Conifold Transitions and Possible New Dualities in 4D N=1 Theories

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Motivation

- Geometric transitions in the moduli space of 4D $N = 2$ string vacua have been much studied. [Greene, Morrison, Strominger '95]
- Transitions like the conifold/flop and dualities such as mirror symmetry in $N = 2$ have both a geometric and a field-theoretic description.
- What can be said for 4D $N = 1$ vacua?
- If different geometries lead to same EFT $\rightarrow$ could be a powerful tool to understand string compactification
- Hence, topic of today's talk: Heterotic conifold transitions
- Some hints of Heterotic conifold transitions related to the (0,2) Target Space Duality?

Conifold Transitions

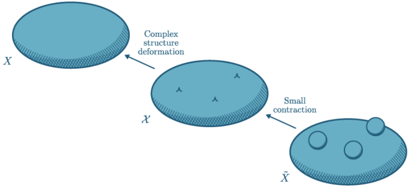
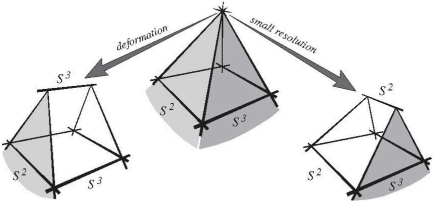


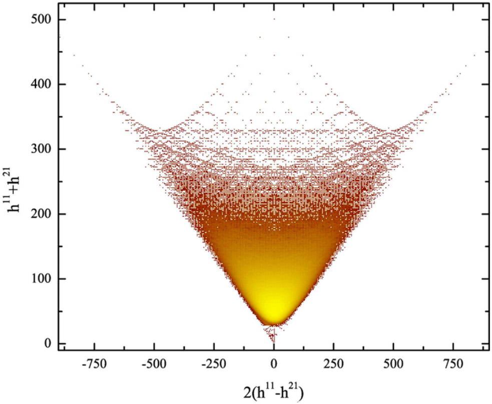
Figure 1: *A schematic depiction of a conifold transition between CY3s, as described in the text.*



$$h_1 \equiv X(x)y_1 + U(x)y_2 = 0$$
$$h_2 \equiv V(x)y_1 + Y(x)y_2 = 0$$

- Nodal limit:
$$h_{nodal} = XY - UV = 0$$
- Deformation:
$$XY - UV = \epsilon(x)$$

Geometric Transitions in Moduli Space



- Topologically different CYs threefolds can be connected by geometric transitions. Are all CY 3-folds connected? [Reid's fantasy]
- Understanding conifolds in $N = 1 \to$ Important insight into landscape of vacua

Known Results for Heterotic Conifolds

- $(0, 2)$ Gauged Linear Sigma Models (GLSM) target space duality [Distler, Kachru '95]
- Possibly dual theories with matching massless singlets (e.g. sum of $h^{1,1}, h^{2,1}$ and bundle moduli) [Blumenhagen, Rahn '11]
- It turns out that the examples above are connected by conifolds. No explanation of the link.
- Geometric procedure of bundles/manifolds connected by conifolds (meeting at the nodal limit). [Anderson, Brodie, Gray '22]

Possible duality and Target Space Duality

- It was shown that geometrically distinct configurations (X, V) and ($\tilde{X}$, $\tilde{V}$) could share a non-geometric phase in their $(0,2)$ GLSM [Distler, Kachru '95]

$$L_W = \int d^2 z d\theta \; [\Gamma^j G_j(X_i) + \langle P_1 \rangle \Lambda^a F_a^1(X_i) + ...]$$

$$\tilde{\Lambda}^a = \frac{\Gamma^j}{p_1}, \quad \tilde{\Gamma}^j = p_1 \Lambda^a$$

- The dual pairs are obtained by exchanging **monad maps** F_a and **CY defining polynomials** G_j
- GLSMs not dual, but target space theories have same massless singlet spectrum.
- The relationship of such pairs to conifolds is a mystery.

Conifold in Heterotic theories

- dH being an exact four-form in the Bianchi identity

$$dH = -\alpha' F \wedge F + \alpha' R \wedge R$$

  implies the anomaly cancellation condition (w/o 5-branes):

$$c_2(T_X) = c_2(V)$$

- We see that the bundle (brane) must also change to compensate for the change in the manifold through the conifold transition:

$$c_2(T_{\tilde{X}}) = c_2(T_X) + [\mathbb{P}^1 s] \Rightarrow c_2(V) + [\mathbb{P}^1 s]$$

- In the presence of a 5-brane wrapping curve $\mathcal{C}$

$$c_2(T_X) = c_2(V) + [\mathcal{C}]$$

An Example

Deformation manifold

$$X_D \;=\; \left[\begin{array}{ccccccc|cc} y_0 & y_1 & y_2 & y_3 & y_4 & x_0 & x_1 & p_1 & p_2 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 & 3 & 5 \end{array}\right]$$

Nodal limit

$$h_{nodal} = XY - UV = 0$$

Resolution

$$X_R \;=\; \left[\begin{array}{ccccccc|cc} y_0 & y_1 & y_2 & y_3 & y_4 & x_0 & x_1 & p_1 & p_2 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 & 4 & 4 \end{array}\right]$$

The deformation side bundle is given by:

$$0 \longrightarrow \mathcal{O}(-5) \xrightarrow{F_a} \mathcal{O}(-1)^{\oplus 5} \longrightarrow V_D \longrightarrow 0$$

Resolution side bundle:

$$0 \longrightarrow \mathcal{O}(-1,-5) \xrightarrow{\tilde{F}_a} \mathcal{O}(0,-1)^{\oplus 3} \oplus \mathcal{O}(0,-2) \oplus \mathcal{O}(-1,0) \longrightarrow V_R \longrightarrow 0$$

Moduli counting with hodge numbers

- 4D gauge singlets arise from Kahler $(h^{(1,1)}(X))$, complex structure $(h^{(2,1)}(X))$, and bundle moduli $(h^1(End(V))$.
- The total moduli on both sides turns out to be the same.

$$D(X,V) = h^{1,1} + h^{2,1} + h^1(End(V))$$

  [Blumenhagen, Rahn '11]
- Here counting done in supergravity limit, to leading order in superpotential

Moduli counting on deformation and resolution sides

- Total moduli = 426

- Deformation side:

  Kahler moduli: $h^{1,1} = 1$

  Complex structure moduli: $h^{2,1} = 101$

  Bundle moduli: $h^1(End(V)) = 324$

- Resolution side:

  Kahler moduli: $h^{1,1} = 2$

  Complex structure moduli: $h^{2,1} = 86$

  Bundle moduli: $h^1(End(\tilde{V})) = 338$

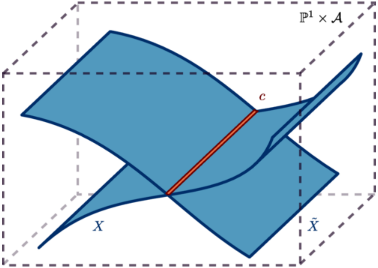
From bundles to NS5-branes

- Observation: The bundle transition is driven by crucial 5-brane transition (via heterotic small instanton transition). [Anderson, Brodie, Gray '22]

$$V \to V_S \oplus \mathcal{I}_{\mathcal{C}}$$

  where $\mathcal{I}_C$ is the ideal sheaf of curve/5-brane

- Simplifies essential structure, as the rest of the bundle spectates.



- We match $h^{1,1}(X) + h^{2,1}(X) + h^0(\mathcal{N}_{\mathcal{C}})$ on both sides of the transition

Moduli Matching beyond counting dimensions

Consider a NS5 brane wrapped on the following curves on the deformation and resolution side respectively:

$$
\begin{aligned}
X_D &= \left[\begin{array}{ccccccc|cc} y_0 & y_1 & y_2 & y_3 & y_4 & x_0 & x_1 & p_1 & p_2 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 & 3 & 5 \end{array}\right] \\
\mathcal{N}_{\mathcal{C}_D} &= \mathcal{O}_{X_D}(1,4)^{\oplus 2}
\end{aligned}
$$

$$h^{1,1} + h^{2,1} + h^0(\mathcal{N}_{C_D}) = 1 + 101 + 38 = 140$$

$$
\begin{aligned}
X_R &= \left[\begin{array}{ccccccc|cc} y_0 & y_1 & y_2 & y_3 & y_4 & x_0 & x_1 & p_1 & p_2 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 & 4 & 4 \end{array}\right] \\
\mathcal{N}_{\mathcal{C}_R} &= \mathcal{O}_{X_R}(1,3) \oplus \mathcal{O}_{X_R}(0,5)
\end{aligned}
$$

$$h^{1,1} + h^{2,1} + h^0(\mathcal{N}_{C_R}) = 2 + 86 + 52 = 140$$

Observation: Systems of defining equations look the same, but play different roles (brane/manifold)

	Deformation	Resolution
Manifold	$P_{1,3}^{(D)}, P_{0,5}^{(D)}$	${P_{1,4}^{\alpha}}^{(R)}$
Brane	${P_{1,4}^{\alpha}}^{(D)}$	$P_{1,3}^{(R)}, P_{0,5}^{(R)}$

Outline of a field mapping?

- We want to understand field mapping of a possible duality
- Problem: Many aspects of the 4d EFT are unknown
- Rough idea:

  $$h^{1,1} + h^{2,1} + h^0(\mathcal{N}_C)$$

  must mix
- E.g. 86 out of 101 C.S. moduli $\rightarrow$ C.S. in resolution, 14 C.S. $\rightarrow$ brane, 1 C.S.$\rightarrow$ extra Kahler modulus, etc.
- Idea: Exchange role of polynomials
- Problem: Swapping defining equations of brane $\leftrightarrow$ manifold is not quite the right map
- Geometric tools to determine/constrain the field map:
  - These geometries (X, V) and $(\tilde{X}, \tilde{V})$ can become the same in the nodal/singular limit
  - Geometric moduli are defined by intricate equivalence classes. Must map physical $\leftrightarrow$ physical degrees of freedom

The problem with exchanging polynomials

	Deformation	Resolution
Manifold	$P_{1,3}^{(D)}, P_{0,5}^{(D)}$	${P_{1,4}^{\alpha}}^{(R)}$
Brane	${P_{1,4}^{\alpha}}^{(D)}$	$P_{1,3}^{(R)}, P_{0,5}^{(R)}$

- What goes wrong with obvious interchange of polynomials?
- Deformation:
  Brane equation: $P_{1,4} \to P_{1,4} + L_{0,1} P_{1,3}$
  does not change the brane equation (*unphysical* fluctuation)
- Resolution:
  Manifold equation: $P_{1,4} \to P_{1,4} + L_{0,1} P_{1,3}$
  is a physical change of the defining manifold.

Tracking the Physical Degrees of Freedom

- Above we showed just one mismatch of physical/unphysical fluctuations.
- Full analysis: Parameterize infinitesimal moduli spaces by equivalence classes, formulate map which takes physical $\leftrightarrow$ physical

Deformation moduli: Manifold

Defining polynomials of the deformation manifold:

$$\begin{aligned} P^{(D)}_{1,3} &= x_0 \\ P^{(D)}_{0,5} &= \alpha\epsilon_{\alpha\beta} l^\alpha q^\beta + P'_{0,5} \end{aligned}$$

$$\begin{aligned} H^1(TX_D) &: \begin{cases} \delta P_{0,5} \sim \delta P_{0,5} + hP^{(D)}_{0,5} + l^i_{0,1}\partial_{y^i} P^{(D)}_{0,5} \\ \delta P_{1,3} \sim \delta P_{1,3} + mP^{(D)}_{1,3} + l^i_{0,1}\partial_{y^i} P^{(D)}_{1,3} \\ \quad +l_{1,3}\partial_{x^0} P^{(D)}_{1,3} + l_{1,0}\partial_{x^1} P^{(D)}_{1,3} \end{cases} \\ H^1(TX^\vee_D) &: \mathbb{C} \end{aligned}$$

($\sim$ denotes equivalence class).
We obtain the above equivalence classes from the standard adjunction and Euler sequences. Unphysical changes are scaling and coordinate transformations.

Deformation moduli: Brane

Brane defining polynomials:

$$P^{\alpha(D)}_{1,4} = l^{\alpha} x_0 + q^{\alpha} x_1$$

From the Koszul sequence, we get :

$$\delta P^{\alpha}_{1,4} \sim \delta P^{\alpha}_{1,4} + A^{\alpha}_{\beta} P^{(D)\beta}_{1,4} + L^{\alpha}_{0,1} P^{(D)}_{1,3}$$

Allowing fluctuations in the curve wrapped by the 5-brane we have

$$H^0(\mathcal{N}_{\mathcal{C}_D}|_{\mathcal{C}_D}) : \delta P^{\alpha}_{1,4} \sim \delta P^{\alpha}_{1,4} + A^{\alpha}_{\beta} P^{(D)\beta}_{1,4} + L^{\alpha}_{0,1} P^{(D)}_{1,3} + l^{i}_{0,1} \partial_{y^i} P^{(D)\alpha}_{1,4} + l_{1,3} \partial_{x^0} P^{(D)\alpha}_{1,4} + l_{1,0} \partial_{x^1} P^{(D)\alpha}_{1,4}$$

Highlighted terms don't appear in the equivalence class of $P^{\alpha}_{1,4}$ on the resolution side if we exchange brane and manifold equations.

Resolution moduli: Manifold

Resolution manifold defining polynomials:

$$P^{\alpha(R)}_{1,4} = l^{\alpha} x_0 + q^{\alpha} x_1$$

Manifold moduli:

$$\begin{aligned} H^1(TX_R) &: \delta P^{\alpha}_{1,4} \sim \delta P^{\alpha}_{1,4} + A^{\alpha}_{\beta} P^{(R)\beta}_{1,4} \\ &+ l^{i}_{0,1} \partial_{y^i} P^{(R)\alpha}_{1,4} + l_{1,3} \partial_{x^0} P^{(R)\alpha}_{1,4} + l_{1,0} \partial_{x^1} P^{(R)\alpha}_{1,4} \\ H^1(TX_R^{\vee}) &: \mathbb{C}^2 \end{aligned} \tag{1}$$

$L^{\alpha}_{0,1} P^{(D)}_{1,3}$ missing. So a change $\propto$ it is physical on this side.

Resolution moduli: Brane

Resolution brane defining polynomials :

$$\begin{aligned} P_{1,3}^{(R)} &= x_0 \\ P_{0,5}^{(R)} &= \alpha\epsilon_{\alpha\beta} l^\alpha q^\beta + P'_{0,5} \end{aligned}$$

Brane moduli:

$$H^0(\mathcal{N}_{\mathcal{C}_R}|_{\mathcal{C}_R}) : \begin{cases} \delta P_{0,5} \sim \delta P_{0,5} + h P_{0,5}^{(R)} + B P_{\text{nodal}}^{(R)} \\ \quad + L_{0,1}^{\alpha}\left(q^{(R)\alpha}\right) + l_{0,1}^{i} \partial_{y^i} P_{0,5}^{(R)} \\ \delta P_{1,3} \sim \delta P_{1,3} + m P_{1,3}^{(R)} + l_{0,1}^{i} \partial_{y^i} P_{1,3}^{(R)} + \\ \quad l_{1,3} \partial_{x^0} P_{1,3}^{(R)} + l_{1,0} \partial_{x^1} P_{1,3}^{(R)} \end{cases}$$

Highlighted terms don't appear in equivalent $P_{0,5}$ polynomial on the deformation side $\rightarrow$ a physical change on def. side

Mapping depends on Nodal limit and an ambiguity

We let the defining equations on both sides of the transition to be the same:

$$
\begin{aligned}
P_{1,4}^{(R)\alpha} &= P_{1,4}^{(D)\alpha} = P_{1,4}^{\alpha} = l^{\alpha} x_0 + q^{\alpha} x_1 \\
P_{1,3}^{(R)} &= P_{1,3}^{(D)} = P_{1,3} = x_0 \\
P_{0,5}^{(R)} &= P_{0,5}^{(D)} = \alpha \epsilon_{\alpha\beta} l^{\alpha} q^{\beta} + P'_{0,5}
\end{aligned}
$$

- Since we split $P_{0,5}$ as a nodal part ($\propto q^{\beta}$) and a remainder, any changes in $P_{1,4}^{\alpha}$ and $P_{0,5}$ are correlated. It can be seen by correlated changes: $\delta P_{0,5}^{(D)} \propto L_{0,1} q_{0,4}^{\alpha}$ and $\delta P_{1,4}^{\alpha\,(R)} \propto L_{0,1} x_0$ being physical degrees of freedom on both sides.
- However, note that there is no unique way to split $P_{0,5}$ into (linear quotient)$\times q^{\beta}$ and remainder $P'_{0,5}$

Degrees of Freedom	Physical	Unphysical
Def. Manifold	$\delta P_{0,5} \propto L^{\alpha}_{0,1} q^{\alpha}$	scaling and coord. redefinition
Def. Brane	$\delta P^{\alpha}_{1,4} \propto {P^{\alpha}_{1,4}}'$	$\delta P^{\alpha}_{1,4} \propto L^{\alpha}_{0,1} x_0$ scaling and coord. redef
Res. Manifold	$\delta P^{\alpha}_{1,4} \propto L^{\alpha}_{0,1} x_0$	scaling coord. redefinition
Res. Brane	$\delta P_{0,5} \propto P'_{0,5}$	$\delta P_{0,5} \propto L^{\alpha}_{0,1} q^{\alpha}$ scaling and coord. redef

What about the extra Kähler modulus on the Resolution side?

Change the background nodal quintic polynomial as:

$$\delta P_{0,5}^{nodal} = \alpha \epsilon_{\alpha\beta} l^{\alpha} q^{\beta}$$

- We see that $P_{1,4}^{\alpha}$ defining equations on resolution remain unchanged as we have just rescaled the constituent polynomials. On the deformation side it changes the relative scale between $P_{0,5}^{nodal}$ and $P'_{0,5}$
- This extra degree of freedom maps on the resolution side to the extra Kähler modulus.
- Something peculiar about the scaling α is that when it becomes large ($\alpha \to \infty$) we approach the nodal limit, and this implies that the Kähler modulus $T \to 0$. We however don't know the functional form of the relation.

Matching Yukawa couplings in perturbative superpotential

- The couplings in our E_6 example above is of the form $\mathbf{27^3}$ and $\overline{\mathbf{27}}^{\mathbf{3}}$. These are complicated functions of complex structure and bundle moduli. If a map between the complex structure and bundle moduli is known we can match the couplings.
- They can provide consistency check that moduli map is correct (i.e. duality holds).
- Yukawa coupling is obtained by finding $H^3(\wedge^3 V)$. There is a map: $H^1(V) \times H^1(V) \times H^1(V) \to H^3(\wedge^3 V) = H^3(\mathcal{O}) = \mathbb{C}$.

In a modification of the above example, we find
$H^3(\wedge^3 V_D)$:

$$P_{15} \sim P_{15} + Ap_{(5)} + \sum_{a=1}^{3} B_a m^a_{(4)} + Dm_{(3)}.$$

$H^3(\wedge^3 V_R)$:

$$x_1^3 P_{0,15} \sim x_1^3 P_{0,15} + x_1^3 \tilde{A} m_{(0,5)} + x_1^3 \sum_{a=1}^{2} \tilde{B}_a p^a_{(0,4)} + x_1^3 \tilde{B}_3 m_{(0,4)} + x_1^3 \tilde{D}_2 m^2_{(0,3)}.$$

The resolution side Yukawa coupling is the same as the deformation side, but multiplied by x_1^3

Note that, since terms proportional to q^α are quotiented out from the above equivalence classes the ambiguity of the non-unique splitting of $P_{0,5}$ does not affect the Yukawa matching.

Conclusion

What did we show?

- Proposed a way to go beyond just massless spectrum matching, by tracking degrees of freedoms of functions in the superpotentials of the two possibly dual theories.
- In this example we matched 147,440 Yukawa couplings on both sides utilizing the moduli map found. This gives a nontrivial check that our proposed moduli map works.
- In our canonical example Yukawa coupling match bypassed the ambiguity introduced in the moduli map by the non-unique way of splitting $P_{0,5} = \alpha \epsilon_{\alpha\beta} l^{\alpha} q^{\beta} + P'_{0,5}$.

Future directions

- We would like to understand the N=1 4D supersymmetric gauge theory mechanism of a conifold transition. (like Greene, Morrison, Strominger '95).
- We will demonstrate a complete general moduli map between manifold, bundle, and Kähler degrees of freedom in upcoming work.
- We would also like to show how the mapping works when the non-perturbative terms ($W_{\mathcal{C}} = \left(\sum_i^{n_{\mathcal{C}}} \mathrm{Pfaff}_{C_i}\right) e^{-\int_{\mathcal{C}} J+iB}$) in the superpotential are non-zero.
- Can we propose a rule or criteria for all transitions in the moduli space that have the same EFT?

Thank You