Entanglement Hamiltonians for chiral fermions

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Outline

- The resolvent - why we care
- Green's functions for chiral fermions
- Entanglement Hamiltonians for chiral fermions
  - Majorana: NS, R
  - Dirac: generic, periodic BC
- Entanglement entropy

Generalities

Suppose that we can partition a system into V and its complement, and the Hilbert space decomposes into $\mathcal{H}_V \otimes \mathcal{H}_{V\,\mathrm{complement}}$.

Starting with some state in $\mathcal{H}$ (which can be pure, or mixed) and tracing over the degrees of freedom in the complement of V yields a reduced density matrix ρ_V.
This will be used to compute correlation functions of operators defined in V only $\langle O \rangle_V = Tr(\rho_V O)$.

(von Neumann) Entanglement entropy $S_V = -Tr(\rho_V \ln \rho_V)$ is a measure of entanglement for bi-partite pure states. It is non-zero if the original (pure) state was entangled/non-separable.
A more refined measure of entanglement is the entanglement Hamiltonian: $\rho_V = \mathcal{N} e^{-H_V}$.

What are we after

What is the effect of the boundary conditions on entanglement?

What is the effect of zero-modes on entanglement?

We will consider a simple, yet interesting enough system to address these questions: 1+1 dimensional chiral fermions.

The spatial direction is a circle.

Entanglement Hamiltonians for Free Fermions

Consider a system of spinless free fermions on a lattice: $\{\psi_i, \psi_j\} = \delta_{ij}$. Then, given the correlation function

$$G_{ij} \equiv \langle \psi_i \psi_j^\dagger \rangle$$

all higher order correlators (by Wick's theorem) can be expressed in terms of G_{ij}, e.g.

$$\langle \psi_i \psi_j \psi_k^\dagger \psi_l^\dagger \rangle = G_{jk} G_{il} - G_{ik} G_{jl}.$$

Then on a subset of lattice sites, labelled $V = \{m, n, \dots\}$, the correlator is computed with the help of the density matrix

$$\tilde{G}_{mn} = Tr(\rho_V \psi_m \psi_n^\dagger)$$

and, more generally, for any operator in V

$$\langle O \rangle_V = Tr(\rho_V O_V)$$

To satisfy the factorization,

$$\rho_V \equiv \mathcal{N} \exp(-H_v) = \mathcal{N} \exp(-\sum_{m,n} h_{mn} \psi_m^\dagger \psi_n)$$

where H_v is the entanglement Hamiltonian.

The entanglement Hamiltonian may be diagonalized by some fermion transformation $\psi_i = \sum_k \phi_k(i) a_k$ with $\{a_k, a_l^\dagger\} = \delta_{kl}$:

$$H_V = \sum_k \epsilon_k a_k^\dagger a_k$$

which means that

$$h_{mn} = \sum_k \epsilon_k \phi_k(m) \phi_k^*(n)$$

Then $\mathcal{N}$ is determined from the normalization condition $Tr(\rho_V) = 1$ and

$$\tilde{G}_{mn} = \sum_k \frac{1}{\exp(\epsilon_k) + 1} \phi_k^*(m) \phi_k(n).$$

Given the relationship between the e-vales of h_V and the correlator $\tilde{C}$ then (Peschel, 2003)

$$h_V = \ln(\tilde{G}^{-1} - 1).$$

The kernel of the entanglement Hamiltonian can be expressed in integral form as Casini, Huerta 2009

$$h_V = -\int_{\frac{1}{2}}^{\infty} d\beta \Big(L(\beta) + L(-\beta) \Big)$$

where $L(\beta)$ is the resolvent

$$L(\beta) = (\tilde{G} + \beta - \tfrac{1}{2})^{-1}$$

More precisely, the correlation $\tilde{G}$ can be written in terms of the un-projected correlator G as

$$\tilde{G} = P_V G P_V$$

and where P_V is a spatial projector on V.
To find the entanglement Hamiltonian we then have to compute the resolvent $L(\beta)$

$$L(\beta) = (P_V G P_V + \beta - 1/2)^{-1}$$

Green's functions as projectors

Here we are addressing 1+1 dimensional chiral fermions, Majorana and Dirac, on a spatial circle $x \sim x + 2\pi R$.

For Majorana fermions, the Lagrangian is

$$\mathcal{L} = \frac{i}{2}\psi(\partial_t + \partial_x)\psi$$

and we have two possible boundary conditions:

- Neveu-Schwarz/anti-periodic: $\psi(x + 2\pi R) = -\psi(x)$ which dictates the mode expansion

$$\psi(t,x) = \frac{1}{2\pi R}\sum_k b_k \exp(-ik(t-x)/R), k \in \mathbf{Z} + \tfrac{1}{2}, \quad b_{-k} = b_k^\dagger,$$

$$b_k|\Omega\rangle = 0, \quad \text{for} \quad k > 0.$$

- Ramond/periodic: $\psi(x + 2\pi R) = \psi(x)$ which dictates the mode expansion

$$\psi(t,x) = \frac{1}{2\pi R}\sum_k b_k \exp(-ik(t-x)/R), k \in \mathbf{Z}, \quad b_{-k} = b_k^\dagger,$$

$$b_k|\Omega\rangle = 0, \quad \text{for} \quad k > 0.$$

- NS Green's function as a projector:

$$G^{\rm NS}(x,y) = \langle \Omega|\psi(x+i0^+)\psi(y)|\Omega\rangle = \exp(i(x-y)/(2R))n(x,y)$$

$$n(x,y) \equiv \langle x|n|y\rangle = 1/(2\pi R)\sum_{k=0}^{\infty}\exp(ik(x-y+i0^+)/R)$$

$$= \frac{1}{2\pi R}\frac{1}{1-\exp(i(x-y+i0^+)/R)}$$

Then, acting on the space of single-particle states spanned by the momentum eigenstates $|k\rangle$, with

$$\langle x|k\rangle = (1/\sqrt{2\pi R})\exp(ikx/R),$$

n is a projector onto non-negative $(k \geq 0)$ momentum modes. With U_α a unitary operator which induces a shift of momenta

$$U_\alpha|k\rangle = |k+\tfrac{1}{2}\rangle$$

the NS Green's function $G^{NS}(x,y) = \langle x|G^{NS}|y\rangle$ can be written as

$$G^{\rm NS} = U_{\alpha=-1/2}\, n\, U^{-1}_{\alpha=-1/2}$$

Upshot: (somewhat sketchily, we'll come back to this).
To find the entanglement Hamiltonian in the NS sector we begin by finding the resolvent

$$N(\beta) \equiv (P_V n P_V + \beta - 1/2)^{-1}$$

From $N(\beta)$ we get the resolvent in the NS sector by using the gauge transform/spectral flow

$$L(\beta) = U_{\alpha=1/2} N U^{-1}_{\alpha=1/2}$$

and lastly we perform the β integral $\int d\beta (L(\beta) + L(-\beta))$ to obtain the EH.

What about the Ramond sector? There is a zero-energy mode which complicates the story.

We'll come back to this...

For now, there is another low-hanging fruit, chiral Dirac fermions.

Dirac fermions and their Green's functions

For the Dirac fermions, with Lagrangian

$$\mathcal{L} = \frac{i}{2}\Psi^\dagger(\partial_t + \partial_x)\Psi,$$

we find that we can impose more general BC

$$\Psi(x + 2\pi R, t) = e^{i2\pi\alpha}\Psi(x,t), \qquad \alpha \in [0,1).$$

The mode expansion of the chiral (right-movers) Dirac fermions is

$$\Psi(t,x) = \frac{1}{\sqrt{2\pi R}}\sum_k b_k \exp(-ik(t-x)/R)), \quad k \in \mathbf{Z} + \alpha$$

and the ground state $|\Omega\rangle$ is defined by

$$b_k|\Omega\rangle = 0, \quad \text{for} \quad k > 0,$$
$$b^\dagger_{-k}|\Omega\rangle = 0 \quad \text{for} \quad k > 0,$$

As long as $\alpha \neq 0$, the Green's function takes the same form as discussed before, with a more general α:

$$G^\alpha_\Psi(x,y) \equiv \langle\Omega|\Psi(x + i0^+)\Psi^\dagger(y)|\Omega = \exp(i\alpha(x-y)/R)n(x,y)$$

So, we can write as before

$$G_\Psi = U_\alpha n U_\alpha^{-1}.$$

What if $\alpha = 0$?

If $\alpha = 0$, then there is a zero mode $|k\rangle = 0$ and the ground state is degenerate:

$$b_0|\mathrm{empty}\rangle = 0, \qquad \mathrm{b}_0|\mathrm{occupied}\rangle = |\mathrm{empty}\rangle$$
$$b_0^\dagger|\mathrm{empty}\rangle = |\mathrm{occupied}\rangle, \qquad \mathrm{b}_0^\dagger|\mathrm{occupied}\rangle = 0$$

We will consider the case of a statistical mixture

$$\rho = \tfrac{1}{2}|\mathrm{occupied}\rangle\langle\mathrm{occupied} + \tfrac{1}{2}|\mathrm{empty}\rangle\langle\mathrm{empty}|$$

which would arise if we start from a finite-temperature Fermi Dirac distribution and lower the temperature to zero.
Then, the Green's function on this mixture (where all $|k\rangle$ states for $k < 0$ are occupied and there is 50% probability that the zero mode is occupied) is

$$G_\Psi^{\alpha=0} = n - \tfrac{1}{2}|0\rangle\langle 0|$$

The Ramond sector of chiral Majorana fermions

We come back now to the Majorana fermions $\psi^\dagger = \psi$.

In the R sector there is a zero-mode b_0, with

$$\{b_0, b_0\} = 1.$$

The minimal non-trivial Hilbert space rep of the Clifford algebra is 2-dimensional. The ground state is again degenerate.

Our main assumption (following Peschel) is that the Green's function determines all subsequent correlators and we can obtain the entanglement Hamiltonian from the Green's function by computing the appropriate resolvent.

However, the Green's function in the R sector is the same

$$G^{\mathrm{R}}(x,y) = \frac{1}{2\pi R}\left[-\tfrac{1}{2} + \sum_{k=0}^{\infty} \exp(ik(x-y+i0^+)/R)\right]$$

$$G^{\mathrm{R}}(x,y) = n(x,y) - \frac{1}{4\pi R},$$

regardless on which linear combination of the two ground states we evaluate it on, or whether we start with a mixture.

The EH we compute are for states which preserve parity (have a vanishing vev for an odd number of Majorana operators, e.g. $\langle b_0 \rangle = 0$).

Entanglement Hamiltonian for chiral Dirac fermions

Using the results of Peschel, and Casini-Huerta, the reduced density matrix in the subset V of S^1 defines the EH

$$\rho_V = \mathcal{N} e^{-H_V} = \mathcal{N} \exp\left(- \int_V dxdy h_V(x,y) \Psi(x)^\dagger \Psi(y) \right)$$

If $\alpha \neq 0$, the kernel of the EH is in terms determined from the resolvent

$$L^\alpha(\beta) = U_\alpha N U_\alpha^{-1}, \;\; \alpha \neq 0$$

$$N(\beta) = (P_V n P_V + \beta - 1/2)^{-1}$$

$$U_\alpha^{-1} h_V U_\alpha = -\ln((P_V n P_V)^{-1} - 1) = -\int_{1/2}^{\infty} d\beta \Big(N(\beta) + N(-\beta) \Big)$$

So, we must endeavor to find the resolvent $N(\beta)$. Then we'll sort out $\alpha = 0$.

A Riemann-Hilbert problem

To find the resolvent $N(\beta) = (P_V n P_V + \beta - 1/2)^{-1}$ we use the fact that n is a projector.
Start with

$$K(x, y) = f(x)n(x, y)g(x).$$

Suppose we want to compute $(1 + K)^{-1}$.
Then this reduces to a RH problem: For

$$X(x) \in S^1, \qquad X(x) = 1 + f(x)g(x),$$

we want to find X_+, X_- s.t.

$$X(x) = X_-^{-1} X_+(x), \qquad X_+/X_- \text{ has only positive/negative } k \text{ modes}$$

Assuming this is done then

$$(1 + K)^{-1} = 1 - fX_+^{-1} n X_- g$$

where f, g act multiplicatively on x-space: $\langle x|f|y\rangle = f(y)\delta(x - y)$.
For us, f, g are equal to each other $f(x) = g(x) = \Theta_v(x)$ and equal to the characteristic function of the subset V.

Let's check!

$$(1+K)(1+K)^{-1} = (1+fng)(1 - fX_+^{-1}nX_-gX_-g) \stackrel{?}{=} 1$$

A bit of algebra:

$$fng + fX_+^{-1}nX_-g - fn\boxed{gf}X_+^{-1}nX_-g \stackrel{?}{=} 0$$

Substitute $gf = X_-^{-1}X_+ - 1$:

$$f(n - X_+^{-1}nX_- - \boxed{nX_-^{-1}n}X_- + \boxed{nX_+^{-1}n}X_-)g \stackrel{?}{=} 0$$

Use next $nX_+^{-1}n = X_+^{-1}n$ and $nX_-^{-1}n = nX_-^{-1}$.
It works!
So, in our case, all is left to do is find the X_-, X_+ functions, given that $f(x), g(x)$ are the equal to the characteristic function on V.

Suppose that we would be looking at the characteristic function on an interval on the real axis.
Then we can write

$$\Theta_{V=(a,b)\subset\mathbf{R}} = \frac{1}{2\pi i}\left(\ln \frac{x-a-i0^-}{x-b-i0^-} - \ln \frac{x-a+i0^+}{x-b+i0^+} \right)$$

Since we are after X_+/X_- s.t. $X_-^{-1}X_+ = 1 - \# fg = 1 - \#\Theta_V$, by taking the log we find

$$\ln(1-\#\Theta_V) = \Theta_V \ln(1-\#)$$

and so

$$-\ln(X_-) + \ln(X_+) = \ln(1-\#)\left[\frac{1}{2\pi i}\left(\ln \frac{x-a-i0^-}{x-b-i0^-} - \ln \frac{x-a+i0^+}{x-b+i0^+} \right) \right]$$

Of course, we need to do this for a set of disjoint intervals, and we need to do it on S^1. Easy!

A quick look at $X_\pm$, for $V \cup (a_j, b_j) \subset S^1$:

$$\ln X_\pm = ih(\beta) \sum_j \ln \frac{e^{\frac{i}{R}(x \pm i\epsilon)} - e^{ia_j}}{e^{\frac{i}{R}(x \pm i\epsilon)} - e^{ib_j}} \equiv ih(\beta) Z^\mp$$

where

$$h(\beta) = 1 - \# = 1 - \frac{1}{\beta - 1/2} = \frac{\beta + 1/2}{\beta - 1/2}.$$

We still have do the integral over β of

$$N(\beta) = \frac{1}{\beta - 1/2}\left(1 - \frac{1}{\beta - 1/2} f X_+^{-1} n X_- g\right)$$

whose kernel we have just computed:

$$\langle x|N(\beta)|y\rangle = \frac{\delta(x-y)}{\beta - 1/2} - \frac{1}{\beta^2 - 1/4} e^{-ih(\beta)Z^+(x) + ih(\beta)Z^+(y)} n(x,y)$$

Dirac $\alpha = 0$/ Majorana in the Ramond sector

Before we get there, what is the resolvent $L(\beta)$ if $\alpha = 0$?
If $\alpha = 0$, the Green's function wasn't equal to some gauge transformed projector n. Instead,

$$G_\Psi^{\alpha=0} = n - \tfrac{1}{2}|0\rangle\langle 0|$$

We can think of this as the zero-mode being responsible for a rank one perturbation of the problem we already solved.
We can do a Schwinger-Dyson expansion of

$$L^{\alpha=0}(\beta) = \frac{1}{P_V n P_V - \beta + 1/2 - \frac{1}{2} P_V|0\rangle\langle 0|P_V}$$

$$L^{\alpha=0}(\beta) = N(\beta)\frac{1}{1 - \frac{1}{2} N P_V|0\rangle\langle 0|P_V}$$

and resum!

$$L^{\alpha=0}(\beta) = N(\beta) + \frac{N(\beta)P_V|0\rangle\langle 0|P_V N(\beta)}{2 - \langle 0|P_V N(\beta) P_V|0\rangle}$$

As a bonus, we have just found a way to discuss excited states of the type

$$G_{\rm new} = n + |a\rangle\langle a|$$

Their resolvent will be

$$(P_V G_{\rm new} P_V + \beta - \tfrac{1}{2})^{-1} = N(\beta) + \frac{N(\beta)P_V|a\rangle\langle a|P_V N(\beta)}{1 + \langle a|P_V N(\beta) P_V|a\rangle}$$

Side comment:

$$n - |k=1\rangle\langle k=1|$$

is a genuine excited state since n is a projector onto non-negative momentum modes, but

$$n + |k=-1\rangle\langle k=-1|$$

is not.

Back to the Majorana fermions and their EH

Punch line: crucial factor of 1/2 difference :

$$h_V^{\mathrm{Majorana}} = \frac{1}{2} \ln \left((P_V G P_V)^{-1} - 1 \right)$$

Why? Consider the Majorana fermions defined on a lattice $\{\psi_i, \psi_j\} = \delta_{ij}$. The reduced density matrix is

$$\rho_V^{\mathrm{Majorana}} = \mathcal{N} \exp(-\sum_{m,n} h_{mn} \psi_m \psi_n)$$

and the entanglement Hamiltonian kernel h_{mn} is an antisymmetric matrix (no longer hermitian).
The correlation functions in the subset V can be shown to equal

$$\tilde{G}_{mn} = \langle \psi(m) \psi(n) \rangle = \left(\frac{1}{1 + \exp(-2 h_V)} \right)_{mn}$$

So, the EH kernel of the Majoranas is given in terms of the resolvent as

$$h_V^{\mathrm{Majorana}} = \frac{1}{2}\int_{1/2}^{\infty} d\beta \Big(L(\beta) + L(-\beta) \Big)$$

and where $L(\beta)$ is the same as for the Dirac fermions

$$L(\beta) = (P_v G^{\mathrm{Majorana}} P_v + \beta - 1/2)^{-1}.$$

Does this make sense? We'll see that it does - e.g. in computing the entanglement entropy we expect that the EE for the Dirac fermions be twice that of the Majoranas (one Dirac= two Majorana fermions).

Entanglement Hamiltonian(s)

Time to compute some integrals!

The easy ones first: $\alpha \neq 0$ (we can do both generic Dirac BC and Majorana NS chiral fermions at the same time).
Combining $L(\beta) + L(-\beta)$ yields

$$h^{\text{Dirac}}_{V,\alpha\neq 0} = 2\pi \int_{-\infty}^{\infty} dh \boxed{e^{i\alpha(x-y)/R}} n_{PV}(x,y) e^{-ih(Z^+(x)-Z^+(y))}$$

where $n_{PV} = n - \frac{1}{2}$.
We find

$$h^{\text{Dirac}}_{V,\alpha\neq 0} = 4\pi^2 e^{i\alpha(x-y)/R} n_{PV}(x,y)\delta(Z^+(x) - Z^+(y))$$

Evaluating the solutions $y_l(x)$ of $Z(x) = Z(y)$, there will be a trivial solution $y = x$ which yields a local contribution to the entanglement Hamiltonian and a set of non-trivial solutions which give rise to non-local contributions.

$$H^{\mathrm{NS}}_{V,\mathrm{loc.}} = \pi i \int_V dx \frac{1}{Z'(x)} \psi(x) \partial_x \psi(x),$$

$$H^{\mathrm{Dirac}}_{V,\alpha\neq 0,\mathrm{loc.}} =$$
$$-2\pi i \int_V \mathrm{dx} \Psi^\dagger(x) \left(\frac{1}{|Z'|} \frac{d}{\mathrm{dx}} - \frac{(1-2\alpha)}{2iR|Z'|} - \left(\frac{1}{2|Z'|} \right)' \right) \Psi(x),$$

as well as a bi-local contribution with kernel:

$$h^{\mathrm{Dirac}}_{V\ \mathrm{bi\text{-}loc.}} = 2\pi \sum_{l; y_l(x) \neq x} \frac{e^{i\alpha \frac{(x-y)}{R}} |Z'(x)|^{-1}}{R\left(1 - e^{\frac{i(x-y)}{R}}\right)} \delta(x - y_l(x))$$

Side comment(s): the local terms in the NS and Dirac ($\alpha \neq 0$ case) reproduce results derived earlier (Myers et al, Klich, Pand0-Zayas, DV, Wong 2013) using a path integral approach.
Schematically, the reduced density matrix can be viewed as a propagator which evolves BC for a field from the upper lip of a cut along V to the lower lip.
In a CFT and for spherical entangling regions this yieds the EH in terms of

$$EH \sim \int_V \beta(x) T_{00}(x)$$

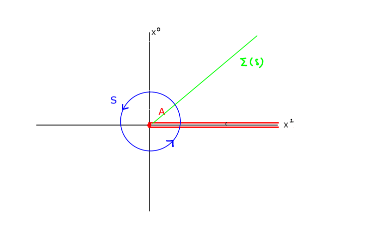
where $\beta(x)$ is an entanglement (inverse) temperature.

Why?

$\rho \sim \mathcal{T} e^{-\int dsK(s)}$.

For the half-line K is the generator of rotations/boosts, and so it is s-independent.

s evolution :



$H_{\mathrm{half-line}} = 2\pi K = 2\pi \int_V dx\, x T_{00}$ where we evaluated K on the slice $s = 0$.

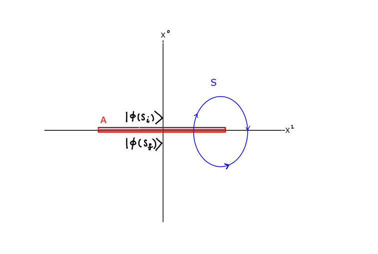
$$\beta_{\mathrm{half-line}}(x) = 2\pi x$$

Then, in a CFT we can use conformal transf to map the half-line to an interval (or a circle).

E.g. $z \to \frac{w-u}{w-v}$ is a mapping to the interval, and

$$\beta_{\text{interval}}(x) = 2\pi \frac{(x-u)(x-v)}{x-v}$$

s evolution :



The inverse temperature is the result of the conformal mapping.

The expression for the local term of the Majorana NS EH has precisely this form:

$$H^{\mathrm{NS}}_{V,\mathrm{local}} = \pi i \int_V dx \frac{1}{Z'(x)} \psi(x)\partial_x\psi(x),$$

and $\beta(x)$ read off from this expression reproduces our previous result (IK,LPZ, DV, GW, 2013).

$T_{00} \sim i\psi(x)\partial_x\psi(x)$ and
$\beta(x) \sim \frac{1}{Z'(x)}$, more concretely this is the entanglement temperature when V is an interval of S^1:

$$\beta(x) = 4\pi R \csc\frac{a-b}{2R}\sin\frac{a-x}{2R}\sin\frac{b-x}{2R}.$$

In the Dirac ($\alpha \neq 0$) case, the local term can be interpreted as

$$H_{V,\text{local}}^{\text{Dirac},\alpha\neq 0} = -2\pi i \int_V \beta(x)(T_{00} - \mu(x)\Psi^\dagger(x)\Psi(x))$$

where

$$\beta(x) = 2\pi|Z'(x)|^{-1} = 4\pi R \csc\frac{a-b}{2R}\sin\frac{a-x}{2R}\sin\frac{b-x}{2R},$$
$$\mu = \frac{1-2\alpha}{2R}$$

act as a local entanglement (inverse) temperature and chemical potential.
Note: the stress-energy tensor should be taken hermitean

$$T_{00} = \langle\frac{-i\Psi^\dagger(\partial_x\Psi) + i(\partial_x\Psi^\dagger)\Psi}{2}\rangle$$

Zero-mode and entanglement Hamiltonian(s)

To account for the zero-mode contribution we had to sum the rank one perturbation, from a Schwinger-Dyson series. The result was

$$L^{\alpha=0}(\beta) = N(\beta) + \frac{N(\beta)P_V|0\rangle\langle 0|P_V N(\beta)}{2 - \langle 0|P_V N(\beta) P_V|0\rangle}$$

In position space the zero-mode contribution is

$$\langle x|L^{\mathrm{R}}_{\text{zero-mode}}(\beta)|y\rangle = \frac{1}{2\pi R}\frac{\int_V dzdz'\langle x|N(\beta)|z\rangle\langle z'|N(\beta)|y\rangle}{2 - \frac{1}{2\pi R}\int_V dzdz'\langle z|N(\beta)|z'\rangle}.$$

Evaluating the integrals yields

$$\langle x|L^{\mathrm{R}}_{\text{zero-mode}}(\beta)|y\rangle = \frac{2\sinh^2(\pi h)e^{ih(Z(y)-Z(x))}}{\pi R(1+e^{\frac{l_V h}{R}})},$$

where l_V is the total length of V: $l_V = \sum_i (b_i - a_i)$.

To get the EH we need to do the β-integral: $\int d\beta_{1/2}^{\infty}(L(\beta)+L(-\beta)$.
The zero mode induced contribution for the Majorana (Dirac $\alpha=0$ case has an extra factor of 2) fermion is:

$$H^{\mathrm{R}}_{V\,\text{zero-mode}} = \frac{1}{2R}\int_{-\infty}^{\infty} dh \frac{1}{1+e^{\frac{l_v h}{R}}} e^{ih(Z(y)-Z(x))}$$
$$= \sum_l \frac{\pi}{2|Z'(x)|R}\delta(x-y_l(x)) + p.v.\frac{\pi i}{2l_v}\frac{1}{\sinh\left(\frac{\pi R}{l_v}(Z(y)-Z(x))\right)}$$

NOTE: the local contribution for the Dirac fermion cancels the chemical potential $\mu = 1/2-(\alpha=0)=1/2$ shift from the local non-zero mode piece such that

$$H^{\mathrm{Dirac}}_{V,\mathrm{local}}(\alpha=0) = H^{\mathrm{Dirac}}_{V,\mathrm{local}}(\alpha=\tfrac{1}{2})$$

The zero-mode also contributes to non-local terms (even for the one-interval case).

Entanglement Entropy

What can we say about the EE? $S_V = -Tr(\rho_V \ln(\rho_V))$.
Using that $\rho_V = \mathcal{N} e^{-H_V}$,

$$\begin{aligned} S_V &= -\ln \mathcal{N} + Tr(\rho_V \int \Psi^\dagger \cdot h \cdot \Psi) = -\ln \mathcal{N} + Tr(P_V G_\Psi P_V h_V) \\ &= \ln(Tr(e^{-H_V}) + Tr\left(\boxed{P_V G_\Psi P_V} \ln((P_V G_\Psi P_V)^{-1} - 1) \right) \\ &= \ln(\textstyle\prod_k (1 + e^{-\epsilon_k})) + Tr(\tilde{G} \ln(1 - \tilde{G})) - Tr(\tilde{G} \ln \tilde{G}) \\ &= Tr(1 - \tilde{G}) \ln(1 - \tilde{G}) - Tr \tilde{G} \ln \tilde{G} \end{aligned}$$

where the relation between ϵ_k and the e-values of $\tilde{G}$ was used ($g_k = (1 + \epsilon_k)^{-1}$).
Bottom line: the resolvent can be used again, this time to compute the entanglement entropy.

An integral form (Casini & Huerta 2009) for the EE:

$$S_V = \int_{1/2}^{\infty} d\beta \; Tr\left[(\beta - \tfrac{1}{2})(L(\beta) - L(-\beta)) - \frac{2\beta}{\beta + 1/2}\right]$$

The difference between the Majorana/Dirac R and NS chiral fermions EE:

$$\delta S_{\rm Majorana} = \tfrac{l_V}{4R} \int_0^\infty dh \, \tanh(\tfrac{l_V h}{2R})(\coth(h\pi) - 1).$$

As an asymptotic expanion in the ratio $\frac{l_V}{2\pi R}$ gives:

$$\delta S_{\rm Majorana} \sim \tfrac{1}{2} \sum_{n=1}^{\infty} \frac{l_V^{2n}}{(2\pi R)^{2n}} \frac{\left(2^{2n} - 1\right) B_{2n}\zeta(2n)}{2n}.$$

This result was obtained before by Herzog& Nishioka (2013) who noted that the sum is not convergent.

NOTE The integral form of $\delta S_{\rm Majorana}$ is perfectly well defined!

2nd NOTE: For $l_V = 2\pi R$, $\delta S_{\rm Majorana} = 1/2 \ln(2)$ and $\delta S_{\rm Dirac} = \ln(2)$.

Boundary entropy - Affleck &Ludwig (1991).

Future directions

- Excited states
- Non-chiral fermions, non-relativistic fermions, paired states
- Is there a way to account for parity odd states?
- Chiral bosons
- Higher dimensions?