Tree-level graviton scattering amplitudes in worldline formalism

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Motivation

In string theory, scattering amplitudes between various asymptotic spin states are related. E.g. for bosonic strings

$$\begin{aligned} V_{\text{closed}} &= \; : \exp(ik \cdot X + \epsilon \cdot \partial_z X + \tilde{\epsilon} \cdot \partial_{\bar{z}} X) : \Big|_{multilinear} \\ &= V_{\text{open}} \times V_{\text{open}} \end{aligned}$$

In particular $V_{\text{graviton}} = (V_{\text{gauge}})^2$. This leads to relationships such as

$$3 - point\ graviton\ amplitude = (3 - point\ gauge\ amplitude)^2$$

and KLT (Kawai-Lewellen-Tye) relations, and, more recently to the double-copy relations between gravity and gauge theory scattering amplitudes.

We found that similar relationships exist between vertex operators which create various spin particles when acting on a given spin worldline

spin	0	1	2
0	$e^{ik\cdot x}$	$\frac{1}{2}\epsilon_\mu \dot{x}^\mu e^{ik\cdot x}$	$\frac{1}{2}\epsilon_{\mu\nu}\dot{x}^\mu \dot{x}^\nu e^{ik\cdot x}$
1	NA	$\frac{1}{2}\epsilon_\mu(\dot{x}^\mu + ik_\nu S^{\mu\nu})e^{ik\cdot x}$	$\frac{1}{2}\epsilon_{\mu\nu}\dot{x}^\mu(\dot{x}^\nu + ik_\rho S^{\nu\rho})e^{ik\cdot x}$
2	NA	NA	$\frac{1}{2}\epsilon_{\mu\nu}(\dot{x}^\mu + ik_\rho S^{\mu\rho})(\dot{x}^\nu + ik_\sigma \tilde{S}^{\nu\sigma})$ $e^{ik\cdot x}$

Table: The linearized vertex operators of different interactions

where $S^{\mu\nu} = 2\bar{\psi}^{[\mu}\psi^{\nu]}$ accounts for the spin degrees of freedom on the worldline.

First-quantized fields: worldlines

Bern and Kosower (1992): infinite tension limit of string amplitudes
Strassler (1992) recovered their master formula and rules by rewriting the one-loop amplitudes as a path integral over point-particle coordinates.

This is the beginning of the "worldline formalism".

First-quantized fields: worldlines

Consider a spin 0 (scalar) particle.
The position-space propagator can be cast as a path integral over a "worldline":

$$
\begin{aligned}
G(X', X) &= \langle X' | \frac{1}{\Box - m^2 + i\epsilon} | X \rangle \\
&= \int_0^\infty dT \langle X' | \exp(iT(\Box - m^2 + i\epsilon)) | X \rangle \\
&= \int_0^\infty dT \int_{-\infty}^{\infty} \frac{d^4 p}{(2\pi)^4} e^{ip\cdot(X'-X)} e^{-iT(p^2+m^2-i\epsilon)} \\
&= \int_0^\infty dT \int_{x^\mu(0)=X^\mu}^{x^\mu(T)=X'^\mu} \mathcal{D}x(\tau) \int \mathcal{D}p(\tau) e^{-i\int_0^T d\tau (p^2(\tau)+m^2-p\cdot\dot{x}-i\epsilon)} \\
&= \int_0^\infty dT \int_{x^\mu(0)=X^\mu}^{x^\mu(T)=X'^\mu} \mathcal{D}x(\tau) \exp\left(i \int_0^T d\tau \mathcal{L}_0[\dot{x}] \right) .
\end{aligned}
$$

Accounting for the interactions with a background field, the propagator becomes the "dressed propagator" $\Gamma[X', X]$.

Worldline Formalism: Path Integral form of the dressed propagator

$$\Gamma[X',X] = \int_0^\infty dT \int_{x^\mu(0)=X^\mu}^{x^\mu(T)=X'^\mu} \mathcal{D}x(\tau)\ \exp\left[-\int_0^T d\tau \left(\mathcal{L}_0[\dot{x}] - V[x,\dot{x}]\right)\right]$$

Perform a plane-wave expansion for the background field

$$V = \sum_{i=2}^{N-1} V_i e^{ik_i\cdot x(\tau)}$$

The interaction with $N-2$ background particles is given by

$$\Gamma_N[X',X] = \int_0^\infty dT \int_{x(0)=X'}^{x(T)=X} \mathcal{D}x(\tau)\ e^{-\int_0^T d\tau \mathcal{L}_0} \prod_{i=2}^{N-1} \left(\int_0^T d\tau_i\ V_i(\tau_i)\right)$$

Then $\Gamma_N[p,p']$ will be of the form $\Gamma_N[p,p'] = \frac{1}{p^2+m^2}\mathcal{A}_N\frac{1}{p'^2+m^2}$
$\mathcal{A}_N$ yields the scattering amplitude.

Transition amplitude to dressed propagator

$$
\begin{aligned}
K(X', T; X, 0) &= \langle X' | U(T, 0) | X \rangle \\
&= \langle X', T || U_I(T; 0) | X, 0 \rangle \\
&= \int_{x(0)=X}^{x(T)=X'} \mathcal{D}x(\tau)\; e^{-\int_0^T d\tau \mathcal{L}_0} \prod_{i=2}^{N-1} \left(\int_0^T d\tau_i \; V_i(\tau_i) \right)
\end{aligned}
$$

$$
\boxed{\Gamma[X, X'] = \int_0^\infty dT\, K(X', T; X, 0)}
$$

Dressed propagator to scattering amplitudes

$$\Gamma(X,X') = \int_0^\infty dT \langle X', T | U_I(T;0) | X, 0\rangle$$

For an N-point-function, expand the interacting picture U_i in the background potential keep the multilinear term Γ_N.
Example: 3-point function:

$$\begin{aligned}\Gamma_3[p,p'] &= \int_0^\infty dT \int_0^T d\tau_2 \langle p', T | e^{i(k_2\cdot x(\tau_2))} | p, 0\rangle \\ &= \int_0^\infty d\tau_{32} \int_0^\infty d\tau_{21} \langle p' | e^{-H_0\tau_{32}} V(k_2, \tau = 0) e^{-H_0\tau_{21})} | p\rangle \\ &= \frac{1}{p^2+m^2} \mathcal{A}_3 \frac{1}{p'^2+m^2}\end{aligned}$$

Note that the T integral and one integral over the interaction potential yielded the free propagators.

4-point scattering amplitudes

Consider a 4-point amplitude, with p and p' on-shell

$$\begin{aligned}\mathcal{A}_4 &= \int_0^{+\infty} d\tau_{32} \langle p| V_3\ e^{-H_0 \tau_{32}}\ V_2 |p'\rangle + (2 \leftrightarrow 3) \\ &= \langle p| e^{-H_0 \tau_{43}}\ V_3 (\int_0^{+\infty} d\tau_{32}\ e^{-H_0 \tau_{32}})\ V_2\ e^{-H_0 \tau_{21}} |p'\rangle + (2 \leftrightarrow 3) \\ &= \int_{-\infty}^{+\infty} d\tau_{32} \langle \mathcal{T}\{ V_4(\tau_4) V_3(\tau_3) V_2(\tau_2) V_1(\tau_1) \} \rangle\end{aligned}$$

where $\tau_1 < \tau_{2,3} < \tau_4$ and where V_4 and V_1 are two vertex operators that create $|p\rangle$ and $|p'\rangle$ when acting on the vacuum. We can also set $\tau_2 = 0$.

Scattering amplitudes in worldline formalism

In general, the scattering amplitudes are of the form

$$\mathcal{A}_N = \lim_{\substack{\tau_N \to +\infty \\ \tau_1 \to -\infty}} \left(\prod_{i=3}^{N-1} \int_{-\infty}^{\infty} d\tau_i\right) \langle \mathcal{T}\{V_N(\tau_N) V_{N-1}(\tau_{N-1}) \ldots V_2(0) V_1(\tau_1)\}\rangle$$

Similar to Feynman diagrams in QFT, we can represent the expression with a specific ordering of $\{\tau_i\}$ diagrammatically as

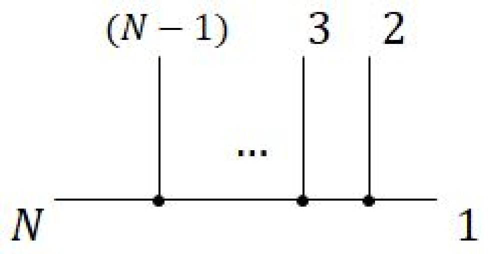


Figure: A part of $\mathcal{A}_N$, with ordering $\tau_{N-1} > \tau_{N-2} \ldots > \tau_3 > 0$

$\mathcal{A}_\mathcal{N}$ has legs $3, 4 \ldots (N-1)$ freely sliding on the worldline.

This expression is similar to how scattering amplitudes are computed in string theory. To evaluate it we only need the 2-point-function of the bosonic coordinates (on the infinite line) is

$$\langle x^\mu(\tau) x^\nu(\tau') \rangle = -\frac{1}{2}\eta^{\mu\nu}|\tau - \tau'| \,.$$

For particles with spins, we will add fermions to the worldline action.

$N = 2S$ where N=supersymmetry in the w.l. and S=spin.

The fermion correlation functions (on the infinite line) are

$$\langle \bar{\psi}^a(\tau) \psi^b(\tau') \rangle = \eta^{ab}\Theta(\tau - \tau') \,.$$

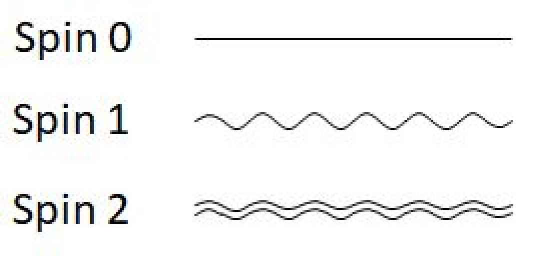


Figure: Worldlines of particles with different spins

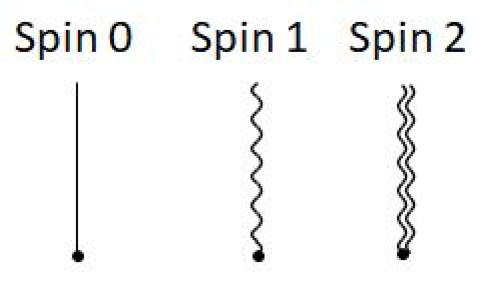


Figure: Linear vertex operators of particles with different spins

Scalar QED

Start with the worldline action for a scalar interacting with a background photon field,

$$S = \int d\tau \left(\frac{1}{2}\dot{x}^2(\tau) - \frac{i}{2}\dot{x}^\mu(\tau) A_\mu(x(\tau)) \right)$$

This gives the photon vertex operator

$$V_j(\tau) = -\frac{i}{2}(\epsilon_j \cdot \dot{x}(\tau)) e^{ik_j \cdot x(\tau)}, \; j = 2, 3 \ldots (N-1)\,.$$

Using $\langle e^A e^B \rangle = e^{\langle AB \rangle}$ the 3-point function is

$$\begin{aligned}
\mathcal{A}_3 &= \langle V_3(+\infty) V_2(0) V_1(-\infty) \rangle \\
&= \langle e^{ik_3 \cdot x(+\infty)} \; (-\frac{i}{2}) \epsilon_2 \cdot \dot{x}(0) e^{ik_2 \cdot x(0)} \; e^{ik_1 \cdot x(-\infty)} \rangle \\
&= -\frac{1}{2}\epsilon_2 \cdot (-\frac{1}{2}k_3 + \frac{1}{2}k_1) e^{\sum_{i>j} \frac{1}{2} k_i \cdot k_j (\tau_i - \tau_j)} \\
&= -\frac{1}{4}\epsilon_2 \cdot (k_1 - k_3)\,,
\end{aligned}$$

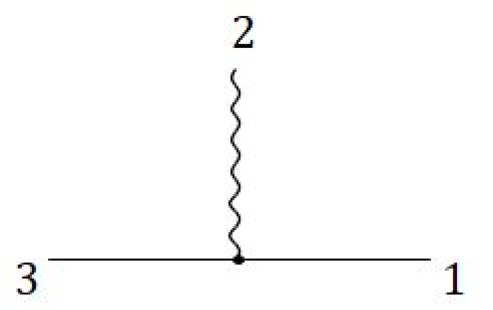


Figure: 3-point amplitude of scalar QED

$$
\begin{aligned}
\mathcal{A}_4 &= \int_{-\infty}^{\infty} d\tau \langle \mathcal{T}\{V_4(+\infty)V_3(\tau)V_2(0)V_1(-\infty)\}\rangle \\
&= \frac{-1}{4}\int_{-\infty}^{\infty} d\tau \langle \mathcal{T}\{e^{ik_4\cdot x(+\infty)}\epsilon_3\cdot\dot{x}(\tau)e^{ik_3\cdot x(\tau)}\epsilon_2\cdot\dot{x}(0)e^{ik_2\cdot x(0)}\; e^{ik_1\cdot x(-\infty)}\}\rangle \\
&= \frac{1}{4}(\epsilon_3\cdot k_4)(\epsilon_2\cdot k_1)\int_0^{+\infty} d\tau \langle e^{ik_4\cdot x(+\infty)}\; e^{ik_3\cdot x(\tau)}\; e^{ik_2\cdot x(0)}\; e^{ik_1\cdot x(-\infty)}\rangle \\
&+ \frac{1}{4}(\epsilon_2\cdot k_4)(\epsilon_3\cdot k_1)\int_{-\infty}^{0} d\tau \langle e^{ik_4\cdot x(+\infty)}\; e^{ik_2\cdot x(0)}\; e^{ik_3\cdot x(\tau)}\; e^{ik_1\cdot x(-\infty)}\rangle \\
&- \frac{1}{4}(\epsilon_3\cdot\epsilon_2)\int_{-\infty}^{+\infty} d\tau\; \delta(\tau)\langle \mathcal{T}\{e^{ik_4\cdot x(+\infty)}\; e^{ik_3\cdot x(\tau)}\; e^{ik_2\cdot x(0)}\; e^{ik_1\cdot x(-\infty)}\}\rangle \\
&= -\frac{1}{2s}(\epsilon_3\cdot k_4)(\epsilon_2\cdot k_1) - \frac{1}{2u}(\epsilon_2\cdot k_4)(\epsilon_3\cdot k_1) - \frac{1}{4}(\epsilon_3\cdot\epsilon_2)\,,
\end{aligned}
$$

where $s = -(k_1+k_2)^2$, $u = -(k_1+k_3)^2$ and $t = -(k_1+k_4)^2$ are Mandelstam variables, and we used the on-shell conditions for all four particles.

As a general feature of the worldline formalism, the two factors
$\int_0^{+\infty} d\tau \langle e^{ik_4\cdot x(+\infty)} e^{ik_3\cdot x(\tau)} e^{ik_2\cdot x(0)} e^{ik_1\cdot x(-\infty)} \rangle$
and
$\int_{-\infty}^{0} d\tau \langle e^{ik_4\cdot x(+\infty)} e^{ik_2\cdot x(0)} e^{ik_3\cdot x(\tau)} e^{ik_1\cdot x(-\infty)} \rangle$
yield the poles $-2/s$ and $-2/u$ respectively.

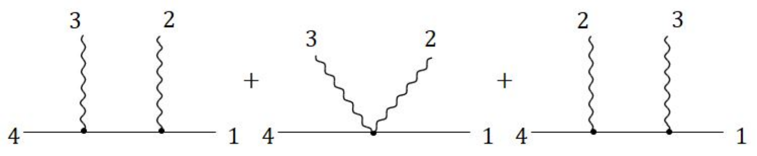


Figure: 4-point amplitude of scalar QED

Coupling to gravity

The Euclidean classical action of the $O(N)$ spinning particle in a curved background is given by (Howe et al 1988, Bastianelli et al 2011)

$$\int d\tau \left[\frac{1}{2}g_{\mu\nu}(\dot{x}^\mu\dot{x}^\nu+b^\mu c^\nu+a^\mu a^\nu)+\frac{1}{2}\psi_{ia}D_\tau\psi_i^a+\alpha R_{abcd}\psi_i^a\psi_i^b\psi_j^c\psi_j^d\right]$$

where $i=1\ldots N$, $D_\tau\psi_i^a=\partial_\tau\psi_i^a+\dot{x}^\mu\omega_\mu^{ab}\psi_{ib}$ and where the Grassmann-even a^μ and the Grassmann-odd b^μ, c^μ are ghosts introduced to make up for the $\sqrt{-g}$ factor in the general-covariant path integral measure $\int \mathcal{D}x^\mu\sqrt{g(x)}$ (van Nieuwenhuizen, Bastianelli).

Regularization and renormalization needed

In computing in the worldline formalism one encounters products of distributions $\delta(\tau - \tau')\Theta(\tau - \tau')$ which result from contractions of the type $\langle \dot{X}\dot{X}\rangle\langle X\dot{X}\rangle$ etc. These expressions are defined through a regularization scheme. Then one needs to add counterterms(scheme dependent) to the action in order to get matching results with the quantum field theory corresponding to the first-quantized worldline action.

The counterterms depend on the regularization scheme. If we use dimensional regularization, the counterterms will be in a general-covariant form $V_{DR} = \beta R$. Then, the worldline action in dimensional regularization is:

$$
\begin{aligned}
S_{DR} &= \int d\tau \left[\frac{1}{2} g_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu + b^\mu c^\nu + a^\mu a^\nu) + \frac{1}{2}\psi_{ia} D_\tau \psi_i^a \right. \\
&\quad \left. + \alpha R_{abcd} \psi_i^a \psi_i^b \psi_j^c \psi_j^d + \beta R \right]
\end{aligned}
$$

Renormalization and fixing of the counterterm coeff.

Previous results in the literature have been derived starting from the $O(N)$ supersymmetric spinning particle action. These concern the so-called transition amplitude, which is the probability amplitude for the particle in some initial state, specified by $x^\mu(\tau{=}0) = X^\mu, \psi^i(\tau{=}0) = \Psi^i$, to evolve at some later time $\tau = T$ into a final state specified by $x^\mu(\tau{=}T) = X'^\mu, \bar{\psi}'^i(\tau{=}T) = \bar{\Psi}'^i$:

$$K(X', \bar{\Psi}'; X, \Psi) = \langle X', \bar{\Psi}' \,| e^{iT\hat{H}} |\, X, \Psi \rangle \,,$$

The counterterm β was previously determined by Bastianelli et al. (2011) by computing the small-T expansions of the matrix element of $\exp(-TH)$ acting coherent fermionic states as in the transition amplitude and matching with the path integral computation, using dim reg.

Our approach

We fix the coupling with background gravity and the counterterms by computing the 3-point vertex with the background graviton off-shell. Later we verify these coefficients by computing 4-point scattering amplitudes. By matching with the corresponding field theory tree diagrams (e.g. minimally coupled scalar or Yang-Mills theory in a curved background) we obtained the following results:

N	α	β
0	NA	$-\frac{1}{8}$
2	$-\frac{1}{8}$	$-\frac{1}{8}$

Previous results have $\beta = 0$ for $N = 2$.

Scalar coupled to gravity

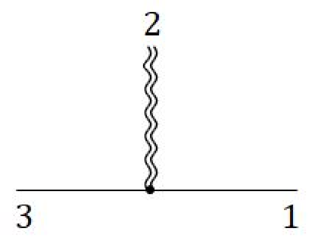
Vertex operators creating the asymptotic states:

$$V_1(\tau_1) = e^{ik_1\cdot x(\tau_1)}$$
$$V_3(\tau_3) = e^{ik_3\cdot x(\tau_3)}$$

Vertex operator for the emission of an off-shell graviton:

$$V_2^{\mu\nu}(\tau_2) = -\frac{1}{2}\dot{x}^{\mu}(\tau_2)\dot{x}^{\nu}(\tau_2)e^{ik_2\cdot x(\tau_2)} - \beta(R^{(1)})^{\mu\nu}\,,$$

$(R^{(1)})^{\mu\nu}$ is defined by the linearized expansion of the background Ricci scalar: $R[\eta_{\mu\nu} + h_{\mu\nu}] = h_{\mu\nu}(R^{(1)})^{\mu\nu} + \mathcal{O}[h^2]$.
$\Rightarrow (R^{(1)})^{\mu\nu} = \eta_{\mu\nu}k^2 - k_\mu k_\nu$



When $\beta = -\frac{1}{8}$,

$$\begin{aligned}
\mathcal{A}_3(k_1, k_3; k_2) &= \langle V_3(\tau_3) V^{\mu\nu}(\tau_2) V_1(\tau_1) \rangle \\
&= -\frac{1}{2}(-1)\frac{1}{4}(k_3 - k_1)^\mu (k_3 - k_1)^\nu \\
&\quad +\frac{1}{8}(-1)[k_2^\mu k_2^\nu - \eta^{\mu\nu} k_2^2] \\
&= -\frac{1}{8}[(k_3 + k_1)^\mu (k_3 + k_1)^\nu \\
&\quad -\eta^{\mu\nu}(k_1 + k_3)^2 - (k_3 - k_1)^\mu (k_3 - k_1)^\nu] \\
&= -\frac{1}{4}(k_1^\mu k_3^\nu + k_3^\mu k_1^\nu - \eta^{\mu\nu} k_1 \cdot k_3)
\end{aligned}$$

This matches the QFT vertex $h_{\mu\nu} T^{\mu\nu}$ for a minimally coupled scalar. (This was noted in Mogull, Plefka and Steinhoff 2010.) 4-point functions are reproduced as well.

Photon coupled to gravity

The following vertex operators are needed

$$
\begin{aligned}
V_1(\tau_1) &= \epsilon_{1\mu}\bar{\psi}^\mu(\tau_1)e^{ik_1\cdot x(\tau_1)} \\
V_3(\tau_3) &= \epsilon_{3\mu}\psi^\mu(\tau_3)e^{ik_3\cdot x(\tau_3)} \\
V_2^{\mu\nu}(\tau_2) &= -\frac{1}{2}\dot{x}^\mu(\tau_2)(\dot{x}^\nu(\tau_2)+ik_{2\sigma}S^{\nu\sigma}(\tau_2))e^{ik_2\cdot x(\tau_2)} \\
&\quad -\alpha[R^{(1)}_{abcd}S^{ab}S^{cd}]^{\mu\nu}-\beta(R^{(1)})^{\mu\nu} \qquad (1) \\
&= -\frac{1}{2}\dot{x}^\mu(\tau_2)(\dot{x}^\nu(\tau_2)+ik_{2\sigma}S^{\nu\sigma}(\tau_2)) \\
&\quad +4\alpha[R^{(1)}_{ad}\bar{\Psi}^a\Psi^d]^{\mu\nu}-\beta(R^{(1)})^{\mu\nu}\,, \qquad (2)
\end{aligned}
$$

$[R^{(1)}_{ad}\bar{\Psi}^a\Psi^d]^{\mu\nu}$ is the coefficient of $h_{\mu\nu}$ of the linearized $R_{ad}[\eta_{\mu\nu}+h_{\mu\nu}]\bar{\Psi}^a\Psi^d$. In going from (1) to (2) we normal-ordered the spin.

When $\alpha = -\frac{1}{8}, \beta = -\frac{1}{8}$, the vertex operator becomes

$$
\begin{aligned}
\mathcal{A}_3 \;=\; & \frac{1}{4}(k_3 - k_1)^\mu \left(\frac{1}{2}(k_3 - k_1)^\nu (\epsilon_3 \cdot \epsilon_1) - \epsilon_3^\nu \epsilon_1 \cdot k_3 + \epsilon_1^\nu \epsilon_3 \cdot k_1 \right) \\
& + \frac{1}{4}\Big(\epsilon_3 \cdot k_1 \epsilon_1^\mu (k_1 + k_3)^\nu + \epsilon_1 \cdot k_3 \epsilon_3^\mu (k_1 + k_3)^\nu \\
& \quad - \epsilon_3 \cdot k_1 \epsilon_1 \cdot k_3 \eta^{\mu\nu} - (k_1 + k_3)^2 \epsilon_3^\mu \epsilon_1^\nu \Big) \\
& - \frac{1}{8}\Big((k_3 + k_1)^\mu (k_3 + k_1)^\nu - \eta^{\mu\nu}(k_3 + k_1)^2 \Big)(\epsilon_1 \cdot \epsilon_3)\,,
\end{aligned}
$$

which matches the field theory. The 4-point amplitude is also reproduced.

Worldline action for gravitons

- A free, massless spin 2 particle is described by an $N = 4$ supersymmetric worldline action which is also $O(4)$ symmetric (Howe 1988).
- Spin S free particles are described through $N = 2S$ supersymmetric worldline actions exhibiting $O(N)$ symmetry (Bastianelli et al.2011)
- Coupling with background gravity imposes restrictions for the background geometry if worldline supersymmetry is to be preserved:

  - Howe, Penati, Pernici and Townsend (1988) concluded that $N = 4$ supersymmetry constrains the background curvature to vanish;

  -Kuzenko and Yarevskaya (1995) showed that $N \geq 4$ supersymmetry could be preserved in an anti de-Sitter background;

  -Bastianelli et al(2008) : N-supersymmetric particle can be consistently coupled with a conformally flat background).
- Recently, Bonezzi et al (2018) used BRST to construct the on-shell background graviton emission vertex from a graviton worldline. Nonetheless, an action describing the coupling of higher spin ($S \geq 2$) particles with generic background gravity is unknown.

Our approach : constrain the worldline action by requiring that the term linear in the off-shell background field yields a 3-point vertex that matches the 3-point QFT vertex. In particular, we found that to reproduce general relativity's cubic graviton vertex, interpreted as the emission of an off-shell graviton from the worldline, the coupling to background gravity must break the $O(4)$ symmetry to $O(2) \times O(2)$.

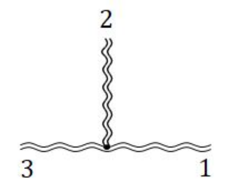
$$
\begin{aligned}
\int d\tau \Bigg[& \frac{1}{2} g_{\mu\nu} (\dot{x}^\mu \dot{x}^\nu + b^\mu c^\nu + a^\mu a^\nu) \\
& + \frac{1}{2} \bar{\Psi}_a \partial_\tau \Psi^a + \frac{1}{2} \Psi_a \partial_\tau \bar{\Psi}^a + \frac{1}{2} \bar{\tilde{\Psi}}_a \partial_\tau \tilde{\Psi}^a + \frac{1}{2} \tilde{\Psi}_a \partial_\tau \bar{\tilde{\Psi}}^a \\
& + \frac{1}{2} \dot{x}^\mu \omega_{\mu ab} (S^{ab} + \tilde{S}^{ab}) + 2\alpha_1 R_{abcd} S^{ab} \tilde{S}^{cd} \\
& + \alpha_2 R_{abcd} (S^{ab} S^{cd} + \tilde{S}^{ab} \tilde{S}^{cd}) + \beta R \Bigg] ,
\end{aligned}
$$

$$
\boxed{\alpha_1 = -\alpha_2 = -1/8 \qquad \beta = \tfrac{3}{8}}
$$

Graviton worldline and emitted off-shell graviton

Here we have (since the ghosts don't contribute to the 3-point vertex, we simply ignore them):

$$
\begin{aligned}
V_1(\tau_1) &= \epsilon_{1\mu\nu}\bar{\Psi}^\mu(\tau_1)\bar{\tilde{\Psi}}^\nu(\tau_1)e^{ik_1\cdot x(\tau_1)} \\
V_3(\tau_3) &= \epsilon_{3\mu\nu}\Psi^\mu(\tau_3)\tilde{\Psi}^\nu(\tau_3)e^{ik_3\cdot x(\tau_3)} \\
V_2^{\mu\nu}(\tau_2) &= -\frac{1}{2}(\dot{x}^\mu(\tau_2) + ik_{2\rho}S^{\mu\rho}(\tau_2))(\dot{x}^\nu(\tau_2) + ik_{2\sigma}\tilde{S}^{\nu\sigma}(\tau_2))e^{ik_2\cdot x(\tau_2)} \\
&\quad + \frac{1}{2}[R^{(1)}_{ad}(\bar{\Psi}^a\Psi^d + \bar{\tilde{\Psi}}^a\tilde{\Psi}^d)]^{\mu\nu} \\
&\quad - \frac{3}{8}(R^{(1)})^{\mu\nu} .
\end{aligned}
$$



We can now compute the 3-point function

$$\begin{aligned}
\mathcal{A}_3 \;=\; & \frac{1}{2}\left(\frac{1}{2}(k_3-k_1)^\mu(\epsilon_3\cdot\epsilon_1) - \epsilon_3^\mu \epsilon_1\cdot k_3 + \epsilon_1^\mu \epsilon_3\cdot k_1\right) \\
& \qquad \times\left(\frac{1}{2}(k_3-k_1)^\nu(\epsilon_3\cdot\epsilon_1) - \epsilon_3^\nu \epsilon_1\cdot k_3 + \epsilon_1^\nu \epsilon_3\cdot k_1\right) \\
& -\frac{1}{2}(\epsilon_3\cdot\epsilon_1)\Bigg(\epsilon_3\cdot k_1 \epsilon_1^\mu (k_1+k_3)^\nu + \epsilon_1\cdot k_3 \epsilon_3^\mu (k_1+k_3)^\nu \\
& \qquad -\epsilon_3\cdot k_1 \epsilon_1\cdot k_3 \eta^{\mu\nu} - (k_1+k_3)^2 \epsilon_3^\mu \epsilon_1^\nu\Bigg) \\
& +\frac{3}{8}(\epsilon_3\cdot\epsilon_1)^2\left((k_3+k_1)^\mu(k_3+k_1)^\nu - \eta^{\mu\nu}(k_3+k_1)^2\right),
\end{aligned}$$

Glimpse of double copy

For the emission of an on-shell graviton we have:

$$V_2^{\mu\nu}(\tau_2) = -\frac{1}{2}(\dot{x}^\mu(\tau_2) + ik_{2\rho}S^{\mu\rho}(\tau_2))(\dot{x}^\nu(\tau_2) + ik_{2\sigma}\tilde{S}^{\nu\sigma}(\tau_2)),$$

(the linearized Ricci tensor vanishes and the linearized term $R_{abcd}S^{ab}\tilde{S}^{cd}$ leads to the squaring we see above if $\alpha_1 = -\frac{1}{8}$.)

$$\boxed{V_2^{\mu\nu}(\tau_2) = -2\bar{V}^\mu(\tau_2)\bar{V}^\nu(\tau_2)}$$

where $\bar{V}^\mu(\tau) = -\frac{1}{2}(\dot{x}^\mu(\tau) + ik_{2\rho}S^{\mu\rho}(\tau))$ is the vertex operator for gauge boson self-interaction.

This leads directly to the double copy relation between the 3-point amplitudes:

$$\langle V_3(\tau_3)V_2(\tau_2)V_(\tau_1)\rangle_{gravity} = -2\langle\bar{V}_3(\tau_3)\bar{V}_2(\tau_2)\bar{V}_(\tau_1)\rangle^2_{gauge\ \ boson}.$$

Higher N-point

New features:

- The emission vertices may become non-linear "pinch operators"
- The $a.b.c$ ghosts will play a role
- We need to use dim-reg to get well-defined expressions

$$\int d^D\tau \frac{1}{2} g_{\mu\nu}\partial_I x^\mu \partial_I x^\nu + \dots$$

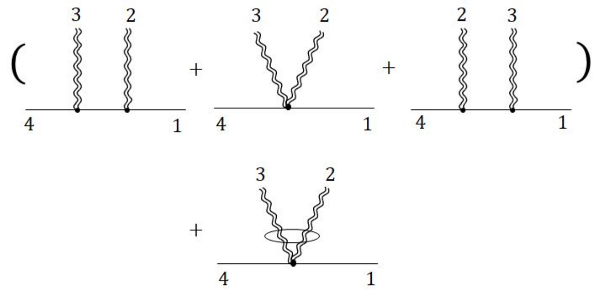
$$\langle x^\mu(\tau) x^\nu(\tau')\rangle = \eta^{\mu\nu}\Delta(\tau,\tau'),\ \ \Box_D \Delta(\tau,\tau') = -\delta^D(\tau-\tau'),\ \ \partial_{IJ}\Delta \neq \partial_I\partial_I\Delta$$

$$\int d\tau_{32}\ddot{\Delta}(\tau_{32})\dot{\Delta}(\tau_{32})e^{\Sigma} \longrightarrow \int d^D\tau \partial_I\partial_J\Delta\partial_I\Delta e^{\Sigma} = -\tfrac{1}{2}\int d^D\tau(\partial_J e^{\Sigma})(\partial_I\Delta)^2$$

- If we place the emitted gravitons on-shell, we need to add diagrams called "lower-trees" which essentially ensure that the gravitons are solving their equations of motion.

Example: 4-point scattering amplitude from a scalar worldline and two on-shell gravitons

$$
\begin{aligned}
V_{23}(\tau)\Big|_{pinch} &= \frac{1}{8}R^{(2)} \\
&= \frac{1}{8}\left[\frac{3}{4}(\partial_\mu h_{\alpha\beta})^2 - \frac{1}{2}(\partial^\alpha h^{\beta\mu})(\partial_\beta h_{\mu\alpha})\right] \\
&= \left[\frac{3t}{32}(\epsilon_2\cdot\epsilon_3)^2 + \frac{1}{8}(\epsilon_2\cdot k_3)(\epsilon_3\cdot k_2)(\epsilon_2\cdot\epsilon_3)\right] e^{i(k_2+k_3)\cdot x(\tau)} .
\end{aligned}
$$



$$
\begin{aligned}
A_{4,sg} &= \lim_{\substack{\tau_4 \to +\infty \\ \tau_1 \to -\infty}} \int_{-\infty}^{+\infty} d\tau_3 \langle \mathcal{T}\{V_4(\tau_4) V_3(\tau_3) V_2(0) V(\tau_1)\}\rangle \\
&\quad + \lim_{\substack{\tau_4 \to +\infty \\ \tau_1 \to -\infty}} \langle V_4(\tau_4) V_{23}(\tau) V_1(\tau_1)\rangle \\
&= \int_{-\infty}^{+\infty} d\tau \langle V_4(+\infty) \mathcal{T}\{V_3(\tau) V_2(0)\} V_1(-\infty)\rangle \\
&\quad + \langle V_4(+\infty) V_{32}(0) V_1(-\infty)\rangle \\
&\quad = -\frac{1}{2}\left[\frac{1}{s}\left((\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_4) - \frac{s}{2}(\epsilon_2 \cdot \epsilon_3)\right)^2\right. \\
&\quad \left. + \frac{1}{u}\left((\epsilon_3 \cdot k_1)(\epsilon_2 \cdot k_4) - \frac{u}{2}(\epsilon_2 \cdot \epsilon_3)\right)^2\right].
\end{aligned}
$$

Glimpse of double copy

Let's make use of the spinor helicity formalism get rid of the terms containing $(\epsilon_2 \cdot \epsilon_3)$ by appropriately choosing the reference twistors $|\pm\rangle$ and $|\pm]$.

E.g. if particles 2 and 3 have the same helicity $(\epsilon_{2,3}^{(+)} \propto |-\rangle[k_{2,3}|,\ \epsilon_{2,3}^{(-)} \propto |k_{2,3}\rangle[+|)$ then $\epsilon_2 \cdot \epsilon_3 = 0$. And if the helicities are opposite we can still arrange for $\epsilon_2 \cdot \epsilon_3 = 0$ by choosing e.g. $|-\rangle \propto |k_2\rangle$ if 3 has negative helicity.

Now, compare $A_{4,sg}$ that with the corresponding scalar QED case,

$$A_{4,sb} = \frac{-1}{4}\left[\int_0^{+\infty} d\tau \langle e^{ik_4\cdot x(+\infty)} \epsilon_{3\mu} \dot{x}^{\mu}(\tau) e^{ik_3\cdot x(\tau)} \epsilon_{2\bar{\mu}} \dot{x}^{\bar{\mu}}(0) e^{ik_2\cdot x(0)} e^{ik_1\cdot x(-\infty)}\rangle \right.$$

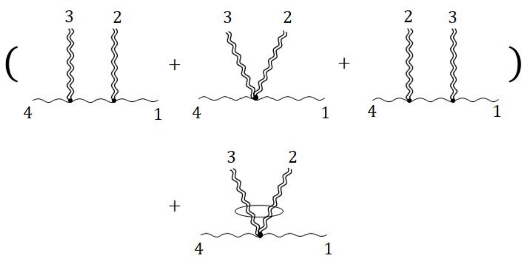
$$\left. + \int_{-\infty}^{0} d\tau \langle e^{ik_4\cdot x(+\infty)} \epsilon_{2\bar{\mu}} \dot{x}^{\bar{\mu}}(0) e^{ik_2\cdot x(0)} \epsilon_{3\mu} \dot{x}^{\mu}(\tau) e^{ik_3\cdot x(\tau)} e^{ik_1\cdot x(-\infty)}\rangle \right]$$

$$= -\frac{1}{2}\left[\frac{1}{s}(\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_4) + \frac{1}{u}(\epsilon_3 \cdot k_1)(\epsilon_2 \cdot k_4)\right]$$

Here is the double copy structure again:

$$
\begin{aligned}
A_{4,sg} &= -\frac{n_{s,sg}}{2s} - \frac{n_{u,sg}}{2u} \\
A_{4,sb} &= -\frac{n_{s,sb}}{2s} - \frac{n_{u,sb}}{2u} \\
n_{s,sg} &= n_{s,sb}^2 \\
n_{u,sg} &= n_{u,sb}^2 \,.
\end{aligned}
$$

4-point photon w.l. and 2 on-shell gravitons



$$
\begin{aligned}
V_{bg,i}(\tau_i) &= -\frac{1}{2}\epsilon_{\mu\nu,i}(\tau_i)\dot{x}^{\mu}(\tau_i)[\dot{x}^{\nu}(\tau_i) + ik_{i\sigma}S^{\nu\sigma}(\tau)]e^{ik_i\cdot x(\tau_i)} \\
V_{sb,i}(\tau_i) &= -\frac{1}{2}\epsilon_{\mu,i}\dot{x}^{\mu}e^{ik_i\cdot x(\tau_i)} \\
V_{bb,i}(\tau_i) &= -\frac{1}{2}\epsilon_{\nu,i}[\dot{x}^{\nu}(\tau_i) + ik_{i\sigma}S^{\nu\sigma}(\tau)]e^{ik_i\cdot x(\tau_i)} \,,
\end{aligned}
$$

Glimpse of double copy

Ignoring the $e^{ik_i \cdot x(\tau_i)}$ factor, we have the following relation

$$V_{bg,i}(\tau_i) = V_{sb,i}(\tau_i) V_{bb,i}(\tau_i) \,.$$

The final amplitudes have the structure

$$\begin{aligned}
A_{4,bg} &= \frac{n_{s,bg}}{2s} + \frac{n_{u,bg}}{2u} \\
A_{4,sb} &= \frac{n_{s,sb}}{2s} + \frac{n_{u,sb}}{2u} \\
A_{4,bb}(1243) &= \frac{n_{s,bb}}{2s} + \frac{n_{u,bb}}{2u} \\
n_{s,bg} &= n_{s,sb}\ n_{s,bb} \\
n_{u,bg} &= n_{u,sb}\ n_{u,bb} \,,
\end{aligned}$$

where $A_{4,bb}(1243)$ is the color-ordered 4-point amplitude of gauge bosons.

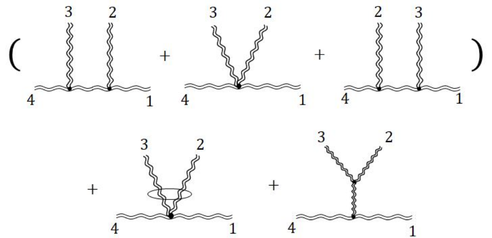
Graviton 4-point

For the graviton worldline, we need the following vertex operators to create the asymptotic states (Bonezzi 2018)

$$V_1(\tau_1) = \epsilon_{1\mu\nu}\bar{\Psi}^\mu(\tau_1)\bar{\tilde{\Psi}}^\nu(\tau_1)e^{ik_1\cdot x(\tau_1)}$$
$$V_4(\tau_4) = \epsilon_{4\mu\nu}\tilde{\Psi}^\mu(\tau_4)\Psi^\nu(\tau_4)e^{ik_4\cdot x(\tau_4)}\,.$$

The linear vertex operator and the pinch operator are extracted from the $O(2)\times O(2)$ action.

Since now the background field is dynamical, similar to how the 4-point scattering of non-abelian gauge bosons is computed from the worldline, we also have to add a lower-order tree (the t-channel) to the worldline to get the correct four graviton scattering amplitude.



Double copy for MHV

The reason worldline formalism yields the double-copy dirrectly for MHV amplitudes is the helicity structure: In the amplitude there will be $\epsilon_I \cdot k_J$ and $\epsilon_I \cdot \epsilon_J$ terms.
With the same choices for the reference twistors as before there can be only one $\epsilon_I \cdot \epsilon_J$.
This eliminates the 4-point diagrams, such as the $\langle \dot{x}\dot{x} \rangle$ contractions, ghost, and the pinch operator contributions.
At this point only the squaring of the lienar vertex operators matters and the double copy is immediate.

spin	0	1	2
0	$e^{ik\cdot x}$	$\frac{1}{2}\epsilon_\mu \dot{x}^\mu e^{ik\cdot x}$	$\frac{1}{2}\epsilon_{\mu\nu}\dot{x}^\mu \dot{x}^\nu e^{ik\cdot x}$
1	NA	$\frac{1}{2}\epsilon_\mu(\dot{x}^\mu + ik_\nu S^{\mu\nu})e^{ik\cdot x}$	$\frac{1}{2}\epsilon_{\mu\nu}\dot{x}^\mu(\dot{x}^\nu + ik_\rho S^{\nu\rho})e^{ik\cdot x}$
2	NA	NA	$\frac{1}{2}\epsilon_{\mu\nu}(\dot{x}^\mu + ik_\rho S^{\mu\rho})(\dot{x}^\nu + ik_\sigma \tilde{S}^{\nu\sigma})$ $e^{ik\cdot x}$

Conclusions

- We extended the worldline formalism for graviton worldlines and multioule graviton emissions.
- We found that the $O(4)$ symmetry of the free $N = 4$ worldline action is broken when accounting for graviton self-interactions, and we found different counterterms than in the previous literature, A different renormalization condition may be the reason. Our renormalization condition was matching with the QFT 3-point vertex.
- We identified the squaring of the linearized vertex operators as the reason for the double-copy relations among MHV amplitudes.
- Future directions:
  - Worldgraph?
  - A rennaisance of the WF with applications to the classical limit of the scattering amplitudes and effective potentials: think two worldlines interacting by exchanging massless mediators.
  - Are counterterms relevant?
  - Higher-spin worldlines interacting by exchanging gravitons