = |\downarrow\rangle_R \otimes \begin{bmatrix} r \\ g \\ b \end{bmatrix} \qquad \in \mathbf{2}_R \otimes \mathbf{3}_C,$$

- Introduce J, charge conjugate,

$$J \begin{bmatrix} |\uparrow\rangle_R \\ |\downarrow\rangle_R \\ |\uparrow\rangle_L \\ |\downarrow\rangle_L \end{bmatrix} \otimes \begin{bmatrix} \ell \\ r \\ g \\ b \end{bmatrix} \sim \begin{bmatrix} \ell \\ r \\ g \\ b \end{bmatrix} \otimes \begin{bmatrix} |\uparrow\rangle_R \\ |\downarrow\rangle_R \\ |\uparrow\rangle_L \\ |\downarrow\rangle_L \end{bmatrix},$$

- $\forall a \in \mathcal{A}$ with left action on flavor space as before, JaJ^{-1} is the right action on color space.
- Ready to combine the previous result on flavor space and color space.

	$\lvert\uparrow\rangle$	$\lvert\downarrow\rangle$
$\mathbf{2}_R$	1	-1
$\mathbf{2}_L$	0	0

ℓ	r	g	b
-1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$U(1)_R$:

	$\lvert\uparrow\rangle \otimes \mathbf{1}^0$	$\lvert\downarrow\rangle \otimes \mathbf{1}^0$	$\lvert\uparrow\rangle \otimes \mathbf{3}^0$	$\lvert\downarrow\rangle \otimes \mathbf{3}^0$
$\mathbf{2}_L$	0	0	0	0
$\mathbf{2}_R$	1	-1	1	-1

$U(1)_{B-L}$:

	$\lvert\uparrow\rangle \otimes \mathbf{1}^0$	$\lvert\downarrow\rangle \otimes \mathbf{1}^0$	$\lvert\uparrow\rangle \otimes \mathbf{3}^0$	$\lvert\downarrow\rangle \otimes \mathbf{3}^0$
$\mathbf{2}_L$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbf{2}_R$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$

	$\|\uparrow\rangle \otimes \mathbf{1}^0$	$\|\downarrow\rangle \otimes \mathbf{1}^0$	$\|\uparrow\rangle \otimes \mathbf{3}^0$	$\|\downarrow\rangle \otimes \mathbf{3}^0$
$\mathbf{2}_L$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbf{2}_R$	0	-2	$\frac{4}{3}$	$-\frac{2}{3}$

Spectral Action

According to Chamseddine et. al. (hep-th/9606001), one builds the action based on spectral action principle:

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$$S_{spec} = \mathrm{Tr}(f(D_A/\Lambda)).$$

We can expand it as

$$\mathrm{Tr}(f(D_A/\Lambda)) \sim \int_M \mathcal{L}(g_{\mu\nu}, A)\sqrt{g}\,\mathrm{d}^4x.$$

The bosonic action,

$$
\begin{aligned}
S_{Bosonic} =& S_{Higgs} + S_{YM} + S_{Cosmology} + S_{Riemann}, \\
S_{Higgs} =& \frac{f_0 a}{2\pi^2} \int |D_\mu \phi|^2 \sqrt{g}\ d^4x + \frac{-2af_2\Lambda^2 + ef_0}{\pi^2} \int |\phi|^2 \sqrt{g}\ d^4x \\
&+ \frac{f_0 b}{2\pi^2} \int |\phi|^4 \sqrt{g}\ d^4x, \\
S_{YM} =& \frac{f_0}{16\pi^2} \mathrm{Tr}(\mathbb{F}_{\mu\nu}\overline{\mathbb{F}}^{\mu\nu}) \\
=& \frac{f_0}{2\pi^2} \int (g_3^2 G^i_{\mu\nu} \overline{G}^{\mu\nu\, i} + g_2^2 W^i_{\mu\nu} \overline{W}^{\mu\nu\, i} + \frac{5}{3} g_1^2 B_{\mu\nu} \overline{B}^{\mu\nu}) \sqrt{g}\ d^4x
\end{aligned}
$$

where the parameters are

$$
\begin{aligned}
a =& \mathrm{Tr}(M_\nu^* M_\nu + M_e^* M_e + 3(M_u^* M_u + M_d^* M_d)) \\
b =& \mathrm{Tr}((M_\nu^* M_\nu)^2 + (M_e^* M_e)^2 + 3(M_u^* M_u)^2 + 3(M_d^* M_d)^2) \\
c =& \mathrm{Tr}(M_R^* M_R) \\
d =& \mathrm{Tr}((M_R^* M_R)^2) \\
e =& \mathrm{Tr}(M_R^* M_R M_\nu^* M_\nu),
\end{aligned}
$$

f_n is the $(n-1)$th momentum of f.

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Can be calculated from spectral action. Intuitively,

	Cont'	Disc'
Fermion	$\overline{\psi} \partial\!\!\!/ \psi$	$\Psi^\dagger D \Psi$
Boson	$\partial_\mu W \partial^\mu W$	$D^2 W^2$

Output:

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- $m_H \approx 170\ \mathrm{GeV}$, problematic, which is naturally saved by the left-right completion we propose.

Other Fun Facts – 'local twist'

- $[D, u]$ is insensitive to local/global transformation w.r.t. M.
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$$\begin{aligned}
\delta S &= \int \frac{\delta \mathcal{L}}{\delta \phi} \delta\phi + \frac{\delta \mathcal{L}}{\delta \partial \phi} \delta \partial \phi \\
&= \int \frac{\delta \mathcal{L}}{\delta \phi} \delta\phi + \partial \left(\frac{\delta \mathcal{L}}{\delta \partial \phi} \delta\phi \right) - \partial \left(\frac{\delta \mathcal{L}}{\delta \partial \phi} \right) \delta\phi \\
&\overset{EOM}{=} \int \partial \left(\frac{\delta \mathcal{L}}{\delta \partial \phi} \delta\phi \right) \\
&= \int \partial \left(\epsilon \frac{\delta \mathcal{L}}{\delta \partial \phi} \Phi \right) \\
&= \int \partial(\epsilon j) \\
&= \int \partial_\mu(\epsilon) j^\mu + \int \epsilon \partial_\mu j^\mu .
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- $[D, \epsilon^i \sigma^i] \neq 0$ a 'local' symmetry in the discrete direction, with a gauge.

Other Fun Facts – ‘local twist’

$[D, u] = 0$ refers to

$$\begin{aligned} & Du = uD, \\ \Leftrightarrow \quad & D = uDu^\dagger. \end{aligned}$$

In SM, this refers to the VEV shift is invariant under the transformation u.

This describes the transformation of VEV shift, or the symmetry under which vacuum is invariant.

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This describes the transformation of VEV shift, or the symmetry under which vacuum is invariant.

$\sim$ Remaining symmetry,

$\sim$ Breaking chain.

Other Fun Facts – 'local twist'

In the simplest case, $A = \mathbb{H} \oplus \mathbb{H}$, $D = \begin{bmatrix} 0 & M^\dagger \\ M & 0 \end{bmatrix}$ and $M = \begin{bmatrix} 0 & m_u \\ m_d & 0 \end{bmatrix}$.

- Pictorially, the twist between 'left sheet' and 'right sheet'.

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- Pictorially, the twist between 'left sheet' and 'right sheet'.
- But even we make same twists for left and right, we still have a local 'twist term', unless $m_u = m_d$, isospin-like.

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- The separation introduces a second scale $\sim$ EW, from $a_i[D, b_i]$, different from the GUT scale which is led by the fluctuation in the continuous direction $f_i[\partial\!\!\!/, g_i]$.
- When the separation goes to ∞, $m_f \to 0$. This corresponds to the decouple of Higgs sector: left and right stop talking to each other, physically and geometrically.

Back to the Left-Right Completion

- Different realizations.
- For example. NCG/ spectral triple is built using lattice, supersymmetric quantum mechanics operators, Moyal deformed space, etc.
- We have tried a specific realization using superconnection, $su(2|1)$, and the left-right completion of $su(2|2)$.
- Low energy emergent left-right completion, $\sim 4\ TeV$.

(Ufuk Aydemir, Djordje Minic, C.S., Tatsu Takeuchi: Phys. Rev. D **91***, 045020 (2015) [arXiv:1409.7574])*

More About the Left-Right Completion

- Hints for left-right symmetry behind the scene. (*Pati-Salam Unification from NCG and the TeV-scale WR boson*, [arXiv:1509.01606], Ufuk Aydemir, Djordje Minic, C.S., Tatsu Takeuchi)
- Changing the algebra to $(\mathbb{H}_R \oplus \mathbb{H}_L) \otimes (\mathbb{C} \oplus M_3(\mathbb{C}))$ does not change the scale.

$$\frac{2}{3} g_{BL}^2 = g_{2L}^2 = g_{2R}^2 = g_3^2 .$$

Through the mixing of $SU(2)_R \times U(1)_{B-L}$ into $U(1)_Y$, we get

$$\frac{1}{g'^2} = \frac{1}{g^2} + \frac{1}{g_{BL}^2} = \frac{5}{3} \frac{1}{g^2} .$$

$\sim$ LR symmetry breaking at GUT.

Myths and Outlooks

- So far it is a classical theory – only classical $\mathcal{L}$ is given. But it has a GUT feature! Without adding new d.o.f.
- If it just happens at one scale, how to accommodate Wilson picture.
- Quantization of the theory? Loops?
- Relation to the D-brane structure?
- Measure of the Dirac operator?
- ...

Summary (of Fun Facts)

Recipe to cook up a (generalized) gauge theory:

- $(\mathcal{A}, \mathcal{H}, D)$, the spectral triple.
- Take mass matrix as a derivative, trace as the integral.
- Generate the gauge field $A = \sum a[D, b]$.
- Spectral action, $Tr(f(D/\Lambda)) \sim DA + A^2$, as the gauge strength Generalized free fermion action, $\Psi^\dagger D_A \Psi$, for the fermionic part.

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- Separation of the sheets ($m_f \to 0$, second scale, etc.)

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- Separation of the sheets ($m_f \to 0$, second scale, etc.)
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Summary (of Fun Facts)

Recipe to cook up a (generalized) gauge theory:

- $(\mathcal{A}, \mathcal{H}, D)$, the spectral triple.
- Take mass matrix as a derivative, trace as the integral.
- Generate the gauge field $A = \sum a[D, b]$.
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  Generalized free fermion action, $\Psi^\dagger D_A \Psi$, for the fermionic part.

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