Localization on twisted spheres and supersymmetric GLSMs

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Supersymmetric gauge theories in two dimensions

Two-dimensional supersymmetric gauge theories—a.k.a. GLSM—are an interesting playground for the quantum field theorist.

- They exhibit many of the qualitative behaviors of their higher-dimensional cousins.
- Supersymmetry allows us to perform exact computations.
- They provide useful UV completions of non-linear $\sigma$-models, including conformal ones, and of other interesting 2d SCFTs.
- Consequently, they are useful tools for string theory and enumerative geometry:
  - $\mathcal{N}=(2,2)$ susy: IIB string theory compactifications.
  - $\mathcal{N}=(0,2)$ susy: heterotic compactifications.
Consider a GLSM with at least one $U(1)$ factor. We have the complexified FI parameter

$$\tau = \frac{\theta}{2\pi} + i\xi$$

which is classically marginal in 2d.

*Schematically*, expectation values of appropriately supersymmetric local operators $\mathcal{O}$ have the expansion

$$\langle \mathcal{O} \rangle \sim \sum_k q^k Z_k(\mathcal{O}), \quad q = e^{2\pi i\tau}.$$ 

The 2d instantons are *gauge vortices.*
GLSM supersymmetric observables

We consider half-BPS local operators.

In the $\mathcal{N} = (2, 2)$ case, we have two choices (up to charge conjugation):

- $[\tilde{Q}_-, \mathcal{O}] = [\tilde{Q}_+, \mathcal{O}] = 0$, chiral ring.
- $[Q_-, \mathcal{O}] = [\tilde{Q}_+, \mathcal{O}] = 0$, twisted chiral ring.

The so-called “twisted” theories [Witten, 1988] efficiently isolate these subsectors: $B$- and $A$-twist, respectively. We will focus on the latter.

In the $(0, 2)$ case, half-BPS operators commute with a single supercharge and there is no chiral ring, in general. However, some interesting models share properties with the $(2, 2)$ case. We will discuss them in the second part of the talk.
$S^2_{\epsilon \Omega}$ correlators for $(2,2)$ theories

We will consider correlations of twisted chiral ring operators on the $\Omega$-deformed sphere,

$$\langle \mathcal{O} \rangle_{S^2_{\Omega}}.$$

This $\Omega$-background constitutes a one-parameter deformation of the $A$-twist at genus zero.

We will derive a formula for GLSM supersymmetric observables on $S^2_{\Omega}$ of the schematic form:

$$\langle \mathcal{O} \rangle = \sum_k q^k \oint_C d^r \sigma Z^{1\text{-loop}}_k (\sigma) \mathcal{O}(\sigma),$$

valid for any standard GLSM. This results simplifies previous computations [Morrison, Plesser, 1994; Szenes, Vergne, 2003] and generalizes them to non-Abelian theories.
Some further motivations

In field theory:

- These 2d $\mathcal{N} = (2, 2)$ theories appear on the worldvolume of surface operators in 4d $\mathcal{N} = 2$ theories.
- Our 2d setup can also be uplifted to 4d $\mathcal{N} = 1$ on $S^2 \times T^2$.

[C.C., Shamir, 2013, Benini, Zaffaroni, 2015, Gadde, Razamat, Willett, 2015]

In string theory or “quantum geometry”:

- Think in terms of a target space $X_d$ with $\xi \sim \text{vol}(X_d)$. New localization results can give new tools for enumerative geometry.
  [Jockers, Kumar, Lapan, Morrison, Romo, 2012]
- The $(0, 2)$ results are relevant for heterotic string compactifications.
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Curved-space \((2, 2)\) supersymmetry

The first step is to define the theory of interest in \textit{curved space}, while preserving some supersymmetry. A systematic way to do this is by coupling to \textit{background supergravity}. [Festuccia, Seiberg, 2011]

Assumption: The theory possesses a \textbf{vector-like} \(R\)-symmetry, \(R_V = R\).

In that case, we have:

\[ j^{(R)}_\mu, \quad S_\mu, \quad T_{\mu\nu}, \quad j^Z_\mu, \quad j^{\tilde{Z}}_\mu, \]
\[ A^{(R)}_{\mu}, \quad \Psi_\mu, \quad g_{\mu\nu}, \quad C_\mu, \quad \tilde{C}_\mu \]

A supersymmetric background corresponds to a non-trivial solution of the \textit{generalized Killing spinor equations}, \(\delta_\zeta \Psi_\mu = 0\).
Supersymmetric backgrounds in 2d

The allowed supersymmetric background are easily classified.

For $\Sigma$ a closed orientable Riemann surface of genus $g$:

- If $g > 1$, we need to identify $A^{(R)}_\mu = \pm \frac{1}{2} \omega_\mu$. Witten’s A-twist.
- If $g = 1$, this is flat space.
- If $g = 0$, we have two possibilities, depending on

$$\frac{1}{2\pi} \int_\Sigma dA^R = 0, \pm 1$$
Supersymmetric backgrounds on $S^2$

On the sphere, we can have:

$$\frac{1}{2\pi} \int_{S^2} dA^R = 0 \ , \quad \frac{1}{2\pi} \int_{S^2} dC = \frac{1}{2\pi} \int_{S^2} d\tilde{C} = 1$$

This was studied in detail in [Doroud, Le Floch, Gomis, Lee, 2012; Benini, Cremonesi, 2012]. In this case, the $R$-charge can be arbitrary but the real part of the central charge, $Z + \tilde{Z}$, is constrained by Dirac quantization.

The second possibility is

$$\frac{1}{2\pi} \int_{S^2} dA^R = 1 \ , \quad \frac{1}{2\pi} \int_{S^2} dC = \frac{1}{2\pi} \int_{S^2} d\tilde{C} = 0$$

This is the case of interest to us. Note that the $R$-charges must be integers, while $Z, \tilde{Z}$ can be arbitrary.
Equivariant $A$-twist, a.k.a. $\Omega$-deformation

Consider this latter case. We preserve two supercharges if the metric on $S^2$ has a $U(1)$ isometry with Killing vector $V^\mu$. This gives a one-parameter deformation of the $A$-twist:

$$Q^2 = 0, \quad \tilde{Q}^2 = 0, \quad \{Q, \tilde{Q}\} = Z + \epsilon_\Omega L_V.$$

The supergravity background reads:

$$ds^2 = \sqrt{g(|z|^2)}dzd\bar{z}, \quad A^{(R)}_\mu = \frac{1}{2}\omega_\mu, \quad C_\mu = \frac{1}{2}\epsilon_\Omega V_\mu, \quad \tilde{C}_\mu = 0.$$

Using the general results of [C.C., Cremonesi, 2014], we can write down any supersymmetric Lagrangian we want.
GLSMs: Lightning review

Let us consider $2d \mathcal{N} = (2, 2)$ supersymmetric GLSM on this $S^2_\Omega$.

We have the following field content:

- **Vector multiplets** $\mathcal{V}_a$ for a gauge group $G$, with Lie algebra $\mathfrak{g}$.
- **Chiral multiplets** $\Phi_i$ in representations $\mathcal{R}_i$ of $\mathfrak{g}$.

We also have interactions dictated by:

- A superpotential $W(\Phi)$
- A twisted superpotential $\hat{W}(\sigma)$, where $\sigma \subset \mathcal{V}$. 
Assumption: The classical twisted superpotential is linear in $\sigma$:

$$\hat{W} = \tau^I \Tr_I(\sigma).$$

That is, we turn on one FI parameter for each $U(1)_I$ factor in $G$.

The FI term often runs at one-loop:

$$\tau(\mu) = \tau(\mu_0) - \frac{b_0}{2\pi i} \log \left( \frac{\mu}{\mu_0} \right),$$

If $b_0 = 0$, we expect an SCFT in infrared.

This $\hat{W}$ preserves a $U(1)_A$ axial $R$-symmetry, broken to $\mathbb{Z}_{2b_0}$ by an anomaly if $b_0 \neq 0$. 
Examples with $G = U(1)$

**Example 1: $\mathbb{CP}^{n-1}$ model.** With $n$ chirals with $Q_i = 1$, $r_i = 0$. \(\tau\) runs at one-loop \((b_0 = n)\), and there is a dynamical scale:

$$\Lambda = \mu q^{\frac{1}{n}}.$$

For $\xi \gg 0$, target space is $\mathbb{CP}^{n-1}$.

**Example 2: The quintic model.** 5 chirals $x_i$ with $Q_i = 1$, $r_i = 0$, and one chiral $p$ with $Q_p = -5$, $r_p = 2$, with a superpotential

$$W = pF(x_i)$$

$F$ is homogeneous of degree 5.

$b_0 = 0$. For $\xi \gg 0$: quintic $CY_3$ in $\mathbb{CP}^4$. 
Non-Abelian examples

Example 3: Grassmanian models. Consider a $U(N_c)$ vector multiplet with $N_f$ chirals in the fundamental. This non-Abelian GLSM flows to the NL$\sigma$M on the Grassmanian $Gr(N_c, N_f)$. The Grassmanian duality

$$Gr(N_c, N_f) \cong Gr(N_f - N_c, N_f)$$

corresponds to a Seiberg-like duality of the GLSMs.
We can also study new classes of CY manifolds inside Grassmanians (and generalizations thereof). [Hori, Tong, 2006; Jockers, Kumar, Morrison, Lapan, Romo, 2012]

Example 4: The Rødland CY\(_3\) model. Consider \(G = U(2)\) with 7 chirals \(\Phi_i\) in the fundamental with \(r_i = 0\) and 7 chirals \(P_\alpha\) in the \(\det^{-1}\) rep. with \(r_\alpha = 2\). We have the baryons

\[
B_{ij} = \epsilon_{a_1 a_2} \Phi_i^{a_1} \Phi_j^{a_2},
\]

charged under the diagonal \(U(1) \subset U(2)\). Let \(G_\alpha(B)\) be polynomials of degree one in \(B_{ij}\). We have a superpotential

\[
W = \sum_{\alpha=1}^{7} P_\alpha G_\alpha(B)
\]

The target space for \(\xi \gg 0\) is a complete intersection in the Grassmanian \(G(2, 7)\) known as the Rødland CY\(_3\).
Supersymmetric observables

When $\epsilon_\Omega = 0$, the only local operators (built from elementary fields) which are $Q$-closed and not $Q$-exact are

$$O(\sigma) ,$$

the gauge-invariant polynomials in $\sigma$. Supersymmetry also ensures that the theory is topological. In particular:

$$\partial_\mu \langle O_x \cdots \rangle = \langle \{ Q, \cdots \} \rangle = 0 .$$

When $\epsilon_\Omega \neq 0$, instead:

$$[Q, \sigma] \sim \epsilon_\Omega V^\mu \Lambda_\mu .$$

Thus $\sigma$ is only $Q$-closed at the fixed points of $V$. 
We can insert $O(\sigma)$ at the north or south poles of $S^2_\Omega$:

$$\langle O_N(\sigma) O_S(\sigma) \rangle$$

This is what we shall compute explicitly, as a function of $q$ and $\epsilon_\Omega$.

*Note*: One can write down a supersymmetric local term:

$$S = \int d^2x (F(\omega)R + \cdots) \sim F(\omega)$$

Thus, correlators $\langle O \rangle$ are only defined up to an overall holomorphic function.
Localization on the Coulomb branch

Localizations

Localization principle: For any $\mathcal{O}$ which is $\mathcal{Q}$-closed,

$$\langle \mathcal{O} \rangle = \langle e^{tS_{\text{loc}}} \mathcal{O} \rangle \quad \text{if} \quad S_{\text{loc}} = \{ \mathcal{Q}, \Psi_{\text{loc}} \}.$$ 

Therefore, we can take $t \to \infty$ and localize the path integral on the saddle point configurations of $S_{\text{loc}}$. The question is how to choose $S_{\text{loc}}$.

We can consider two distinct localizations:

- "Higgs branch" localization: Sum over vortices. [Morrison, Plesser, 1994]
- "Coulomb branch" localization: Contour integral.

We will discuss the latter. The contour picks ‘poles’ on the Coulomb branch corresponding to the vortices.
“Coulomb branch” localization

Choose:

\[ \mathcal{L}_{\text{loc}} = \mathcal{L}_{\text{YM}}. \]

Note: We also localize the matter sector with its standard kinetic term.

The saddles are on the Coulomb branch:

\[ \sigma = \text{diag}(\sigma_a), \quad G \rightarrow H = \prod_{a=1}^{\text{rank}(G)} U(1)_a \]

There is a family of gauge field saddles for each allowed (GNO) flux:

\[ k = (k_a) \in \Gamma_{G^\vee} \]
When $\epsilon_\Omega \neq 0$, there is a non-trivial profile for $\sigma$

$$
\sigma = \sigma(|z|^2),
$$

related to the \textbf{gauge flux} by supersymmetry:

$$
f_{1\bar{1}} = -\frac{1}{\epsilon_\Omega \sqrt{g}} \partial |z|^2 \sigma .
$$

The important feature is that:

$$
\sigma_N = \hat{\sigma} - \epsilon_\Omega \frac{k}{2}, \quad \sigma_S = \hat{\sigma} + \epsilon_\Omega \frac{k}{2},
$$

with $\hat{\sigma}$ a constant mode, over which we need to integrate.
In this localization scheme, we also have gaugino zero modes, \( \lambda, \tilde{\lambda} = \text{constant} \).

The path integral reduces to a supersymmetric ordinary integral:

\[
\langle O_{N,S}(\sigma) \rangle \sim \sum_k \int d\lambda d\tilde{\lambda} \int dD \int d^2\hat{\sigma} \ Z_k(\hat{\sigma}, \hat{\bar{\sigma}}, \lambda, \tilde{\lambda}, D) \ O_{N,S}(\sigma_{N,S})
\]

We refrained from integrating over the constant mode of the auxiliary field \( D \) in the vector multiplet.

We have

\[
Z_k = e^{-S_{\text{cl}}} \ Z_{k}^{1-\text{loop}}.
\]

The one-loop term results from integrating out the chiral multiplets and the \( W \)-bosons. It can be computed explicitly by standard techniques.
The integration over the gaugino zero-modes can be performed implicitly by using the residual supersymmetry of $Z_k$. We have

$$
\delta \sigma = 0 , \quad \delta \tilde{\sigma} = \tilde{\lambda} , \quad \delta \tilde{\lambda} = 0 , \quad \delta \lambda = D , \quad \delta D = 0 .
$$

and therefore

$$
\delta Z_k = \left( \tilde{\lambda} \partial_{\tilde{\sigma}} + D \partial_{\lambda} \right) Z_k = 0 \quad \Rightarrow \quad D \partial_{\lambda} \tilde{\lambda} Z_k \bigg|_{\lambda=\tilde{\lambda}=0} = \partial_{\tilde{\sigma}} Z_k \bigg|_{\lambda=\tilde{\lambda}=0}
$$

This crucial step leads to a contour integral on the $\sigma$-plane:

$$
\int d^2 \lambda d^2 \sigma \ Z \sim \int d^2 \sigma \frac{1}{D} \partial_{\tilde{\sigma}} Z \sim \oint d\sigma \frac{1}{D} Z .
$$

This is like in case of the flavored elliptic genus. [Benini, Eager, Hori, Tachikawa, 2013]
The Coulomb branch formula

The remaining steps are similar to previous works [Benini, Eager, Hori, Tachikawa, 2013; Hori, Kim, Yi, 2014]. We find:

\[
\langle \mathcal{O}_{N,S}(\sigma) \rangle = \frac{1}{|W|} \sum_k \oint_{JK} \prod_{a=1}^{\text{rank}(G)} [d\hat{\sigma}_a q_a^{k_a}] \ Z_k^{1-\text{loop}}(\hat{\sigma}) \mathcal{O}_{N,S} \left( \hat{\sigma} + \frac{1}{2} \epsilon \Omega k \right)
\]

- \( |W| \) denotes the order of the Weyl group.
- The contour is determined by a Jeffrey-Kirwan residue.
- The result depends on the FI parameters explicitly and through the definition of the contour.
- The sum is over all fluxes \( k \)'s. However, only some chambers in \( \{k_a\} \) effectively contribute residues.
The Coulomb branch formula

\[ \langle O_{N,S}(\sigma) \rangle = \frac{1}{|W|} \sum_k \oint_{JK} \prod_{a=1}^{\text{rank}(G)} [d\hat{\sigma}_a q^k_a] \ Z_k^{1-\text{loop}}(\hat{\sigma}) O_{N,S} \left( \hat{\sigma} + \frac{1}{2} \epsilon_{\Omega k} \right) \]

- The distinct \( q_a \)'s are a formal device. We have as many actual \( q \)'s as the number of \( U(1) \) factors in \( g \).
  For instance, for \( G = U(N) \) we have \( q_a = q \) for \( a = 1, \cdots, N \).
- The one-loop term reads

\[ Z_k^{1-\text{loop}}(\hat{\sigma}) = \prod_{\alpha \in g} Z_k^W(\alpha(\hat{\sigma})) \prod_{\rho \in \mathcal{R}} Z_k^\Phi(\rho(\hat{\sigma})) \]

from the \( W \)-bosons and chiral multiplets.
The Coulomb branch formula

\[
\langle O_{N,S}(\sigma) \rangle = \frac{1}{|W|} \sum_k \oint_{J_K} \prod_{a=1}^{\text{rank}(G)} [d\hat{\sigma}_a q^k_a] Z_{k}^{1-\text{loop}}(\hat{\sigma}) O_{N,S} \left( \hat{\sigma} + \frac{1}{2} \epsilon_\Omega k \right)
\]

- For chiral multiplet of $U(1)$ charge $Q$ and $R$-charge $r$, we have

\[
Z_{k}^{\Phi}(\hat{\sigma}) = \epsilon_\Omega Q^{k+1-r} \frac{\Gamma \left( Q \frac{\hat{\sigma}}{\epsilon_\Omega} - Q_k^2 + \frac{r}{2} \right)}{\Gamma \left( Q \frac{\hat{\sigma}}{\epsilon_\Omega} + Q_k^2 - \frac{r}{2} + 1 \right)} = \frac{\epsilon_\Omega Q^{k+1-r}}{(Q \frac{\hat{\sigma}}{\epsilon_\Omega} - Q_k^2 + \frac{r}{2})^{Q_k-r+1}}.
\]

- The $W$-boson $W^\alpha$ contributes exactly like a chiral of $R$-charge $r = 2$ and gauge charges $\alpha$.

- Twisted masses $m_i$ for global symmetries can be introduced in the obvious way.
A-model Coulomb branch formula ($\epsilon_\Omega = 0$)

For $\epsilon_\Omega = 0$, the Coulomb branch formula simplifies to:

$$\langle \mathcal{O}(\sigma) \rangle_0 = \frac{1}{|W|} \sum_k \oint_{JK} \prod_{a=1}^{\text{rank}(G)} [d\hat{\sigma}_a q^k_a] \ Z_k^{1-\text{loop}}(\hat{\sigma}) \mathcal{O}(\hat{\sigma})$$

with

$$Z_k^{1-\text{loop}}(\hat{\sigma}) = (-1)^{\sum_{\alpha>0}(\alpha(k)+1)} \prod_{\alpha>0} \alpha(\hat{\sigma})^2 \prod_i \prod_{\rho_i \in \mathcal{R}_i} \rho_i(\hat{\sigma})^{r_i-1-\rho_i(k)}$$

In the Abelian case, this is a known mathematical result by [Szenes, Vergne, 2003] about volumes of vortex moduli spaces. Our physical derivation generalizes it to non-Abelian GLSMs.
**A-model Coulomb branch formula ($\epsilon_\Omega = 0$)**

In favorable cases, one can do the sum over fluxes explicitly:

$$
\langle O(\sigma) \rangle_0 = \frac{1}{|W|} \oint_{JK} \prod_{a=1}^{\text{rank}(G)} d\hat{\sigma}_a \left[ \frac{1}{1 - e^{2\pi i \partial_{\sigma_a} \hat{W}_{\text{eff}}}} \right] Z_0^{-\text{loop}} (\hat{\sigma}) O (\hat{\sigma})
$$

Here $\hat{W}_{\text{eff}}$ is the one-loop effective twisted superpotential. Finally, if the critical locus

$$e^{2\pi i \partial_{\sigma_a} \hat{W}_{\text{eff}}} = 1, \quad \sigma_a \neq \sigma_b \quad (\text{if } a \neq b)$$

consists of isolated points (such as typically happens for massive theories), we can write the contour integral as

$$
\langle O(\sigma) \rangle_0 = \sum_{\hat{\sigma}^* | d\hat{W} = 0} \frac{Z_0^{-\text{loop}} (\hat{\sigma}^*) O (\hat{\sigma}^*)}{H(\hat{\sigma}^*)}, \quad H = \det \partial_{\sigma_a} \partial_{\sigma_b} \hat{W}
$$

This same formula appeared in [Nekrasov, Shatashvili, 2014] and also in [Melnikov, Plesser, 2005].
**U(1) examples**

**Example 1.** In the $\mathbb{C}P^{n-1}$ model, we have

$$
\langle \mathcal{O}_{N,S}(\sigma) \rangle = \sum_{k=0}^{\infty} q^k \int d\hat{\sigma} \prod_{p=0}^{k} \prod_{i=1}^{n} \frac{1}{\hat{\sigma} - m_i - k/2 + p} \mathcal{O} \left( \hat{\sigma} + \frac{k}{2} \right)
$$

with $m_i$ the twisted masses coupling to the $SU(n)$ flavor symmetry.

In the $A$-model limit and with $m_i = 0$, this simplifies to

$$
\langle \mathcal{O}(\sigma) \rangle_{\epsilon \Omega = 0} = \int d\hat{\sigma} \left( \frac{1}{1 - q\hat{\sigma}^{-n}} \right) \frac{\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^n} = \int d\hat{\sigma} \frac{\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^n - q}
$$

This reproduces known results.
Example 2. For the quintic model, we have

\[ \langle \mathcal{O}_N(\sigma) \rangle = \frac{1}{\epsilon \Omega^3} \sum_{k=0}^{\infty} q^k \int ds \prod_{l=0}^{5k} (5s - l) \frac{\prod_{p=0}^{k} (s + p)^5}{\prod_{p=0}^{k} (s + p)^5} \mathcal{O}(\epsilon \Omega s) \]

In the A-model limit, we obtain

\[ \langle \mathcal{O}(\sigma) \rangle_{\epsilon \Omega = 0} = \sum_{k=0}^{\infty} (-5^5 q)^k \int d\hat{\sigma} \frac{5\hat{\sigma} \mathcal{O}(\hat{\sigma})}{\hat{\sigma}^5} = \frac{5}{1 + 5^5 q} \int d\hat{\sigma} \frac{\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^4} \]

For any \( \epsilon \Omega \), we find \( \langle \sigma^n \rangle = 0 \) if \( n = 0, 1, 2 \), and

\[ \langle \sigma^3 \rangle = \frac{5}{1 + 5^5 q}, \quad \langle \sigma^4 \rangle = 10 \epsilon \Omega \frac{5^5 q}{(1 + 5^5 q)^2}, \ldots \]

in perfect agreement with [Morrison, Plesser, 1994].
Non-Abelian examples

For simplicitly, let us focus on $\epsilon_\Omega = 0$, the A-model.

Example 3. For the Grassmanian model, the residue formula gives

$$
\langle O \rangle_0 = \sum_{k \in \mathbb{Z}_{\geq 0}} q^k Z_k(O),
$$

with

$$
Z_k = \frac{1}{N_c!} \sum_{k_a | \sum_a k_a = k} \frac{(-1)^{2\rho_W(k)}}{(2\pi i)^{N_c}} \oint d^N_c \sigma \frac{\prod_{a,b=1}^{N_c} (\sigma_a - \sigma_b)}{\prod_{a=1}^{N_c} \prod_{i=1}^{N_f} (\sigma_a - m_i)^{1+k_a}} O(\sigma).
$$

Here $m_i$ are twisted masses, corresponding to a $SU(N_f)$-equivariant deformation of $Gr(N_c, N_f)$.

For $m_i = 0$, the numbers $Z_k$ are the $g = 0$ Gromov-Witten invariants.
Example 3, continued. This simplifies explicit formulas found in the math literature. For instance, one finds [C.C., N. Mekareeya, work in progress]

\[ \langle u_1(\sigma)^P \rangle_0 = \delta_{p,(N_f-N_c)N_c+kN_f} q^k \deg(K_{N_f-N_c,N_c}^k) \]

with \( \deg(K_{N_f-N_c,N_c}^k) \) given by [Ravi, Rosenthal, Wang, 1996]

\[ (-1)^{k(N_c+1)+\frac{1}{2}N_c(N_c-1)} [N_c(N_f-N_c+kN_f)]! \sum_{k_a | \sum_a k_a = k} \sum_{\sigma \in S_{N_c}} \prod_{j=1}^{N_c} \frac{1}{(N_f-2N_c-1+j+\sigma(j)+kjN_f)!}, \]

Example: for \( N_c = 2, N_f = 5 \), we have the non-vanishing correlators:

\[ \langle u_1^6 \rangle_0 = 5, \quad \langle u_1^{11} \rangle_0 = 55 q, \quad \langle u_1^{16} \rangle_0 = 610 q^2, \quad \langle u_1^{21} \rangle_0 = 6765 q^3, \quad \cdots \]

This generalizes to the computation of GW invariants of non-CY target space, and is thus complementary of the techniques of [Jockers, Kumar, Lapan, Morrison, Romo, 2012] valid for conformal models.
Example 4. For the Rødland \( CY_3 \) model, our formula reads

\[
\frac{1}{2} \sum_{k_1,k_2=0}^{\infty} q^{k_1+k_2} \int_{(\hat{\sigma}_a=0)} d\hat{\sigma}_1 d\hat{\sigma}_2 (\hat{\sigma}_1 - \hat{\sigma}_2)^2 \frac{(-\hat{\sigma}_1 - \hat{\sigma}_2)^7(1+k_1+k_2)}{\hat{\sigma}_1^7(1+k_1)\hat{\sigma}_2^7(1+k_2)} O(\hat{\sigma}) .
\]

The observables are polynomials in the gauge invariants

\[
u_1(\sigma) = \text{Tr}(\sigma) = \sigma_1 + \sigma_2 , \quad \nu_2(\sigma) = \text{Tr}(\sigma^2) = \sigma_1^2 + \sigma_2^2 .
\]

The only non-vanishing correlators are given by:

\[
\langle \nu_1(\sigma)^3 \rangle = \frac{42 - 14q}{1 - 57q - 289q^2 + q^3} ,
\]

\[
\langle \nu_2(\sigma)\nu_1(\sigma) \rangle = \frac{14 + 126q}{1 - 57q - 289q^2 + q^3} .
\]
Note:

- The Yukawa $\langle u_1(\sigma)^3 \rangle$ was computed by mirror symmetry in [Batyrev et al., 1998]. The second correlator is a new result.
- More generally, the correlators

$$\langle u_n(\sigma) \cdots \rangle, \quad n > 1,$$

in any non-Abelian GLSM are new results which could not be obtained by previous methods (to the best of my knowledge).
- Many more examples can be considered. In particular, one can study the $PAX/PAXY$ models of [Jockers, Kumar, Morrison, Lapan, Romo, 2012] for determinantal $CY$ varieties.
Generalization to (some) $(0, 2)$ theories with a Coulomb branch

$\mathcal{N} = (0, 2)$ observables

A priori, the above would not generalize to $(0, 2)$ theories with only two right-moving supercharges:

$$\{Q_+, \tilde{Q}_+\} = -4P_\tilde{z}.$$ 

Half-BPS operators are $\tilde{Q}_+$-closed, and generally do not form a ring but a chiral algebra:

$$\mathcal{O}_a(z)\mathcal{O}_b(0) \sim \sum_c \frac{f_{abc}}{z^{s_a+s_b-s_c}} \mathcal{O}_c(z),$$

In some favorable cases with an extra $U(1)_L$ symmetry, there exists a subset of the $\mathcal{O}_a$, of spin $s = 0$, with trivial OPE. These pseudo-chiral rings are known as "topological heterotic rings".

[Adams, Distler, Ernebjerg, 2006]
Theories with a $(2,2)$ locus and $A/2$-twist

In this talk, I will focus on $(0,2)$ supersymmetric GLSMs with a $(2,2)$ locus. Schematically, they are determined by the following $(0,2)$ matter content:

- A vector multiplet $\mathcal{V}$ and a chiral $\Sigma$ in the adjoint of the gauge group $G$, with $\mathfrak{g} = \text{Lie}(G)$.
- Pairs of chiral and Fermi multiplets $\Phi_i$ and $\Lambda_i$, in representations $\mathcal{R}_i$ of $\mathfrak{g}$.

The interactions are encoded in two sets of holomorphic functions of the chiral multiplets:

$$\mathcal{E}_i(\Sigma, \Phi) = \Sigma E_i(\Phi), \quad J_i = J_i(\Phi)$$

By assumption, we preserve an additional $U(1)_L$ symmetry classically, which leads to $\mathcal{E}_i$ linear in $\Sigma$

We also turn on an FI term $\tau^I$ for each $U(1)_I$ in $G$. 
Generalization to (some) (0, 2) theories with a Coulomb branch

Theories with a \((2, 2)\) locus and \(A/2\)-twist

We assign the \(R\)-charges:

\[
R_{A/2}[\Sigma] = 0 , \quad R_{A/2}[\Phi_i] = r_i , \quad R_{A/2}[\Lambda_i] = r_i - 1 ,
\]

which is always anomaly-free.

We can define the theory on \(S^2\) (with any metric) by a so-called half-twist:

\[
S = S_0 + \frac{1}{2} R_{A/2} ,
\]

preserving one supercharge \(\tilde{Q} \sim \tilde{Q}_+\). The \(R\)-charges \(r_i\) must be integers (typically, \(r_i = 0\) or \(2\)).

Incidentally, half-twisting is the only way to preserve supersymmetry on the sphere, unlike for \((2, 2)\) GLSM.
The Coulomb branch of theories with a \((2, 2)\) locus

If we have a generic \(\mathcal{E}_i\) potentials, there is a \textbf{Coulomb branch} spanned by the scalar \(\sigma\) in \(\Sigma\):

\[
\sigma = \text{diag}(\sigma_a) .
\]

The matter fields obtain a mass

\[
M_{ij} = \partial_j \mathcal{E}_i \bigg|_{\phi=0} = \sigma_a \partial_j \mathcal{E}_i^a \bigg|_{\phi=0} .
\]

By gauge invariance, \(M_{ij}\) is block-diagonal, with each block spanned by fields with the same gauge charges. We denote these blocks by \(M_\gamma\).

(On the \((2, 2)\) locus, \(M_{ij} = \delta_{ij} Q_i(\sigma)\).)

Let us introduce the notation

\[
P_\gamma(\sigma) = \det M_\gamma \in \mathbb{C}[\sigma_1, \cdots, \sigma_r] , \quad (r = \text{rank}(G))
\]

which is a homogeneous polynomial of degree \(n_\gamma \geq 1\) in \(\sigma\).
A residue formula for $A/2$-model correlators on $S^2$

All the fields are massive on the Coulomb branch, and the localization argument can be carried out similarly to the $(2, 2)$ case, allowing us to compute the $A/2$-twisted correlators on $S^2$ with an half-twist:

$$\langle \mathcal{O}(\sigma) \rangle_{A/2} = \sum_k \frac{1}{|W|} \sum_k \int_{\text{JKG}} \prod_{a=1}^{\text{rank}(G)} [d\sigma_a q^k_a] \ Z^{1-\text{loop}}_k(\sigma) \mathcal{O}(\sigma),$$

with

$$Z^{1-\text{loop}}_k(\sigma) = (-1)^{\sum_{\alpha>0}(\alpha(k)+1)} \prod_{\alpha>0} \alpha(\sigma)^2 \ \prod_{\gamma} \prod_{\rho_{\gamma} \in \mathcal{R}_\gamma} (\det M_{(\gamma, \rho_{\gamma})})^{r_{\gamma}-1-\rho_{\gamma}(k)}.$$ 

Here we have a new residue prescription generalizing the Jeffrey-Kirwan residue relevant on the $(2, 2)$ locus.

In the Abelian case, this reproduces previous results of [McOrist, Melnikov, 2007].
The Jeffrey-Kirwan-Grothendieck residue

In the \((2, 2)\) case, the Jeffrey-Kirwan residue determined a way to pick a middle-dimensional contour in

\[
\mathbb{C}^r - \bigcup_{i \in I} H_i, \quad I = \{i_1, \ldots, i_s\} \ (s \geq r), \quad H_i = \{\sigma_a | Q_i(\sigma) = 0\},
\]

when the integrand has poles on \(H_i\) only.

For generic \((0, 2)\) deformations, we have an integrand with singularities on more general divisors of \(\mathbb{C}^r\):

\[
D_\gamma = \{\sigma_a | P_\gamma(\sigma) = 0\},
\]

which intersect at the origin only.
The Jeffrey-Kirwan-Grothendieck residue

To define the relevant Jeffrey-Kirwan-Grothendieck (JKG) residue, we introduce the data \( P = \{P_\gamma\} \) and \( Q = \{Q_\gamma\} \) of divisors \( D_\gamma \) and associated gauge charges \( Q_\gamma \). The residue is defined by its action on the holomorphic forms:

\[
\omega_S = d\sigma_1 \wedge \cdots \wedge d\sigma_r P_0 \prod_{b \in S} \frac{1}{P_b},
\]

with \( S = \{\gamma_1, \cdots, \gamma_r\} \), which is

\[
\text{JKG-Res}[\eta] \omega_S = \begin{cases} 
\text{sign} \left( \det(Q_S) \right) \text{Res}(0) \omega_S & \text{if } \eta \in \text{Cone}(Q_S), \\
0 & \text{if } \eta \notin \text{Cone}(Q_S),
\end{cases}
\]

with \( \text{Res}(0) \) the (local) Grothendieck residue at the origin.
Generalization to (some) $(0,2)$ theories with a Coulomb branch

The Jeffrey-Kirwan-Grothendieck residue

The Grothendieck residue itself is defined as:

$$\text{Res}_{(0)} \omega_S = \frac{1}{(2\pi i)^r} \oint_{\Gamma_\epsilon} d\sigma_1 \wedge \cdots \wedge d\sigma_r \frac{P_0}{P_{\gamma_1} \cdots P_{\gamma_r}},$$

with the real $r$-dimensional contour:

$$\Gamma_\epsilon = \{ \sigma \in \mathbb{C}^r \mid |P_{\gamma_1}| = \epsilon_1, \cdots, |P_{\gamma_r}| = \epsilon_r \},$$

and it is eminently computable.

Finally, we should take $\eta = \xi_{\text{eff}}^{\text{UV}}$ to cancel the “boundary contributions” from infinity on the Coulomb branch.
Example: \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) with deformed tangent bundle

Consider a theory with gauge group \( U(1)^2 \), two neutral chiral multiplets \( \Sigma_1, \Sigma_2 \) and four pairs of chiral and Fermi multiplets:

\[
\Phi_i, \Lambda_i \ , \ i = 1, 2 \quad Q_i = (1, 0) ,
\Phi_j, \Lambda_j \ , \ j = 1, 2 \quad Q_j = (0, 1) ,
\]

with holomorphic potentials \( J_i = J_j = 0 \) and

\[
\mathcal{E}_i = \sigma_1(A\phi)_i + \sigma_2(B\phi)_i \ , \quad \mathcal{E}_j = \sigma_1(C\phi)_j + \sigma_2(D\phi)_j .
\]

with \( A, B, C, D \) arbitrary \( 2 \times 2 \) constant matrices. This realizes a deformation of the tangent bundle to the \textbf{holomorphic bundle} \( E \)
described by the cokernel:

\[
0 \longrightarrow \mathcal{O}^2 \xrightarrow{(A \quad B)} \mathcal{O}(1, 0)^2 \oplus \mathcal{O}(0, 1)^2 \longrightarrow E \longrightarrow 0
\]
Generalization to (some) \((0, 2)\) theories with a Coulomb branch

\(\mathbb{C}P^1 \times \mathbb{C}P^1\), continued.

We have two sets \(\gamma = 1, 2\):

\[
\text{det } M_1 = \text{det}(A\sigma_1 + B\sigma_2) , \quad \text{det } M_2 = \text{det}(C\sigma_1 + D\sigma_2). 
\]

The Coulomb branch residue formula gives

\[
\langle \sigma_1^{p_1} \sigma_2^{p_2} \rangle = \sum_{k_1, k_2 \in \mathbb{Z}} q_1^{k_1} q_2^{k_2} \oint_{JKG} d\sigma_1 d\sigma_2 \frac{\sigma_1^{p_1} \sigma_2^{p_2}}{(\text{det } M_1)^{1+k_1}(\text{det } M_2)^{1+k_2}}
\]

This can be checked against independent mathematical computations of sheaf cohomology groups.

This result also implies the "quantum sheaf cohomology relations":

\[
\text{det } M_1 = q_1 , \quad \text{det } M_2 = q_2 ,
\]

in the \(A/2\)-ring. This can also be derived from a standard argument on the Coulomb branch. [McOrist, Melnikov, 2008]
Conclusions

- We studied $\mathcal{N} = (2, 2)$ supersymmetric GLSMs on the $\Omega$-deformed sphere, $S^2_\Omega$.
- We derived a simple Coulomb branch formula for the $S^2_\Omega$ observables.
- When $\epsilon_\Omega = 0$, this gives a simple, general formula for $A$-twisted GLSM correlation functions.
  - Some correlators could not be computed with other methods, such as the ones involving $\text{Tr}(\sigma^n)$ in a non-Abelian theory.
  - Even when other methods are possible (e.g. mirror symmetry), the Coulomb branch formula is much simpler.
- The formula is valid in any phase in FI parameter space (away from boundaries), geometric or not.
- Surprisingly, it generalizes off the $(2, 2)$ locus, leading to very interesting new results for some $(0, 2)$ models and the corresponding heterotic geometries.