6D SCFTs and Group Theory

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Based On

- 1502.05405/hep-th
  - with Jonathan Heckman, David Morrison, and Cumrun Vafa
- 1506.06753/hep-th
  - with Jonathan Heckman
- 1601.04078/hep-th
  - with Jonathan Heckman, Alessandro Tomasiello
- 1605.08045/hep-th
  - with David Morrison
- 1612.06399/hep-th
  - with Noppadol Mekareeya, Alessandro Tomasiello
- work in progress
  - with Fabio Apruzzi, Jonathan Heckman
Outline

I. Classification of 6D SCFTs
   i. Tensor Branches/Strings
   ii. Gauge Algebras/Particles

II. 6D SCFTs and Homomorphisms
   i. $\mathfrak{su}(2) \rightarrow \mathfrak{g}_{ADE}$
   ii. $\Gamma_{ADE} \rightarrow E_8$

III. 6D SCFTs and Automorphism Groups
   i. Automorphism Groups
   ii. Geometric Phases
The Big Picture

Group Theory 6D SCFTs Geometry
What is a 6D SCFT?

- $S$=supersymmetric (8 or 16 supercharges)
- $C$=conformal symmetry
- $FT$=Field theory in 5+1 dimensions
Why Study 6D SCFTs?

- Nahm: Maximal SCFT dimension is six
- Degrees of freedom \(\neq\) particles (but it’s a QFT!)
- QFT of M5-branes is a 6D SCFT
- Compactification \(\Rightarrow\) 5D/4D/3D/2D Theories
Focus: \((1, 0)\) SCFTs

Conformal Symmetry: \(\mathfrak{so}(6, 2)\)

Supersymmetry: 8 \(Q\)'s and 8 \(S\)'s

R-symmetry: \(\mathfrak{su}(2)_R\)
Studied since the 1990’s’s

Many groups:

Witten ’95; Strominger ’95; Ganor and Hanany ’96;
Seiberg and Witten ’96; Bershadsky and Johansen ’96;
Brunner and Karch ’96; Blum and Intriligator ’97;
Intriligator ’97; Hanany Zaffaroni ’97;
+

But: Even now, still viewed as “mysterious”...
Classification of 6D SCFTs
Classification of 6D SCFTs

- 6D SCFTs can be classified via F-theory
- Nearly all F-theory conditions can be phrased in field theory terms
- 6D SCFTs = Generalized Quivers
Classification of 6D SCFTs

• Looks like chemistry

"Atoms" for $3 \leq n \leq 12$

$c.f. \text{ Morrison and Taylor '12}$

$A_N$

$D_N$

$E_6$

$E_7$

$E_8$

"Radicals"

$E_6$

$E_7$

$E_8$
What is F-theory?

Vafa '96

IIB: $\mathbb{R}^{9,1}$ with position-dependent coupling $\tau = C_0 + i e^{-\Phi}$
6D Theories and F-theory

Vafa '96, Vafa Morrison, I/II '96

All known 6D theories have F-theory avatar

IIB: $\mathbb{R}^{5,1} \times B_2$ with pos. dep. coupling $\tau(z_B)$

F-theory on $\mathbb{R}^{5,1} \times CY_3$

$T^2 \rightarrow CY_3$

$\downarrow$

$B_2$
Geometric Picture

Base $B_2$

Singularities in base $\Rightarrow$ strings ($D3 / \mathbb{P}^1$)

Singularities in fiber $\Rightarrow$ particles (7-brane on $\mathbb{P}^1$)
Tensionless Strings in F-theory

- Realized by D3-brane on collapsing $\mathbb{CP}^1$
  
  Tension $= \text{Vol}(\mathbb{CP}^1) \to 0$
SCFT Limit

Start: A smooth base $B_2$

End: To get a CFT, sim. contract curves of $B_2$
Strings from D3 on a $\mathbb{P}^1$

$-\Sigma \cap \Sigma = \text{String Charge}$

(which must be integer $> 0$)

$\mathbb{R}^{5,1} \times \text{Base } B_2$
Dirac Pairing for String Charge Lattice

Intersection Matrix $\longleftrightarrow$ Dirac Pairing

$$
\Omega_{IJ} = \begin{pmatrix}
-4 & 1 & 0 \\
1 & -1 & 1 \\
0 & 1 & -4
\end{pmatrix}
$$

$\Omega_{IJ}$ negative definite $\Leftrightarrow$ Curves contractible
Example: All $(2, 0)$ Theories

Witten ’95, Strominger ’95

Type IIB on $\mathbb{C}^2/\Gamma_{ADE}$

Resolution Involves:
Bouquet of $\mathbb{CP}^1$'s

$\mathbb{CP}^1_i \cap \mathbb{CP}^1_j = -\text{Dynkin}_{ij}$

Note: $\mathbb{CP}^1_i \cap \mathbb{CP}^1_i = -2$

$A_N$

$D_N$

$E_6$

$E_7$

$E_8$
Blow-down Operations

\[ m \ 1 \ n \rightarrow m - 1 \ n - 1 \]

\[ 1 \ n \rightarrow n - 1 \]
Coarse Classification of Bases*

Heckman, Morrison, Vafa ’13

(2,0) SCFT \iff \Gamma \subset SU(2)

(1,0) SCFT \Rightarrow \Gamma \subset U(2)

*Bases related by blow-downs/blow-ups have same $\Gamma \subset U(2)$
Coarse Classification of Bases

Heckman, Morrison, Vafa ’13

\[
x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_r
\]

\[
\frac{p}{q} = x_1 - \frac{1}{x_2 - \ldots - \frac{1}{x_r}}
\]

\[
\Gamma : (z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)
\]

\[
B_2 = \mathbb{C}^2 / \Gamma
\]
Complete Classification of Bases

The Base Quivers have a very simple structure!

\[ G_1 \subseteq G_2 \subseteq \cdots \subseteq G_m \supseteq \cdots \supseteq G_{k-1} \supseteq G_k \]
Particles from D7’s on a $\mathbb{P}^1$

$3 \leq n \leq 12 \Rightarrow$ always have gauge fields
(elliptic fiber is singular: Morrison Taylor ’12)
Minimal Gauge Algebras
Fiber Enhancements

\[ \mathfrak{so}_8 \rightarrow 3 \rightarrow 3 \rightarrow 3 \rightarrow 3 \rightarrow \mathfrak{su}_3 \]
+ 8_v, 8_s, 8_c
+ 2 spinors

\[ \mathfrak{so}_7 \rightarrow 3 \rightarrow 4 \rightarrow 4 \rightarrow 4 \rightarrow \mathfrak{e}_6 \]
+ 2 fundamentals

\[ \mathfrak{e}_7 \rightarrow 4 \rightarrow 3 \rightarrow 3 \rightarrow 3 \rightarrow \mathfrak{so}_8 \]
+ 2 fundamentals
6D SCFTs and Homomorphisms
6D SCFTs and Homomorphisms

- Large classes of 6D SCFTs have connections to structures in group theory
- The correspondence has been verified explicitly
M5-Branes Probing $\mathbb{C}^2/\Gamma_{ADE}$

$\mathbb{R}^{5,1}$

$\mathbb{C}^2/\Gamma_{ADE}$

$N$ M5s

$\times \times \times \times \times \times \times$

$\bullet \bullet \bullet \bullet \bullet \bullet$

$g$ $g$ $g$

$g$ $2$ $2$ $\cdots$ $2$ $2$ $g$

$N-1$
M5-Branes Probing $\mathbb{C}^2/\Gamma_{ADE}$

$A_{k-1}$:

$\begin{align*}
A_{k-1} : & \quad \begin{array}{ccc}
\text{su}_k & \text{su}_k & \text{su}_k \\
\text{su}_k & 2 & 2 & 2 & \text{su}_k
\end{array} \\
\end{align*}$

$D_k$:

$\begin{align*}
D_k : & \quad \begin{array}{ccc}
\text{so}_{2k} & \text{so}_{2k} \\
\text{so}_{2k} & 2 & 2 & \text{so}_{2k}
\end{array} \quad \rightarrow \quad \begin{array}{ccc}
\text{sp}_{k-4} & \text{so}_{2k} & \text{sp}_{k-4} & \text{so}_{2k} & \text{sp}_{k-4} \\
\text{so}_{2k} & 1 & 4 & 1 & 4 & 1 & \text{so}_{2k}
\end{array}
\end{align*}$

$E_6$:

$\begin{align*}
E_6 : & \quad \begin{array}{ccc}
\text{e}_6 & \text{e}_6 \\
\text{e}_6 & 2 & \text{e}_6
\end{array} \quad \rightarrow \quad \begin{array}{ccc}
\text{e}_6 & \text{e}_6 & \text{e}_6 \\
\text{e}_6 & 1 & 3 & 1 & 6 & 1 & 3 & 1 & \text{e}_6
\end{array}
\end{align*}$
Nilpotent Deformations

- Matrix of normal deformations $\Phi$ characterizes positions of 7-branes
- View intersection points of $\mathbb{CP}^1$ in base as marked points
- Let adjoint field $\Phi$ have singular behavior at marked points $\Rightarrow$ Hitchin system coupled to defects:

$$\partial_A \Phi = \sum_p \mu_{\mathbb{C}}^{(p)} \delta(p) \quad F + [\Phi, \Phi^\dagger] = \sum_p \mu_{\mathbb{R}}^{(p)} \delta(p)$$
Nilpotent Deformations

• Split \( \mu_\mathbb{C} = \mu_s + \mu_n \), consider nilpotent part \( \mu_n \), get \( \mathfrak{su}_2 \) algebra:

\[
J_+ = \mu_\mathbb{C} \quad J_- = \mu_\mathbb{C}^\dagger \quad J_3 = \mu_\mathbb{R}
\]

• Adjoint vevs \( \Phi \sim \mu_\mathbb{C} \frac{dz}{z} \)

\[\Rightarrow\] Classified by \( \text{Hom}(\mathfrak{su}(2), \mathfrak{g}) \)

(equivalently, by nilpotent orbits \( J_+ \))
6D SCFTs and $\text{Hom}(\mathfrak{su}(2), A_{k-1})$ labeled by partitions of $k$: 
Partial Ordering of Nilpotent Orbits

\[ O_\mu \geq O_\nu \iff \bar{O}_\mu \supset O_\nu \]

\[ \iff \mu \geq \nu \]

\[ \iff \sum_{i=1}^{m} \mu_i^T \geq \sum_{i=1}^{m} \nu_i^T \quad \forall m \]
Renormalization Group Flows

$\mathcal{T}_{UV}$

High Energy \quad Short Distance

$\mathcal{T}_{IR}$

Low Energy \quad Long Distance
Partial Ordering of Theories

- Can define a partial ordering on theories using RG flows:

\[ T_1 \geq T_2 \iff \exists \text{ flow} \]
Nilpotent Orbit Ordering Matches RG Ordering!
$6D$ SCFTs and $\text{Hom}(\mathfrak{su}(2), D_k)$

Diagram:
- $1^8 : [SO(8)]^{sd_8}, \ldots [SO(8)]$
- $2^2, 1^4 : [SU(2) \times SU(2) \times SU(2)]^{sd_8}, \ldots [SO(8)]$
- $2^1 : [Sp(2)]^{sd_7}, \ldots [SO(8)]$
- $3, 3^5 : [Sp(2)]^{sd_7}, \ldots [SO(8)]$
- $2^1 : [Sp(2)]^{sd_7}, \ldots [SO(8)]$
- $3, 2^2, 1 : [SU(2)]^{sd_8}, \ldots [SO(8)]$
- $3^2, 1^2 : [SU(2)]^{sd_8}, \ldots [SO(8)]$
- $4^2, 2^1 : [SU(2)]^{sd_7}, \ldots [SO(8)]$
- $5, 1^3 : [SU(2)]^{sd_7}, \ldots [SO(8)]$
- $4^2, 2 : [SU(2)]^{sd_7}, \ldots [SO(8)]$
- $5, 3 : [SU(2)]^{sd_3}, \ldots [SO(8)]$
- $7, 1 : [SU(2)]^{sd_5}, \ldots [SO(8)]$
6D SCFTs and $\text{Hom}(\mathfrak{su}(2), E_6)$
Nilpotent Orbits and Global Symmetries

• Consider nilpotent orbit $O_\mu \in \mathfrak{g}$
• Let $F(\mu)$ be subgroup of $G$ commuting with nilpotent element
• Claim: $F(\mu)$ is the global symmetry of the 6D SCFT associated with $\mu$
• E.g.

\[
\begin{array}{cccc}
\text{su}_2 & 4 & 1 & 3 & 1 & 6 & \ldots \\
\text{e}_6 & \text{su}_3 & \text{e}_6 \\
\text{so}_7 & 2 & 1 & 6 & \ldots \\
\end{array}
\]
6D SCFTs and $\text{Hom}(\Gamma_{ADE}, E_8)$

- Consider M5-branes probing Horava-Witten wall and $\mathbb{C}^2/\Gamma_{ADE}$ singularity

<table>
<thead>
<tr>
<th>$N$ M5s</th>
<th>$E_8$ Wall</th>
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Boundary data $\sim$ flat $E_8$ connections on $S^3/\Gamma_{ADE}$
6D SCFTs and $\text{Hom}(\Gamma_{\text{ADE}}, E_8)$

- For trivial boundary data, get 6D SCFT:

  $\begin{array}{c}
  \text{e}_8 \\
  1 \\
  2 \\
  2 \\
  \cdots \\
  2 \\
  \text{g}
  \end{array}$

- For non-trivial boundary data, global symmetry is broken to a subgroup

  $\begin{array}{c}
  \text{g}_L \\
  1 \\
  2 \\
  2 \\
  \cdots \\
  2 \\
  \text{g}
  \end{array}$
6D SCFTs and $\text{Hom}(\Gamma_{\text{ADE}}, E_8)$

Flat $E_8$ connections on $S^3/\Gamma_{\text{ADE}} \leftrightarrow \text{Hom}(\Gamma_{\text{ADE}}, E_8)$

E.g. $\Gamma_{A_2}$, $\text{Hom}(\mathbb{Z}_3, E_8)$:

- $e_8$
- $e_7$
- $\text{so}_{14}$
- $e_6$
- $\text{su}_9$
6D SCFTs and Homomorphisms

- Large classes of 6D SCFTs have connections to structures in group theory
- The correspondence has been verified explicitly

$\text{Hom}(\mathfrak{su}(2), g)$

$\text{Hom}(\Gamma_g, E_8)$
6D SCFTs and Automorphism Groups
6D SCFTs and Automorphism Groups

The Dirac pairing $\Omega$ of a 6D SCFT has an associated automorphism group $\text{Aut}(\Omega)$, which is calculable.
Automorphism Groups

Given $\Omega \in GL(n, \mathbb{Z})$, define $\text{Aut}(\Omega)$ by

$$\text{Aut}(\Omega) = \{ \mu \in GL(n, \mathbb{Z}) | \mu^T \Omega \mu = \Omega \}.$$
Automorphism Groups of 6D SCFTs

For 6D SCFT, Dirac pairing $\Omega$, 

$$\text{Aut}(\Omega) = \text{Aut}(\Omega_{\text{end}}) \times \text{Aut}(\mathbb{I}_k)$$

Dirac pairing after blowing down all -1 curves

Dirac pairing associated with $k$ blow-downs
Automorphism Groups of 6D SCFTs

E.g.

\[ \text{Aut}(\begin{array}{ccc} 4 & 1 & 4 \\ \end{array}) = \text{Aut}(\begin{array}{cc} 3 & 3 \\ \end{array}) \times \text{Aut}(\mathbb{I}_1) \]

\[ \text{Aut}(\begin{array}{ccc} 1 & 2 & 2 \\ \end{array}) = \text{Aut}(\begin{array}{ccc} 1 & 3 & 1 \\ \end{array}) \]

\[ = \text{Aut}(\mathbb{I}_3) \]
Automorphism Groups of 6D SCFTs

In general,

$$\text{Aut}(\begin{array}{cccccccc} n_1 & 2 & 2 & \cdots & 2 & n_2 & 2 & \cdots & 2 & n_3 & \cdots \end{array}) = \mathbb{Z}_2 \times S_{m_1+1} \times S_{m_2+1} \times \cdots$$

$$\text{Aut}(\mathbb{I}_k) = S_k \times \mathbb{Z}_2^k$$
Outer Automorphisms

For a symmetric endpoint, $\text{Aut}(\Omega)$ contains an additional factor associated with the quiver symmetry:

$$\text{Aut}(\begin{array}{ccc} 2 & 3 & 2 \end{array}) = \mathbb{Z}_2 \times (\mathbb{Z}_2 \times (S_2 \times S_2))$$

Symmetry of quiver from left -2 curve from right -2 curve from left -2 curve from right -2 curve
Green-Schwarz Couplings

• Group elements label distinct choices for Green-Schwarz coupling

\[ \mathcal{L}_6 \supset \int \mu_{IJ} B_I \wedge \text{Tr}(F_J \wedge F_J) \]

\[ I_{GS} \supset \text{Tr}(F_I \wedge F_I) \mu_{JI} \Omega^{-1}_{JK} \mu_{KL} \text{Tr}(F_L \wedge F_L) \]

\[ \mu_{IJ} \in \text{Aut}(\Omega) \Leftrightarrow \text{Dirac Quantization} \]
(2,0) Automorphism Groups

For a (2,0) SCFT,

\[ \text{Aut}(\Omega_g) = \text{Aut}(g) \]

Group elements \[ \rightarrow \] Permutations of M5-branes

\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} \ \leftrightarrow \ \begin{array}{cccc}
1 & 3 & 4 & 2 \\
\end{array} \]
Phases of 6D SCFTs

• For a general (1,0) SCFT, group elements label tensor branch phases:

\[ \sum_{I} \mu_{IJ}^{T} \phi_{I} > 0 \]

• These in turn correspond to geometric phases of the base \( B_2 \).
Summary

- So far…
  - 6D SCFTs have been classified
  - There are remarkable relationships between 6D SCFTs and two classes of homomorphisms
  - Phases of 6D SCFTs are labeled by automorphism groups of their Dirac pairing
Further Research

• In the future…
  • Can we classify full set of 6D RG Flows in terms of group theory data?
  • Can we understand compactifications to lower dimensions?
  • Can we understand these algebraic/geometric correspondences from a purely mathematical perspective?