# Exact solutions of (0,2) Landau-Ginzburg models

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# Motivaiton

Known:

- $\mathcal{N}=(0,0)$: real boson $\phi$, $V(\phi)=\phi^{2n}$
  $\xrightarrow{\text{IR}}$ $n$-th unitary minimal model $\cong \frac{SU(2)_{n-1}\times SU(2)_1}{SU(2)_n}$.
- $\mathcal{N}=(2,2)$: chiral superfield $\Phi$, $W(\Phi)=\Phi^{n+1}$ [Witten]
  $\xrightarrow{\text{IR}}$ $n$-th $\mathcal{N}=2$ minimal model $\cong \frac{SU(2)_{n-1}\times SO(2)_1}{U(1)_{2(n+1)}}$.

Want:

- $\mathcal{N}=(0,2)$ LG ??? $\xrightarrow{\text{IR}}$ (0,2) SCFT ???

# (0,2) Landau-Ginzburg models

| Supercharges: | $Q$ | $\bar{Q}$ |
|---|---|---|
| $U(1)_{\text{rot}}$ | $+1/2$ | $+1/2$ |
| $U(1)_{\text{R}}$ | $+1/2$ | $-1/2$ |

$\{Q, \bar{Q}\} = 2P_+$, superspace: $x^+, x^-, \theta^+, \bar{\theta}^+$.

Multiplets:

- Chiral: $\Phi^b = \phi^b + \psi_+^b \theta^+ + \ldots$, $b = 1 \ldots \mathsf{p}$
- Fermi: $\Psi^a = \psi_-^a + \ldots +$, $a = 1 \ldots \mathsf{q}$

Interaction: $\int d\theta^+ \sum_{a=1}^{\mathsf{q}} \Psi^a \underbrace{J_a(\Phi)}_{\text{holomorphic}}$ +c.c.

$$\rightsquigarrow \sum_a |J_a(\phi)|^2 + \sum_{a,b} \psi_-^a \frac{\partial J_a(\phi)}{\partial \phi^b} \psi_+^b + \text{c.c.}$$

# Finding IR CFT

$$\text{UV data: } \{J_a(\Phi)\}_{a=1\ldots\mathsf{q}} \longrightarrow \text{IR CFT } \mathcal{H} = \bigoplus_{\lambda} \underbrace{\mathcal{H}_\lambda^{\mathsf{L}}}_{\substack{\text{modules of}\\ \text{VOA}^{\mathsf{L}},\ c_{\mathsf{L}}}} \otimes \underbrace{\bar{\mathcal{H}}_\lambda^{\mathsf{R}}}_{\substack{\text{modules of}\\ \mathcal{N}=2\ \text{SVOA}^{\mathsf{R}},\ c_{\mathsf{R}}}}$$

Tools/constraints to determine IR CFT:

- $c$-extrimization $\Rightarrow c_{\mathsf{L}}, c_{\mathsf{R}}$
- Superconformal index
- BPS spectrum ($\supset$ Topological heterotic ring)
- 't Hooft anomalies
- Modular invariance

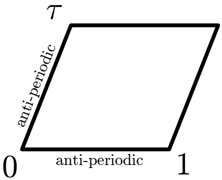
# Modular invariance

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0} \bar{q}^{\bar{L}_0} = \sum_{\lambda} \chi^{\mathsf{L}}_{\lambda}(q) \chi^{\mathsf{R}}_{\lambda}(\bar{q}), \qquad (q = e^{2\pi i \tau})$$

$$\chi^{\mathsf{L}}_{\lambda}(q) = \mathrm{Tr}_{\mathcal{H}^{\mathsf{L}}_{\lambda}}\, q^{L_0}, \qquad \chi^{\mathsf{R}}_{\lambda}(\bar{q}) = \mathrm{Tr}_{\bar{\mathcal{H}}^{\mathsf{R}}_{\lambda}}\, \bar{q}^{\bar{L}_0}$$

$Z(\tau)$ should be modular invariant.

NS sector: $\tau \to -1/\tau$, $\tau \to \tau + 2$.




Gives constraints on possible pairs (VOA$^{\mathsf{L}}$, SVOA$^{\mathsf{R}}$).

(e.g. $(U(1)_5, U(1)_3)$ – modular invariant pairing does not exist)

# $c$-extremization

[Benini-Bobev]

Suppose vacuum is normalizable ($\{\phi \in \mathbb{C}^{\mathtt{p}} \mid J_a(\phi) = 0\}$ is finite)

$$c_{\mathsf{R}} = 3 \sum_{b=1}^{\mathtt{p}} (R[\Phi_b] - 1)^2 - 3 \sum_{a=1}^{\mathtt{q}} (R[\Psi_a])^2$$

$R[\Phi_b], R[\Psi_a]$ – trial R-charges.

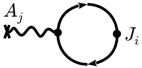
Extremize $c_{\mathsf{R}}$ subject to conditions $R[J_a(\Phi)] + R[\Psi_a] = 1$.

$c_{\mathsf{L}} - c_{\mathsf{R}} = q - p$ (grav. anomaly) $\Rightarrow \quad c_{\mathsf{L}} = \dots$

## 't Hooft anomalies

$U(1)^m$ flavor symmetry (similarly for non-abelian).

$k$ - $m \times m$ anomaly matrix (integral quadratic form on charge lattice $\mathbb{Z}^m$).




$\langle \partial_\mu J_i^\mu \rangle = \sum_j k_{ij} F^j_{\mu\nu} \epsilon^{\mu\nu}$

$$k_{ij} = \sum_{a \in \text{Fermi}} q_i^a q_j^a - \sum_{b \in \text{Chiral}} q_i^b q_j^b$$

$k$ is positive definite $\rightarrow U(1)^m_k$ affine algebra in IR in the left-moving sector.

$$J_i(z) J_j(0) \sim \frac{k_{ij}}{z^2}$$

vertex operators: $e^{\sum_i n^i \int J_i}$, $n \in \mathbb{Z}^m$.

# BPS spectrum

$$\mathcal{H}_{\mathsf{BPS}} \overset{\text{(IR)}}{=} \mathcal{H}|_{\bar{L}_0+\bar{R}_0/2=0} = \bigoplus_{\substack{\lambda\in\ \text{chiral} \\ \text{primaries}}} \mathcal{H}^{\mathsf{L}}_{\lambda}$$

$$\mathcal{H}_{\mathsf{BPS}} \overset{\text{(UV)}}{=} \bar{Q}\text{-cohomology}$$

- Maybe hard to calculate (though in principle possible).
- Simple finite-dimensional subsector: topological heterotic ring

# Topological heterotic ring

[Katz-Sharpe,Adam-Distler-Ernebjerg,Melnikov,...]

Suppose there is a left-moving flavor symmetry $U(1)_{\mathsf{L}} \circlearrowleft \mathcal{H}^{\mathsf{L}}_{\lambda}$ such that in UV: $q_{\mathsf{L}}[\Phi^b] = R[\Phi^b]$, $q_{\mathsf{L}}[\Psi^a] = R[\Psi^a] - 1$.

$$\mathcal{H}_{\mathsf{Top}} \overset{(\mathsf{IR})}{=} \mathcal{H}_{\mathsf{BPS}}|_{L_0 = q_{\mathsf{L}}/2} \; = \; \mathcal{H}|_{\bar{L}_0 + \bar{R}_0/2 = 0,\, L_0 = q_{\mathsf{L}}/2}$$

Can be computed in the UV as Koszul homology:

$$\mathcal{C} = \qquad 0 \xrightarrow{d} \wedge^{\mathsf{q}}\mathcal{E} \xrightarrow{d} \ldots \xrightarrow{d} \wedge^{1}\mathcal{E} \xrightarrow{d} \wedge^{0}\mathcal{E} \xrightarrow{d} 0$$

$$\mathcal{E} = \mathsf{Span}_{\mathbb{C}}\{\bar{\Psi}_a\}_{a=1}^{\mathsf{q}} \otimes \mathbb{C}[\Phi_i] \cong \mathbb{C}[\Phi_i]^{\mathsf{q}}, \qquad d = \sum_{a=1}^{\mathsf{q}} J_a \frac{\partial}{\partial \bar{\Psi}_a},$$

$$\mathcal{H}_{\mathsf{Top}} \overset{(\mathsf{UV})}{=} H_*(\mathcal{C}, d) \equiv \mathrm{Ker}\, d / \mathrm{Im}\, d$$

# Example 1

$$\mathtt{p} = 2,\ \mathtt{q} = 3$$

$$\mathcal{L}_{\mathsf{int}} = \int d\theta^+ \left( \Psi_1 \underbrace{\Phi_1^m}_{J_1} + \Psi_2 \underbrace{\Phi_2^n}_{J_2} + \Psi_3 \underbrace{\Phi_1 \Phi_2}_{J_3} \right) \ + \ \text{c.c.}$$

$$c\text{-extr} \ \Rightarrow \ c_{\mathsf{R}} = 3\,\frac{mn-1}{mn+1},\ c_{\mathsf{L}} = 2\,\frac{2mn-1}{mn+1}$$

$$c_{\mathsf{R}} < 3 \ \Rightarrow \ \mathsf{VOA}^{\mathsf{R}} \ = \ mn\text{-th } \mathcal{N} = 2 \text{ minimal model}$$

$U(1)^2$ flavor symmetry:

| | $q_1$ | $q_2$ |
|---|---|---|
| $\Phi_1$ | $1$ | $0$ |
| $\Phi_2$ | $0$ | $1$ |
| $\Psi_1$ | $-m$ | $0$ |
| $\Psi_2$ | $0$ | $-n$ |
| $\Psi_3$ | $-1$ | $-1$ |

$$k = \begin{pmatrix} m^2 & 1 \\ 1 & n^2 \end{pmatrix}$$

$$\Rightarrow \ \mathsf{VOA}^{\mathsf{L}} \supset U(1)^2_k, \qquad c_{\mathsf{L}} - c_{\mathsf{Sugawara}}(U(1)^2_k) = \underbrace{2\,\frac{mn-2}{mn+1}}_{c(\text{parafermions})} < 2$$

# Example 1, cont.

$$\text{IR CFT} = \Big(\underbrace{\frac{SU(2)_{mn-1}}{U(1)_{2(mn-1)}}}_{\text{parafermions}} \times U(1)_k^2\Big) \otimes \underbrace{\overline{\Big(\frac{SU(2)_{mn-1} \times SO(2)_1}{U(1)_{2(mn+1)}}\Big)}}_{mn\text{-th } \mathcal{N}=2 \text{ minimal model}}$$

$\exists$ modular invariant pairing:

$$Z = \sum_{\alpha=0}^{mn-1} \sum_{\nu \in \mathbb{Z}_{2(mn-1)}} \sum_{a \in \mathbb{Z}_{mn+1}} \chi_{\alpha;\nu}^{SU(2)_{mn-1}/U(1)_{2(mn-1)}} \chi^{U(1)_k^2}_{\underbrace{(ma, n(a+\nu))}_{\in \mathbb{Z}^2/k\mathbb{Z}^2}} \cdot \bar{\chi}_{\alpha;2a+\nu}^{SU(2)_{mn-1} \times SO(2)_1/U(1)_{2(mn+1)}}$$

checks:

- Superconformal index
- Topological heterotic ring

# How to find modular invariant pairing (idea)

cf. [Gannon]

$$\left(\frac{U(1)_A^{m_1}}{U(1)_B^{m_2}}\right) \otimes \overline{\left(\frac{U(1)_C^{m_3}}{U(1)_D^{m_4}}\right)}$$

$\exists$ modular invariant pairing $\Leftrightarrow$ $A \oplus D \overset{\mathbb{Q}}{\sim} C \oplus B$ (as quadratic forms)

$\mathbb{Q}$-linear map $\mapsto$ pairing

(e.g. $U(1)_5 \otimes \overline{U(1)_3}$: $5x^2 = 3y^2 \Rightarrow x = \sqrt{\frac{3}{5}}y \Rightarrow$ no pairing)

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$x = fy$, $f \in \mathrm{Hom}(\mathbb{Q}^{m_2+m_4}, \mathbb{Q}^{m_1+m_3})$, s.t. $(A \oplus D)(x) = (B \oplus C)(y)$.
$Mx = gy$, $M \in \mathbb{Z}$, $g \in \mathrm{Hom}(\mathbb{Z}^{m_2+m_4}, \mathbb{Z}^{m_1+m_3})$, s.t. $(A \oplus D)(gy) = M^2(B \oplus C)(y)$.

$$\chi_{(\mu_1,\mu_3)}^{U(1)_{A\oplus D}^{m_1+m_3}}(gy) = \sum_{\nu_2,\nu_4} \mathcal{A}_{(\mu_1,\mu_3),(\nu_2,\nu_4)} \chi_{(\nu_2,\nu_4)}^{U(1)_{M^2(B\oplus C)}^{m_2+m_4}}(y),$$

$$\chi_{(\mu_2,\mu_4)}^{U(1)_{B\oplus C}^{m_2+m_4}}(My) = \sum_{\nu_2,\nu_4} \mathcal{B}_{(\mu_2,\mu_4),(\nu_2,\nu_4)} \chi_{(\nu_2,\nu_4)}^{U(1)_{M^2(B\oplus C)}^{m_2+m_4}}(y).$$

$$\sum_{\mu_1,\mu_2,\mu_3,\mu_4} \sum_{\nu_2,\nu_4} \mathcal{A}_{(\mu_1,\mu_3),(\nu_2,\nu_4)} \bar{\mathcal{B}}_{(\mu_2,\mu_4),(\nu_2,\nu_4)} \chi_{\mu_1;\mu_2}^{U(1)_A^{m_1}/U(1)_B^{m_2}} \cdot \bar{\chi}_{\mu_3;\mu_4}^{U(1)_C^{m_3}/U(1)_D^{m_4}}$$

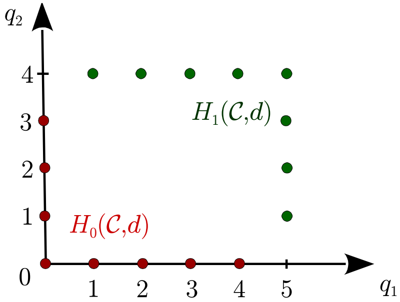
is modular invariant.

# Example 1, cont.

Check that indeed UV and IR calculations of the topological heterotic ring agree: $\mathcal{H}|_{\bar{L}_0+\bar{R}_0/2=0,\, L_0=q_{\mathsf{L}}/2} = H_*(\mathcal{C}, d)$

$$
\begin{aligned}
H_3(\mathcal{C},d) &= 0,\\
H_2(\mathcal{C},d) &= 0,\\
H_1(\mathcal{C},d) &\cong \mathrm{Span}_{\mathbb{C}}\{\bar{\Psi}_3\Phi_1^{m-1}\Phi_2^{b} - \bar{\Psi}_1\Phi_2^{b+1}\}_{b=0}^{n-1} \oplus \mathrm{Span}_{\mathbb{C}}\{\bar{\Psi}_3\Phi_2^{n-1}\Phi_1^{a} - \bar{\Psi}_2\Phi_1^{a+1}\}_{a=0}^{m-2},\\
H_0(\mathcal{C},d) &\cong \mathrm{Span}_{\mathbb{C}}\{\Phi_1^{a}\}_{a=0}^{m-1} \oplus \mathrm{Span}_{\mathbb{C}}\{\Phi_2^{b}\}_{b=1}^{n-1},
\end{aligned}
$$

$(m,n)=(5,4)$



# Example 2

$$\mathtt{p} = 2,\ \mathtt{q} = 2$$

$$\mathcal{L}_{\text{int}} = \int d\theta^+ \left( \Psi_1 \underbrace{(\Phi_1^m + \Phi_2^n)}_{J_1} + \Psi_2 \underbrace{\Phi_1 \Phi_2}_{J_2} \right) \; + \; \text{c.c.}$$

$$\text{IR CFT} \; = \; \Big( \underbrace{\frac{SU(2)_{mn}}{U(1)_{2mn}}}_{\text{parafermions}} \times U(1)_{mn(mn+2)} \Big) \otimes \underbrace{\overline{\Big( \frac{SU(2)_{mn} \times SO(2)_1}{U(1)_{2(mn+2)}} \Big)}}_{mn+1\text{-th } \mathcal{N}=2 \text{ minimal model}}$$

Modular invariant pairing depends on $(m, n)$ individually

(inequivalent $\mathbb{Q}$-linear maps $(x_1, x_2) \mapsto (y_1, y_2)$ such that
$mn(mn+2)\, x_1^2 + 2(mn+2)\, x_2^2 = 2mn\, y_1^2 + y_2^2$)

# Example 2, cont.

$mn = 6$: $\quad 12x_1^2 + x_2^2 \overset{\mathbb{Q}}{\sim} 48y_1^2 + 16y_2^2$

- (m,n)=(1,6)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\mathcal{H} = \bigoplus_{\lambda=0}^{6} \bigoplus_{\alpha\in\mathbb{Z}_{12}} \bigoplus_{s\in\mathbb{Z}_8} \mathcal{H}_{\lambda;\bar{\alpha}}^{SU(2)_6/U(1)_{12}} \otimes \mathcal{H}_{6s+\alpha}^{U(1)_{48}} \otimes \bar{\mathcal{H}}_{\bar{\lambda};2s-\alpha}^{SU(2)_6\times SO(2)_1/U(1)_{16}}$$

- (m,n)=(2,3)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 11 & 5 \\ 30 & -22 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\mathcal{H} = \bigoplus_{\lambda=0}^{6} \bigoplus_{\alpha\in\mathbb{Z}_{12}} \bigoplus_{s\in\mathbb{Z}_8} \mathcal{H}_{\lambda;\bar{\alpha}}^{SU(2)_6/U(1)_{12}} \otimes \mathcal{H}_{6s+5\alpha}^{U(1)_{48}} \otimes \bar{\mathcal{H}}_{\bar{\lambda};2s-5\alpha}^{SU(2)_6\times SO(2)_1/U(1)_{16}}$$

# Example 3

$$\mathcal{L}_{\text{int}} = \int d\theta^+ \sum_{i,j=1}^{N} \Psi^{ij}\Phi_i\Phi_j$$

Nonabelian flavor symmetry: $U(N) \cong \frac{SU(N)\times U(1)}{\mathbb{Z}_N}$.

$$\text{IR CFT} = \left(SU(N)_{N+1} \times U(1)_{N(2N+1)}\right) \otimes \underbrace{\overline{\left(\frac{SO(2N+2)_1 \times SO(N(N+1))_1}{SU(N+1)_N \times U(1)_{(N+1)(2N+1)}}\right)}}_{\text{Kazama-Suzuki coset } [SO(2N+2)/U(N+1)]_{2N+1}}$$

Modular invariant pairing provided by level-rank duality:

$$U(1)_1 \times U(N(N+1))_1 \overset{\text{conformal}}{\supset} \left(SU(N)_{N+1} \times U(1)_{N(2N+1)}\right) \times \left(SU(N+1)_N \times U(1)_{(N+1)(2N+1)}\right)$$

# Summary

- For a number of families of $\mathcal{N} = (0,2)$ LG models we found explicit descritption of the $\mathcal{N} = (0,2)$ SCFT at the IR fixed point in terms of (KS) WZW cosets.
- To determine the IR CFT we use RG protected quantities (BPS spectrum and anomalies) as well as modular invariance
- In particular we give explicit expression of the full partition function of the IR CFT (which is not a supersymetric partition function, it could not be computed by localization)