BPS-State Counting:

Quiver Invariant, Abelianisation & Mutation

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Outline

**Warm-Up**
- A topology Exercise

**Rudiments**
- index and Wall-Crossing
- BPS Quivers

**Quiver Invariants**
- Characterisation of the Higgs Moduli Spaces

**Non-Abelian Quivers**
- Abelianisation
- Mutation
Warm Up

a topology exercise
Chamber 1

\[ \mathbb{P}^{a_2-1} \times \mathbb{P}^{a_3-1} \]

\[ O(1, 1)^{\oplus a_1} \]

Chamber 2

\[ \mathbb{P}^{a_3-1} \times \mathbb{P}^{a_1-1} \]

\[ O(1, 1)^{\oplus a_2} \]

Chamber 3

\[ \mathbb{P}^{a_1-1} \times \mathbb{P}^{a_2-1} \]

\[ O(1, 1)^{\oplus a_3} \]
Chamber 1: $\mathbb{P}^4 \times \mathbb{P}^3$, $O(1, 1) \oplus^6$

Chamber 2: $\mathbb{P}^3 \times \mathbb{P}^5$, $O(1, 1) \oplus^5$

Chamber 3: $\mathbb{P}^5 \times \mathbb{P}^4$, $O(1, 1) \oplus^4$
Cahmber 1  Cahmber 2  Cahmber 3

H•••  H•••  H•••

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26 26 26 26 26 26

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0 2 0 0 2 0

0 0 0 0 0 0

1 1 1

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Rudiments

BPS Index
$\mathcal{N}=2$ Basics

- We consider $\mathcal{N}=2$ Abelian gauge theories.
- States have integer charges: $\gamma \in \mathbb{Z}^{2r} \equiv \Gamma$
- Poincare extends to $\mathcal{N}=2$ super-Poincare: $M$ gets bounded by $|Z|$.

- $M=|Z|$ case: “short” rep, $S_j = [j] \otimes r_{hh}$,
  where $r_{hh} = 2[0] \oplus [1/2]$ is the 4-dim irrep of the odd alg.

- $M>|Z|$ case: “long” rep, $L_j = [j] \otimes r_{hh} \otimes r_{hh}$
For the Hilbert space $\mathcal{H}_\gamma^1 = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}} S_j \oplus n_j(\gamma) \bigoplus \bigoplus_{l \in \frac{1}{2}\mathbb{Z}_{\geq 0}} L_l \oplus m_l(\gamma)$, define the BPS index as:

\[
\Omega(\gamma) := \sum_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}} (-1)^{2j}(2j + 1) n_j(\gamma)
\]

\[
= \text{Tr}'_{\mathcal{H}_\gamma^1}(-1)^{2J_3}
\]

Only genuine short reps contribute to $\Omega(\gamma)$.

The little super-algebra contains $SU(2)_R$ and hence one can define the refined index as:

\[
\Omega(\gamma; y) = \text{Tr}'_{\mathcal{H}_\gamma^1}(-1)^{2J_3} y^{2J_3 + 2J_3} \quad y = 1 \quad \Omega(\gamma) = \text{Tr}'_{\mathcal{H}_\gamma^1}(-1)^{2J_3}
\]
Rudiments

Wall-crossing
Wall-Crossing

- $\Omega(\gamma)$ is invariant under arbitrary deformations of $H^1_\gamma$, but may change under deformations of the theory.
- The index is ill-defined when $H^1_\gamma$ mixes with the multi-ptl spectrum, i.e., if $\gamma$ can split into $\gamma_1$ and $\gamma_2$ s.t. $\gamma_1 + \gamma_2 = \gamma$, $Z_1/Z_2 \in \mathbb{R}^+$.
- Thus, in the parameter space, there appears a wall, across which the BPS index jumps.
Wall-Crossing

• Generic BPS one-particle states as loose bound states of charge centers, balanced by classical forces.

  [Lee, Yi `98; Bak, Lee, Lee, Yi `99; Gauntlett, Kim, Park, Yi `99; Stern, Yi `00; Gauntlett, Kim, Lee, Yi `00]

• The equilibrium distances become infinite as one approaches the wall [Denef `02]:

$$R = \frac{\langle \gamma_1, \gamma_2 \rangle}{2} \frac{|Z_1 + Z_2|}{\text{Im}[\bar{Z}_1 Z_2]}$$
Rudiments

BPS Quivers
BPS Quivers

- BPS states ~ D-branes wrapping various cycles.
- Low-energy D-brane dynamics by a $\mathcal{D}=4$, $\mathcal{N}=1$ quiver gauge theory reduced to the eff. particle world-line.
- E.g. IIB on CY$_3$: one-particle BPS states seen as a D3-brane wrapping a SLag.
- Two pictures arise for the same BPS bound state of branes:
  1. **Set of particles at equilibrium**
  2. **Fusion of D-branes**

related via quiver quantum mechanics [Denef `02]
• U(1) vectors include $\mathbf{x}_v = (x_v^1, x_v^2, x_v^3)$ and bi-fund. chirals include $Z_{vw}^{k=1,...,a_{vw}}$, where $a_{vw} = \langle \gamma_v, \gamma_w \rangle$

• Two phases

(1) **Coulomb:** $\mathbf{x}_v, Z_{vw}^k$

(2) **Higgs:** $\mathbf{x}_v, Z_{vw}^k$
BPS Index

• For large $x_v$-$x_w$, chirals are massive and eff. dynamics leads to

$$K_v = \sum_{w \neq v} \frac{\langle \gamma_w, \gamma_v \rangle}{\|x_w - x_v\|} - \theta_v(u) = 0 \text{ for } \forall v, \text{ with } \theta_v = 2 \text{ Im}[e^{-i\alpha}Z_{\gamma_v}(u)]$$

• By studying the sol^n space $\mathcal{M} = \{x_v \mid K_v = 0, \forall v\} \setminus \mathbb{R}^3$, one can obtain the **Coulomb index** $\Omega_{\text{Coulomb}}(\{\gamma_v\}; y)$

  [de Boer, El-Showk, Messamah, van Den Bleeken `09], [Manschot, Pioline, Sen `11]

• Dialing the coupling to 0, one can describe the system as QM on the variety $\mathcal{M}_H = \{Z_{vw}^k \mid D_v = \theta_v, \forall v\} / \prod_v U(1)$.

• The **Higgs index** is given as:

$$\Omega_{\text{Higgs}}(\{\gamma_v\}; y) = \sum_{p,q} (-1)^{p+q-d} y^{2p-d} h^{p,q}(\mathcal{M}_H)$$
Coulomb vs Higgs

- It has been shown [Denef `02; Sen `11]:
  \[ \Omega_{\text{Coulomb}} = \Omega_{\text{Higgs}} \]

- Multi-center picture has a smooth transition into the fused D-brane picture at a single point.

- The two pictures might become very different if the quivers have a loop [Denef, Moore `07]:
  \[ \Omega_{\text{Coulomb}} \ll \Omega_{\text{Higgs}} \]
Intrinsic Higgs States

• The Higgs phase might in general have more states than the Coulomb phase multi-center states.

• We may call these additional ones “intrinsic” Higgs states.

• Thus, the Higgs index can be written as:

\[ \Omega_{\text{Higgs}} = \Omega_{\text{Coulomb}} + \text{“}\Omega_{\text{Inv}}\text{”} \]

• The intrinsic Higgs states are expected not to experience wall-crossing.
Cyclic Example

- Consider a 3-node quiver with superpotential
  \[ \mathcal{W} (\{ Z_{12}^k \}, \{ Z_{23}^k \}, \{ Z_{31}^k \}) = \sum C_{k_1k_2k_3} Z_{12}^{k_1} Z_{23}^{k_2} Z_{31}^{k_3} \]

- There arise 3 different quiver varieties, in each of which one set of chirals vanishes.

- The moduli space is embedded by F-terms in D-term variety.
Characterisation of $\Omega_{\text{Inv}}$

- Embedding structure $\mathcal{M}_H \hookrightarrow \mathcal{A}$
  $\implies$ Naturally splits the Higgs phase states:

$$H^\bullet(\mathcal{M}_H) = i^* (H^\bullet(\mathcal{A})) \oplus [H^\bullet(\mathcal{M}_H)/i^* (H^\bullet(\mathcal{A}))]$$

$\Omega_{\text{Higgs}}$  $\Omega_{\text{Coulomb}}$  $\Omega_{\text{Inv}}$

[S.-J.L., Z.-L.Wang, P.Yi `12]

(cf.) [Bena, Berkooz, de Boer, El-Showk, van Den Bleeken `12]

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Nonabelian Quivers

Abelianisation
Abelianisation
The Prescription in a Nutshell

[Martin, `00], [Ciocan-Fontanine, Kim, Sabbah, `06]  
(cf.) [Hori, Vafa, `00]
Loopless Quivers

\[ Y = \mu_{G}^{-1}(0)/T \quad \xrightarrow{l} \quad \tilde{X} = \mu_{T}^{-1}(0)/T \]

\[ \xrightarrow{\pi} \]

\[ X = \mu_{G}^{-1}(0)/G \]

**Bridging:** \[ \int_{X} a = \frac{1}{|W|} \int_{\tilde{X}} \hat{a} \wedge e(\Delta) \], where

- \( \pi^{*}a = l^{*}\hat{a} \)
- \( W = \text{Weyl}(G) \)
- \( \Delta = \bigoplus_{\text{root } \alpha} \mathcal{L}_{\alpha} \)
Loopless Quivers

\[ Y = \mu_G^{-1}(0)/T \quad \xrightarrow{\pi} \quad \tilde{X} = \mu_T^{-1}(0)/T \]

Index: \( \Omega(y) = \frac{1}{|W|} \int_{\tilde{X}} \omega_y(T\tilde{X}) \wedge \frac{e(\Delta)}{\omega_y(\Delta)} \),

where \( \omega_y \leftarrow f_{\omega_y}(x) = \frac{x}{1 - e^{-x}} \cdot (ye^{-x} - y^{-1}) \)
Quivers with a Potential

\[ Y = \mu_G^{-1}(0)/T \xrightarrow{\iota} \tilde{X} = \mu_T^{-1}(0)/T \]

\[ \pi \]

\[ M \xrightarrow{F = 0} X = \mu_G^{-1}(0)/G \]

Index: \[ \Omega(y) = \frac{1}{|W|} \int_{\tilde{X}} \omega_y(\mathcal{T}\tilde{X}) \wedge \frac{e(\tilde{\mathcal{N}})}{\omega_y(\tilde{\mathcal{N}})} \wedge \frac{e(\Delta)}{\omega_y(\Delta)} \]

where \[ \omega_y \leftarrow f_{\omega_y}(x) = \frac{x}{(1 - e^{-x})} \cdot (ye^{-x} - y^{-1}) \]
Applications

- Non-Abelian Quiver Invariant
- Partition-sum Structure of the Index
Applications

- Non-Abelian Quiver Invariant
- Partition-sum Structure of the Index
Applications

- Non-Abelian Quiver Invariant
- Partition-sum Structure of the Index

Chamber (a)
\[
\begin{align*}
\Omega_{\text{Higgs}}^{(a)}(y) &= 6 \\
\Omega_{\text{Coulomb}}^{(a)}(y) &= 1
\end{align*}
\]

Chamber (b)
\[
\begin{align*}
\Omega_{\text{Higgs}}^{(b)}(y) &= \frac{1}{y^2} + 7 + y^2 \\
\Omega_{\text{Coulomb}}^{(b)}(y) &= \frac{1}{y^2} + 2 + y^2
\end{align*}
\]

\[\Omega_{\text{inv}} = 5\]
Applications

- Non-Abelian Quiver Invariant
- Partition-sum Structure of the Index
Applications

- Non-Abelian Quiver Invariant
- Partition-sum Structure of the Index
Applications

- Non-Abelian Quiver Invariant
- Partition-sum Structure of the Index

\[ \mathcal{P}_1 = (\{1, 1, 1\}; \{1\}) \]
\[ \mathcal{P}_2 = (\{1, 2\}; \{1\}) \]
\[ \mathcal{P}_3 = (\{3\}; \{1\}) \]
Applications

- Non-Abelian Quiver Invariant
- Partition-sum Structure of the Index

\[ c_1(\mathcal{P}_1; y) \cdot \Omega \left[ \begin{array}{c} 1 \\ \theta_1 \\ 3 \\ \kappa \\ \theta_2 \end{array} \right] (y) + c_2(\mathcal{P}_2; y) \cdot \Omega \left[ \begin{array}{c} 1 \\ \theta_1 \\ 1 \\ \kappa \\ \theta_2 \\ \kappa \end{array} \right] (y) + c_3(\mathcal{P}_3; y) \cdot \Omega \left[ \begin{array}{c} 1 \\ 3\theta_1 \\ 1 \\ 3\kappa \\ 1 \\ \theta_2 \end{array} \right] (y) \]

\[ c(\mathcal{P}; y) \equiv \frac{1}{|\Gamma(\mathcal{P})|} \prod_{v=1}^{N} \prod_{a_v=1}^{l_v} \frac{1}{r_v,a_v} \frac{y - y^{-1}}{y^{r_v,a_v} - y^{-r_v,a_v}} \]
Applications

- Non-Abelian Quiver Invariant
- Partition-sum Structure of the Index
Applications

- Non-Abelian Quiver Invariant
- Partition-sum Structure of the Index

- - - Works in principle for any quivers but practically hard - - -

- Asymptotic behavior?
- Another path towards Non-Abelian Quivers?
Nonabelian Quivers

Mutation
Left and Right Mutations

• Relate the index of a complicated quiver to that of a simpler one via mutation:
  \[ Q = (\{N_i\}; [b_{ij}])_{\zeta_i} \xrightarrow{\mu} \hat{Q} = (\{\hat{N}_i\}; [\hat{b}_{ij}])_{\zeta_i} \]

• With respect to a node \( k \), either Left or Right: \( \mu_k^L \) or \( \mu_k^R \)

• The action on charges \( \gamma_i's \) characterises the mutation:

\[
\begin{align*}
\mu_k^L(\gamma_i) &= \begin{cases} 
-\gamma_k & i = k \\
\gamma_i + [b_{ki}] + \gamma_k & \text{otherwise}
\end{cases} \\
\mu_k^R(\gamma_i) &= \begin{cases} 
-\gamma_k & i = k \\
\gamma_i + [b_{ik}] + \gamma_k & \text{otherwise}
\end{cases}
\end{align*}
\]
Left and Right Mutations

- Relate the index of a complicated quiver to that of a simpler one via mutation:

\[ Q = \left( \{ N_i \} ; [b_{ij}] \right) \zeta_i \xrightarrow{\mu} \hat{Q} = \left( \{ \hat{N}_i \} ; [\hat{b}_{ij}] \right) \hat{\zeta}_i \]

- With respect to a node \( k \), either Left or Right: \( \mu^L_k \) or \( \mu^R_k \)

- The action on charges \( \gamma_i \)'s characterises the mutation:

\[
\mu^L_k(\gamma_i) = \begin{cases} 
-\gamma_k & i = k \\
\gamma_i + [b_{ki}] + \gamma_k & \text{otherwise}
\end{cases}
\]

\[
\mu^R_k(\gamma_i) = \begin{cases} 
-\gamma_k & i = k \\
\gamma_i + [b_{ik}] + \gamma_k & \text{otherwise}
\end{cases}
\]

\[
\mu^L_k(\zeta_i) = \begin{cases} 
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\zeta_i + [b_{ki}] + \zeta_k & \text{otherwise}
\end{cases}
\]

\[
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\end{cases} \\
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-\gamma_k & i = k \\
\gamma_i + [b_{ik}] + \gamma_k & \text{otherwise}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\mu_k^L(N_i) &= \begin{cases} 
-N_k + \sum_j [b_{kj}] + N_j & i = k \\
N_i & \text{otherwise}
\end{cases} \\
\mu_k^R(N_i) &= \begin{cases} 
-N_k + \sum_j [b_{jk}] + N_j & i = k \\
N_i & \text{otherwise}
\end{cases}
\end{align*}
\]
Left and Right Mutations

- Relate the index of a complicated quiver to that of a simpler one via mutation:
  \[ Q = (\{N_i\}; [b_{ij}]) \zeta_i \rightarrow \hat{Q} = (\{\hat{N}_i\}; [\hat{b}_{ij}]) \hat{\zeta}_i \]

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- The action on charges \( \gamma_i \)'s characterises the mutation:

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\end{cases} \\
\mu_k^R(\gamma_i) &= \begin{cases} 
-\gamma_k & \text{if } i = k \\
\gamma_i + [b_{ik}] + \gamma_k & \text{otherwise}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\mu_k(b_{ij}) &= \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k \\
b_{ij} + \text{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise}
\end{cases}
\end{align*}
\]
Triangle Quiver with $\tilde{\mathcal{N}} = (1, 1, N)$
Triangle Quiver with $\vec{N} = (1, 1, N)$

- Trade off between vectors and chirals could be made.
- Would all mutations preserve the Witten index?
Mutation as a viewpoint change in how BPS particles are distinguished from anti-BPS particles

[Alim, Cecotti, Cordova, Espahdodi, Rastogi, Vafa `11]
Figure 2: Four physical chambers of (1, 1, N) triangle quivers, divided by solid lines. These are further divided into ten sub-chambers by relative ordering of the three FI constants; for example, (2, 3, 1) means $\downarrow 2 < \downarrow 3 < 0 < \downarrow 1$. The arrows in the lower-left corner are normal to the respective constant $\downarrow$ lines.

Mutation results in a negative rank of the mutated node, the original quiver must have been in a physically empty chamber with a vanishing Witten index. In this sense, it suffices to consider the original quivers and the chambers thereof such that allowed mutation results in $\mu_k(N_k) \neq 0$, to which cases we will restrict ourselves.

With the index counting enabled by HKY's general formula, we wish to test this mutation idea explicitly by applying to a simplest class of triangle quivers. We will perform numerical test as well as illustrate how HKY formula itself exhibits invariance under such mutations. The latter may be generalized to a larger class of quivers, establishing the mutation invariance rigorously at the level of index theorem.

5.2 A Numerical Check and A Subtlety

Before we plunge into more analytical demonstration in next subsection, let us briefly check the validity of the mutation invariance with a particular example of triangle quivers with ranks (1, 1, 2) and the intersection numbers (4, 5, 7) of figure 4. This will serve to check the aforementioned assertion, regarding invariance of Witten indices of particular chambers as well as non-preservation of Witten indices of “wrong”...
Figure 3:

I. Similarly, the right mutation, allowed in three sub-chambers of figure 2 with most

\[ \mu^L_3 \]

\[ \mu^R_3 \]

and
to those of

An explicit example of mutation. Witten indices are computed for all four

\[ (I) = 50 \]

\[ (II) = 1 \]

\[ (III) = 1 \]

6

\[ (IV) = 1 \]

Because the mutation flips arrow orientations, the roles of

\[ \zeta_1 \]

\[ \zeta_2 \]

\[ \zeta_3 \]

\[ \xi_1 \]

\[ \xi_2 \]

\[ \xi_3 \]

\[ \xi_4 \]

\[ b - N \]

\[ c - N \]

\[ N \]

\[ b \]

\[ c \]

\[ -\xi_3 \]

\[ \hat{\xi}_1 \]

\[ \hat{\xi}_2 \]

\[ \hat{\xi}_3 \]

\[ \hat{\xi}_4 \]

\[ \zeta_1 \]

\[ \zeta_2 \]

\[ \zeta_3 \]

\[ \zeta_4 \]

\[ b c - a \]

\[ bc - a \]

\[ 1 \]

\[ 1 \]

\[ 1 \]

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\[ 1 \]

\[ 1 \]

\[ 1 \]

\[ 1 \]

\[ -\xi_3 \]

\[ \zeta_2 + b\zeta_3 \]

\[ 1 \]

\[ \xi_1 \]

\[ \xi_2 \]

\[ \xi_3 \]

\[ \xi_4 \]

\[ 1 \]

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$$\xi_3 = -\xi_1 - \xi_2$$

are further divided into ten sub-chambers by relative ordering of the three FI constants; these to figure 4. This means

$$\zeta_2 = 0$$

$$\zeta_3 = 0$$

$$\zeta_1 = 0$$

The arrows in the lower-left corner are normal to the respective constant lines.

With the index counting enabled by HKY's general formula, we wish to test the intersection numbers (4, 3, 5) triangle quivers, divided by solid lines. These triangle quivers, divided by solid lines. These

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Figure 2: Four physical chambers of (1, 1, N) triangle quivers, divided by solid lines. These are further divided into ten sub-chambers by relative ordering of the three FI constants; for example, (2, 3, 1+0) means $\downarrow 2 < \downarrow 3 < 0 < \downarrow 1$. The arrows in the lower-left corner are normal to the respective constant $\downarrow$ lines.

Mutation results in a negative rank of the mutated node, the original quiver must have been in a physically empty chamber with a vanishing Witten index. In this sense, it suffices to consider the original quivers and the chambers thereof such that allowed mutation results in $\mu_k(N) = 0$, to which cases we will restrict ourselves.

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Figure 2: Four physical chambers of \((1, 1, N)\) triangle quivers, divided by solid lines. These are further divided into ten sub-chambers by relative ordering of the three FI constants; for example, \((2, 3, 1)\) means \(\zeta_2 < \zeta_3 < 0 < \zeta_1\). The arrows in the lower-left corner are normal to the respective constant lines.

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Can we mutate with respect to node 3?

\[
\mu_{L}^{3}(Q)\mu_{R}^{3}(Q) \text{ must reproduce } \Omega_{Q}(\text{II and III}) (\Omega_{Q}(I) \text{ and } \Omega_{Q}(IV)).
\]
\[ \hat{Q} \equiv \mu_3^L(Q)(\mu_3^R(Q)) \text{ must reproduce } \Omega_Q^{(II)} \text{ and } \Omega_Q^{(III)} \ (\Omega_Q^{(I)} \text{ and } \Omega_Q^{(IV)}). \]

For example, take \( a=7, b=5, c=4 \) and \( N=2 \).
The latter implies that \( \xi_3 = \xi_1 - \xi_2 \) is assumed to be generic but consistent with the gauge symmetry and \( \hat{Q} \equiv \mu_3^L(Q)(\mu_3^R(Q)) \) must reproduce \( \Omega_{Q(II)} \) and \( \Omega_{Q(III)} \) (\( \Omega_{Q(I)} \) and \( \Omega_{Q(IV)} \)).

\[
\begin{align*}
\Omega(\hat{I}) &= ? & \Omega(I) &= ? & \Omega(\hat{I}) &= ? \\
\Omega(\hat{II}) &= ? & \Omega(II) &= ? & \Omega(\hat{II}) &= ? \\
\Omega(\hat{III}) &= ? & \Omega(III) &= ? & \Omega(\hat{III}) &= ? \\
\Omega(\hat{IV}) &= ? & \Omega(IV) &= ? & \Omega(\hat{IV}) &= ?
\end{align*}
\]
The left and the right mutations refer to arrows start and end at the same node, and the 2-cycles refer to two arrows with opposite direction between two nodes. Also, the superpotential is assumed to be generic but consistent with the gauge symmetry and the anomaly cancelation condition of 4d.

When we try to apply the above mutation rule to quivers with loops, it is important to restrict to the set of quivers without 1-cycles nor 2-cycles, where the 1-cycles are assumed to be generic but consistent with the gauge symmetry and the anomaly cancelation condition of 4d.

Furthermore, the analytical structure for Witten index is encoded only implicitly as one needs to extract the intersection numbers in a combinatorial manner.

In principle one can compute all these indices via the Abelianisation. But the toric varieties involved here are of dimension 20-ish, meaning that one needs to deal with such high-rank lattices.

\[ \hat{Q} \equiv \mu_3^L(Q) (\mu_3^R(Q)) \] must reproduce \( \Omega_Q(\text{II}) \) and \( \Omega_Q(\text{III}) \) (\( \Omega_Q(\text{I}) \) and \( \Omega_Q(\text{IV}) \)).
Index of d=1 GLSM via Path Integral

[K.Hori, H.Kim, P.Yi `14]
(cf.) [Benini, Eager, Hori, Tachikawa `13], [Cordova, Chao `14], [Hwang, Kim, Kim, Park `14]

- Compact expression has been obtained:

\[
\Omega(y; \zeta) = \frac{1}{|W|} \text{JK-Res}_\zeta [g(u)d^T u]
\]

where \( u = x_3 + i A_0 \mid_{\text{zero-mode}} \) are the zero modes of Cartan part, and the “integrand” is

\[
g(u) = \prod_A g^{(A)}_{\text{vector}}(u) \prod_I g^{(I)}_{\text{chiral}}(u)
\]

with \( g^{(A)}_{\text{vector}}(u) = \left( \frac{1}{2 \sinh \frac{z}{2}} \right)^{r_A} \prod_{\alpha \in \Delta_A} \frac{\sinh \frac{\alpha(u)}{2}}{\sinh \frac{\alpha(u) - z}{2}} \) and \( g^{(I)}_{\text{chiral}}(u) = -\frac{\sinh q_I(u) + \left( \frac{R_I}{2} - 1 \right) z}{\sinh q_I(u) + \frac{R_I}{2} z} \)
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- The JK-Res is a sum over all co-dim “r” singularities in $(\mathbb{C}^*)^r$ defined as
intersection of hyperplanes via $\{Q_{i_1}, \ldots, Q_{i_r}\}$

$$\text{JK-Res}_\zeta:\{Q_{i_1}, \ldots, Q_{i_r}\} \frac{d^r u}{(Q_1 \cdot u) \cdots (Q_r \cdot u)} = \begin{cases} 
\frac{1}{|\det(Q)|} & \text{if } \zeta \in \text{Span}_+ \langle Q_{i_1}, \ldots, Q_{i_r} \rangle \\
0 & \text{otherwise}
\end{cases}$$
Why JK-acceptable singularities involving the vector multiplet poles must have analysis. This not only reduces poles passing JK test drastically but also tends to eradicate pole passes JK positivity test, the iterated residue becomes order-dependent and degenerate poles where more than \( \kappa \) poses a big combinatorial challenge. This is further aggravated by the presence of type quivers. For general quivers, classifying poles according to JK positivity test we find a quiver with ranks (1, 2, 5) and \( \frac{\xi_1}{\xi_2} \). This example thus demonstrates that the selection of \( \frac{\xi_1}{\xi_2} \) before and after mutation. Perhaps equally noteworthy is the fact that if one starts in disallowed sub-chambers, \( \Theta \) coincides with either of anomaly cancelation condition of 4d \( \mathcal{N} = 1 \) theories.

We find that \( \hat{Q} \equiv \mu_3^L(Q)(\mu_3^R(Q)) \) must reproduce \( \Omega_Q(\text{II}) \) and \( \Omega_Q(\text{III}) \) (\( \Omega_Q(\text{I}) \) and \( \Omega_Q(\text{IV}) \)).

\[
\begin{align*}
\Omega(\text{I}) &= 50, \\
\Omega(\text{II}) &= 1/y^4 + 2/y^2 + 87 + 2y^2 + y^4, \\
\Omega(\text{III}) &= 1/y^6 + 2/y^4 + 4/y^2 + 89 + 4y^2 + 2y^4 + y^6, \\
\Omega(\text{IV}) &= 1/y^6 + 2/y^4 + 4/y^2 + 54 + 4y^2 + 2y^4 + y^6.
\end{align*}
\]

\[
\begin{align*}
\Omega(\hat{\text{I}}) &= 1/y^6 + 2/y^4 + 4/y^2 + 89 + 4y^2 + 2y^4 + y^6, \\
\Omega(\hat{\text{II}}) &= 35, \\
\Omega(\hat{\text{III}}) &= 1/y^4 + 2/y^2 + 37 + 2y^2 + y^4, \\
\Omega(\hat{\text{IV}}) &= 1/y^4 + 2/y^2 + 87 + 2y^2 + y^4.
\end{align*}
\]

\[
\begin{align*}
\Omega(\hat{\text{I}}) &= 1/y^{10} + 2/y^8 + 4/y^6 + 6/y^4 + 8/y^2 + 58 + 8y^2 + 6y^4 + 4y^6 + 2y^8 + y^{10}, \\
\Omega(\hat{\text{II}}) &= 1/y^6 + 2/y^4 + 4/y^2 + 54 + 4y^2 + 2y^4 + y^6, \\
\Omega(\hat{\text{III}}) &= 50, \\
\Omega(\hat{\text{IV}}) &= 50.
\end{align*}
\]
\[ \hat{Q} \equiv \mu_3^L(Q)(\mu_3^R(Q)) \text{ must reproduce } \Omega_{Q(\text{II})} \text{ and } \Omega_{Q(\text{III})} \text{ ( } \Omega_{Q(\text{I})} \text{ and } \Omega_{Q(\text{IV})} \text{).} \]

The JK-Res approach leads to the desired links even analytically.
Summary and Outlook

• d=4 N=2 BPS states were studied via d=1 N=4 Quiver GLSM

• Wall-crossing-sensitive indices have wall-crossing-safe invariants

• The *quiver invariants* of an abelian cyclic quiver theory are naturally characterised as the “middle” cohomology; non-abelian generalisation of the geometric interpretation?

• The moduli space geometry for a non-abelian quiver can be tackled via abelianisation and/or path integral

• Mutation of d=1 quiver theory can only be *selectively* performed to preserve Witten index

• Asymptotics in the large-rank limit and d=4 N=2 BPS black-hole microstates?
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Thank you!