# *Particle-Soliton Degeneracy in 2D QCD*
## *from Spontaneously Broken Non-Invertible Symmetry*

Diego García-Sepúlveda
(University of Chicago)

March 22, 2025

*Southern Regional Mathematical String Theory Meeting*

Particle-Soliton Degeneracies from Spontaneously Broken Non-Invertible Symmetry
[C. Córdova, DGS, N. Holfester. (2403.08883)]

Topological Cosets via Anyon Condensation and Applications to Gapped $QCD_2$
[C. Córdova, DGS. (2412.01877)]

Particle-Soliton Degeneracy in 2D Quantum Chromodynamics
[C. Córdova, DGS, N. Holfester. (2412.21153)]

# *Outline*

- *Introduction and Motivation.*
- *Key concepts: A pedagogical example.*
- *Setting up 2D QCD.*
- *3D TQFT Description of Topological Cosets.*
- *Applications to Gapped 2D QCD.*
- *Future Directions.*

# *Introduction and Motivation*

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- However, the QCD sector is rather difficult too understand due to asymptotic freedom.

    - High-Energy regime well-understood: Weakly coupled as $E \to \infty$.

    - Low-Energy regime mysterious to this day: Strongly coupled as $E \to 0$, and perturbative methods lose their usefulness. We do not understand why QCD is trivially gapped!

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- Valuable to gather intuition about QCD in strongly coupled regime by studying related models.

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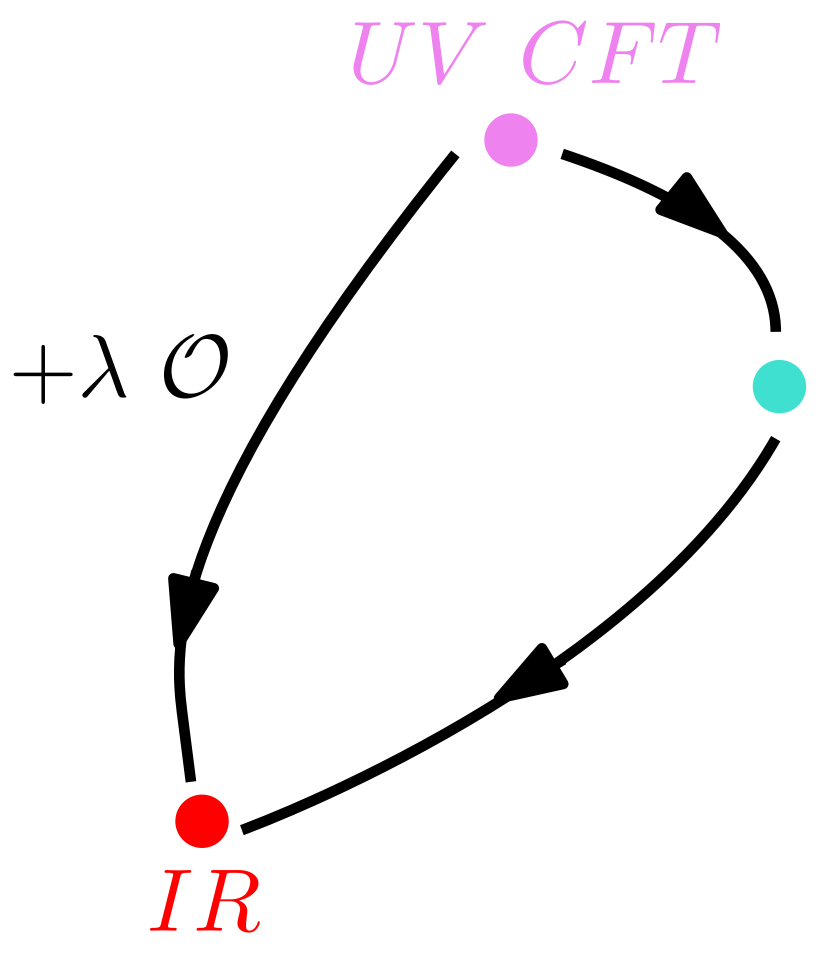
- Recent Advances: Non-Invertible Symmetries help understand important properties in QFT: (de)confinement; vacuum structure; spectra.
- This work: Characterize the spectrum of 2D QCD theories when gapped.

*Introducing key concepts:*

*A pedagogical Example*

# *Modern Paradigm of RG flows*

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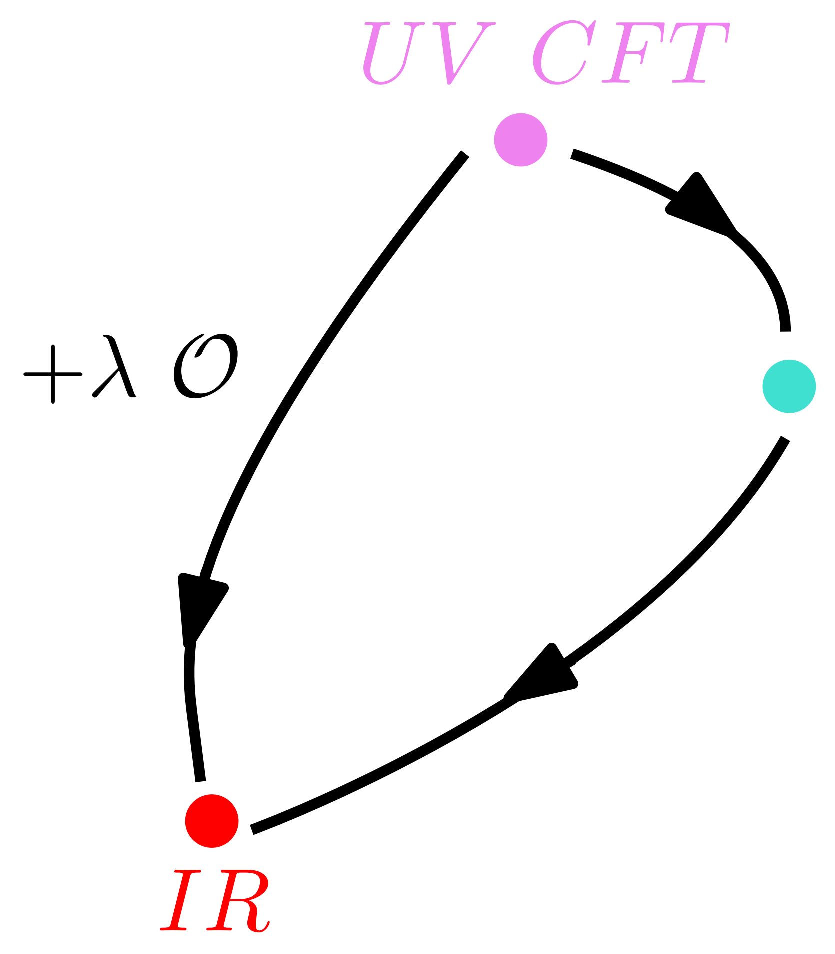


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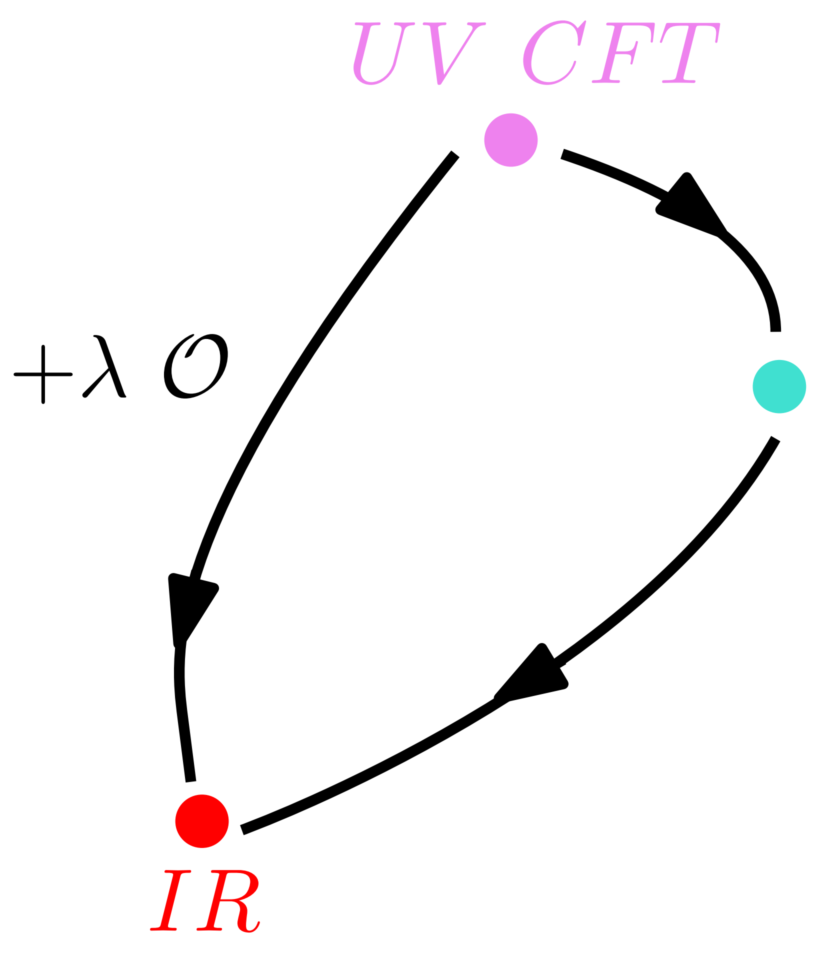


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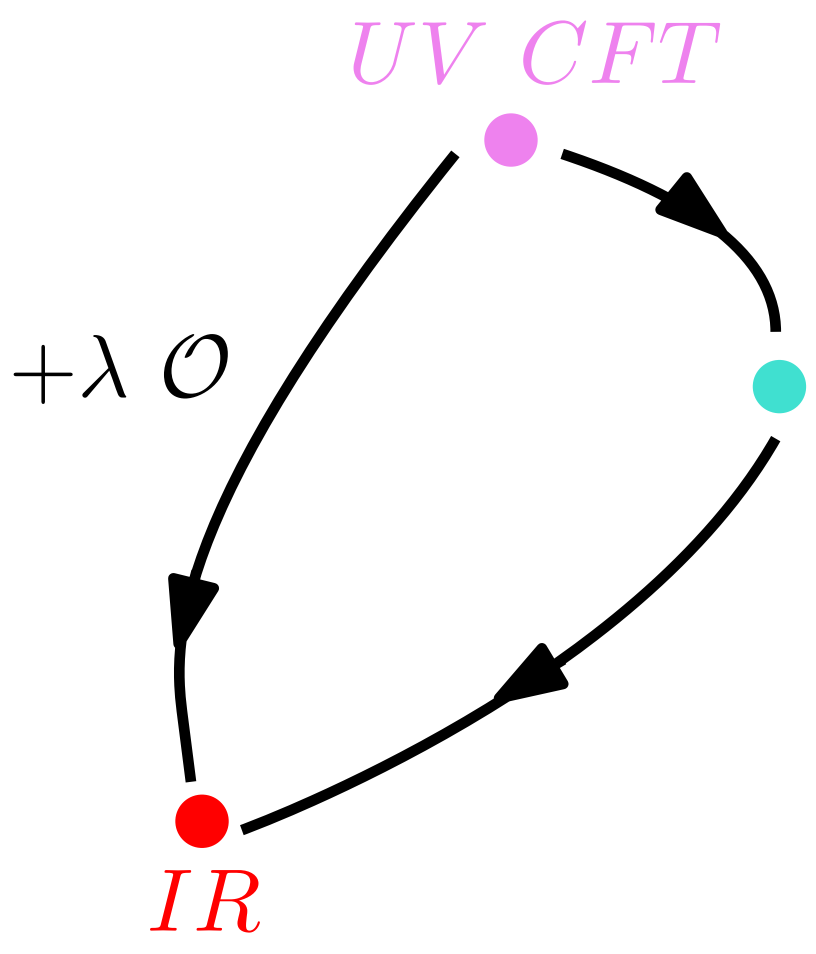


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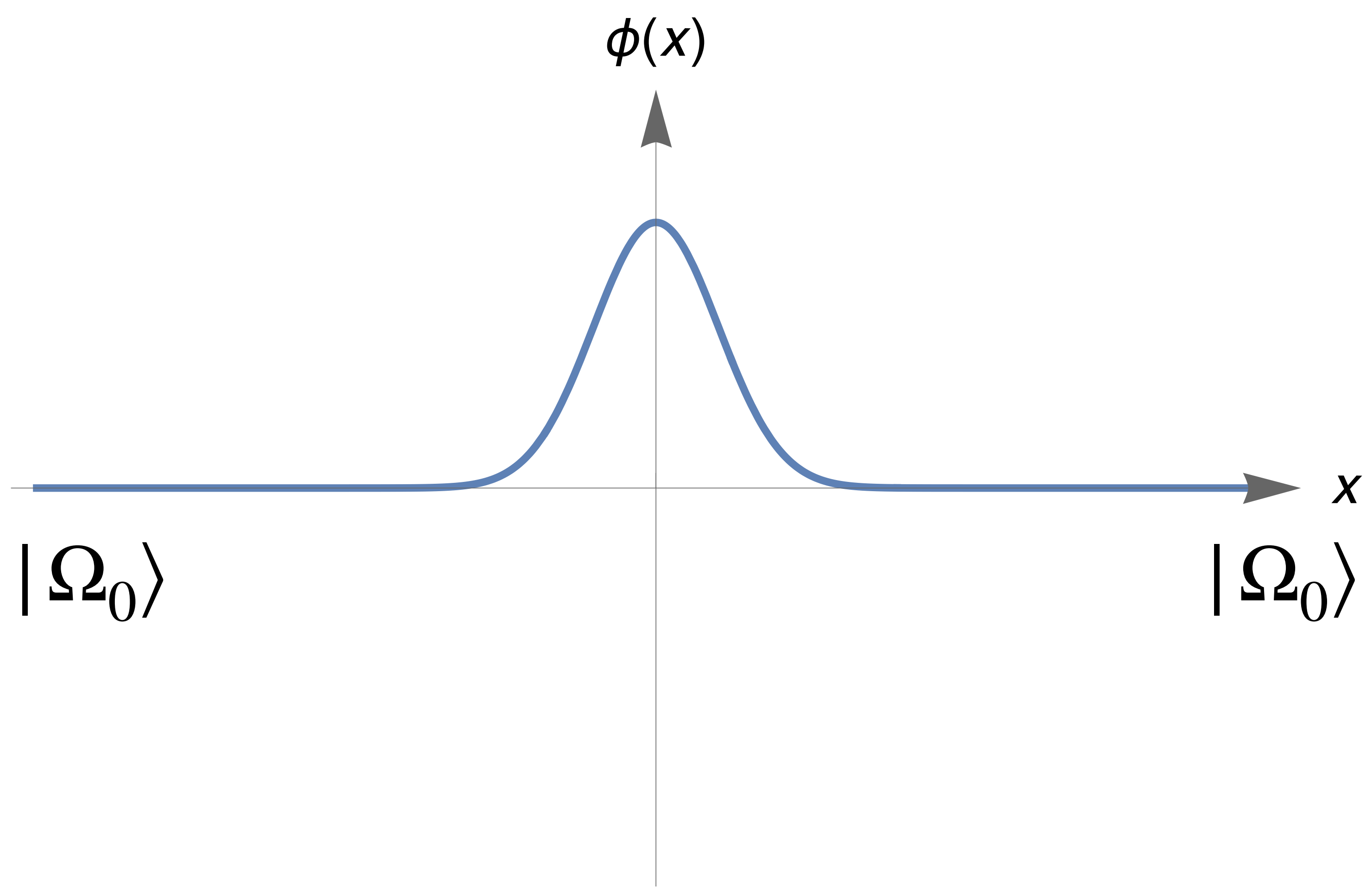
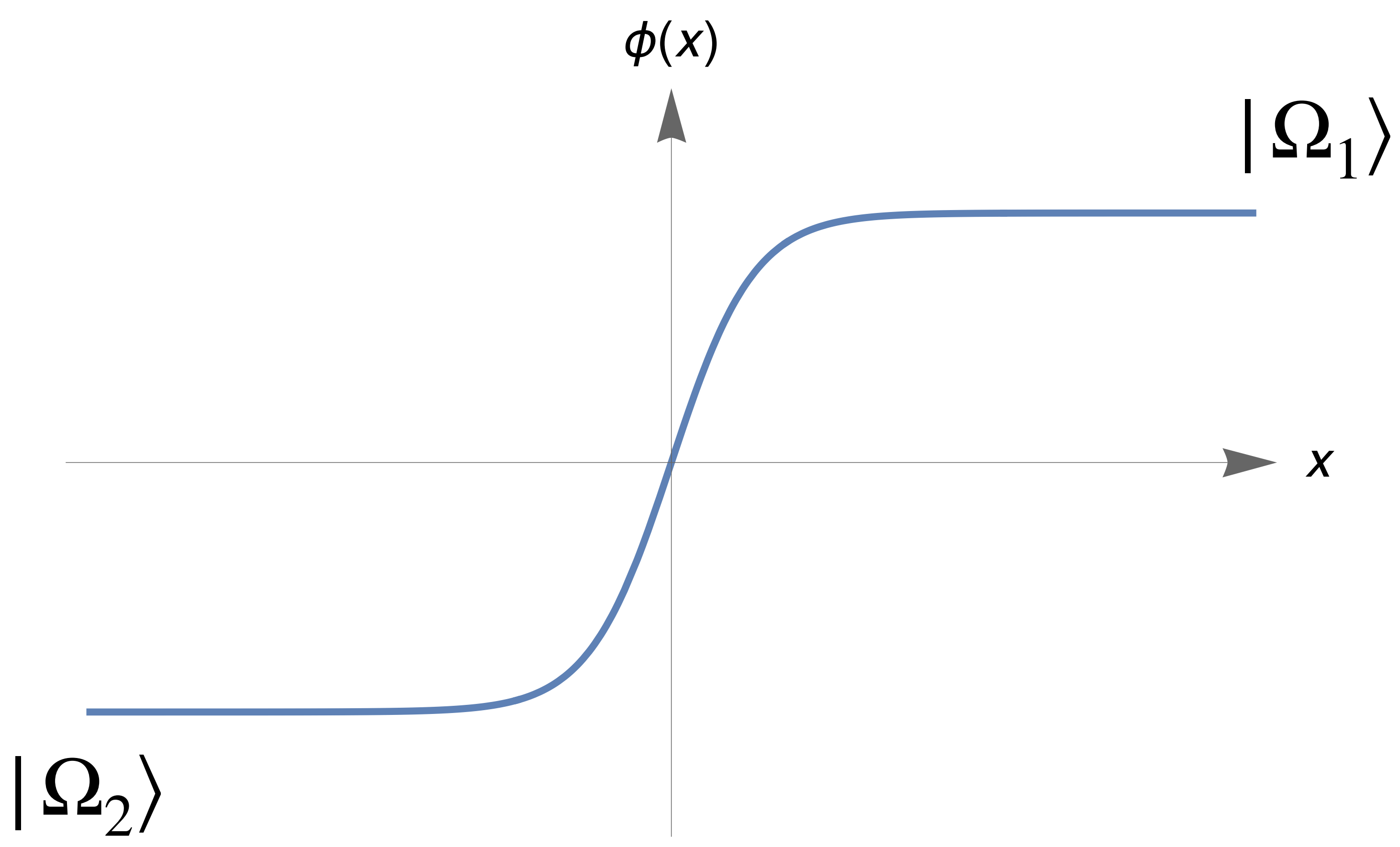
# *Some Key Concepts: Particles and Solitons*

Two qualitative different type of physical excitations:

- **Particles**: Excitations above a single vacuum state.

- **Solitons**: Excitations that interpolate between distinct vacua.




Cannot continuously deform soliton to particle: requires changing boundary condition at infinity.

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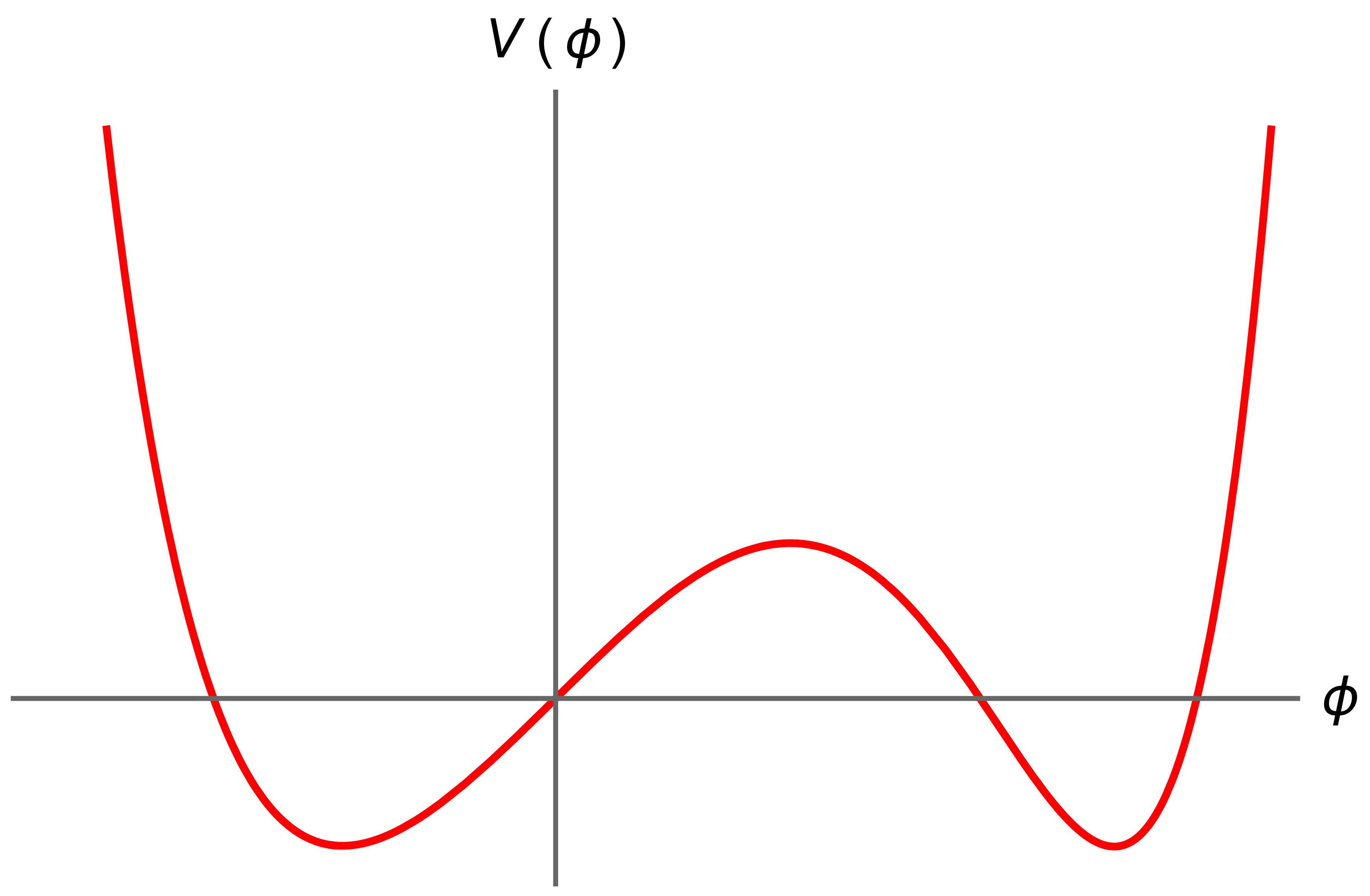
Schematically: Scalar field theory in two spacetime dimensions.
Landau-Ginzburg realization of the model with potential




$$V(\phi) = \phi^6 - 10\lambda^3\phi^3 + 12\lambda^5\phi$$

No reflection symmetry in potential!

[Lassig-Mussardo-Cardy, Zamolodchikov]

- Turning on $\lambda$ sets up a relevant deformation ($\phi_{2,1}$ deformation).
- Spectrum is gapped, and there are two ground states.

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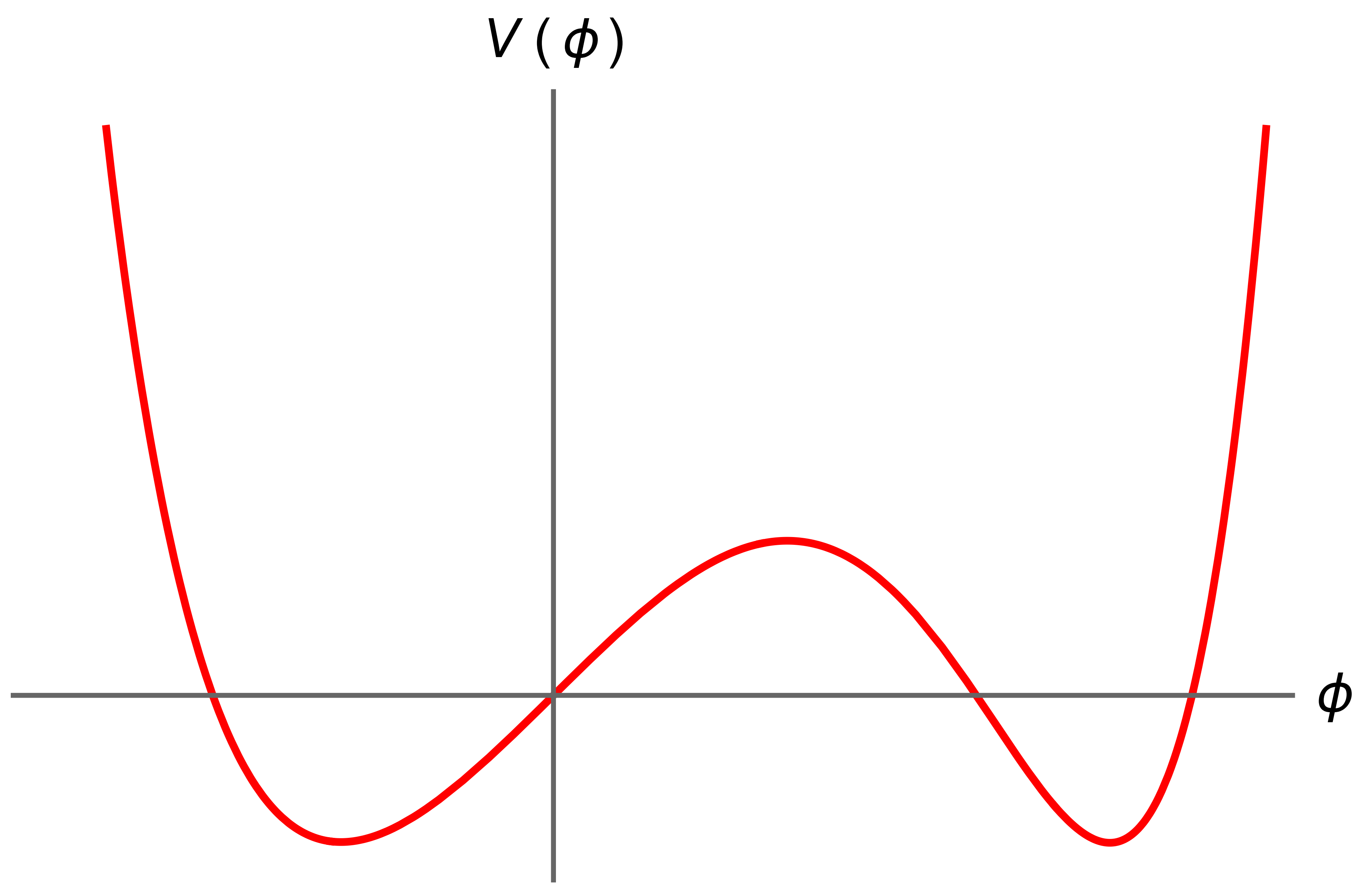


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Massive spectrum of the theory consists of a soliton-antisoliton pair connecting the two vacua, and a single particle state on one vacuum only:

$$|s\rangle \ |\bar{s}\rangle \qquad\qquad |p\rangle$$

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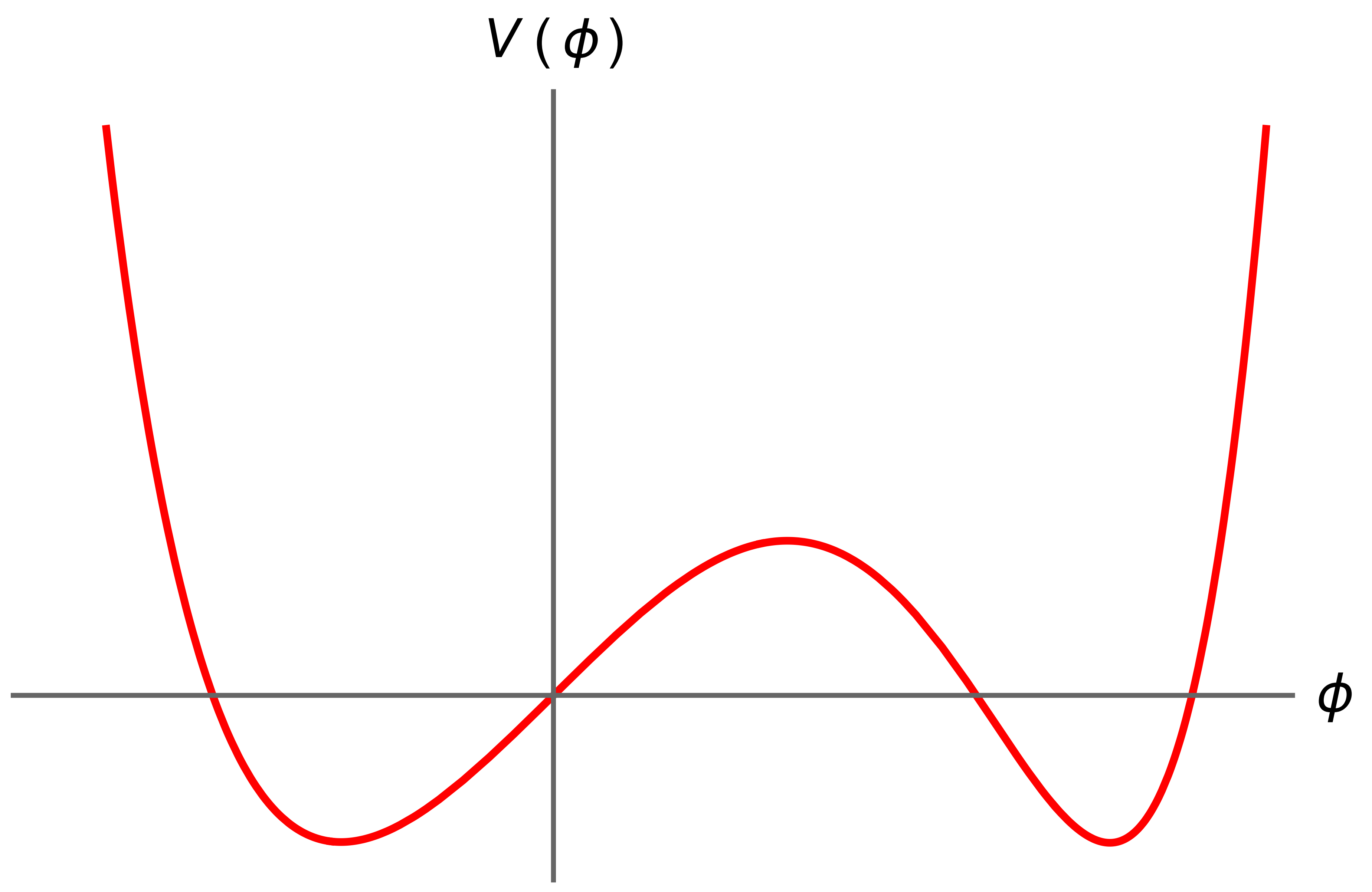


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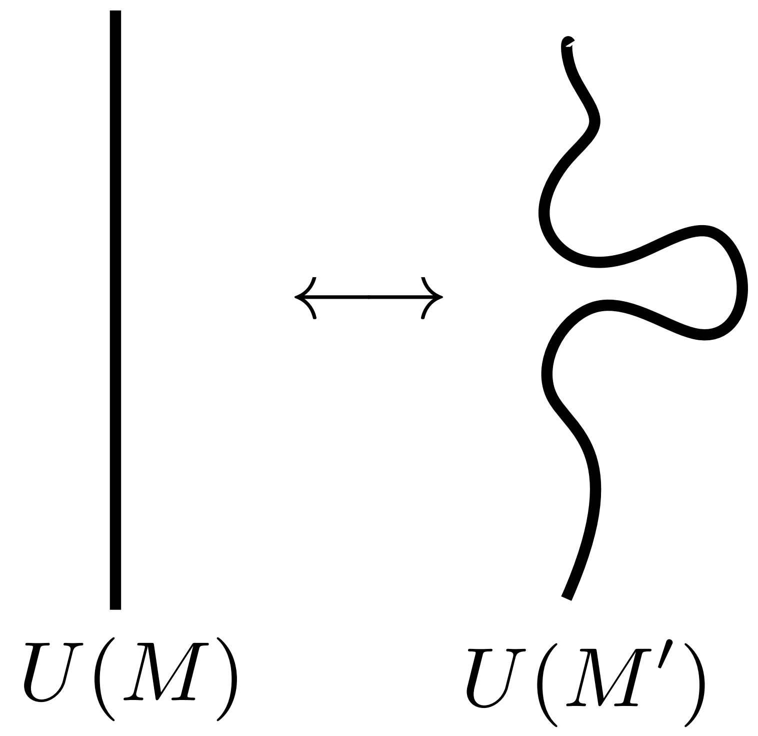
All the states have the same mass:

$$\boxed{m_p = m_s}$$

Natural interpretation to this phenomenon?

# *Generalized Symmetry and Topology*

Contemporary understanding of symmetry in a physical system: A physical excitation in the system which can be continuously deformed at no cost in energy. A topological operator in the theory.
[D. Gaiotto, A. Kapustin, N. Seiberg, B. Willet. 1412.5148].

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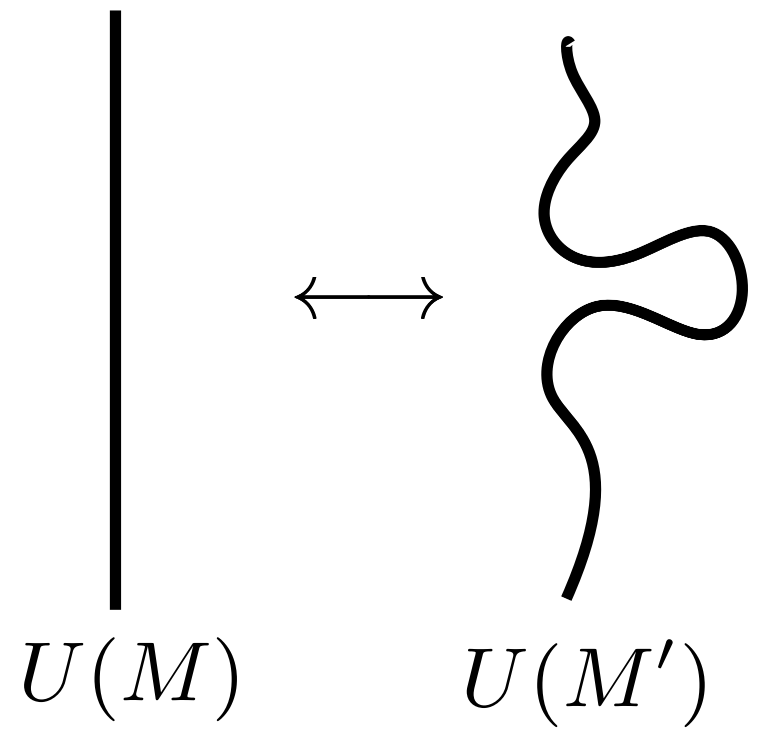


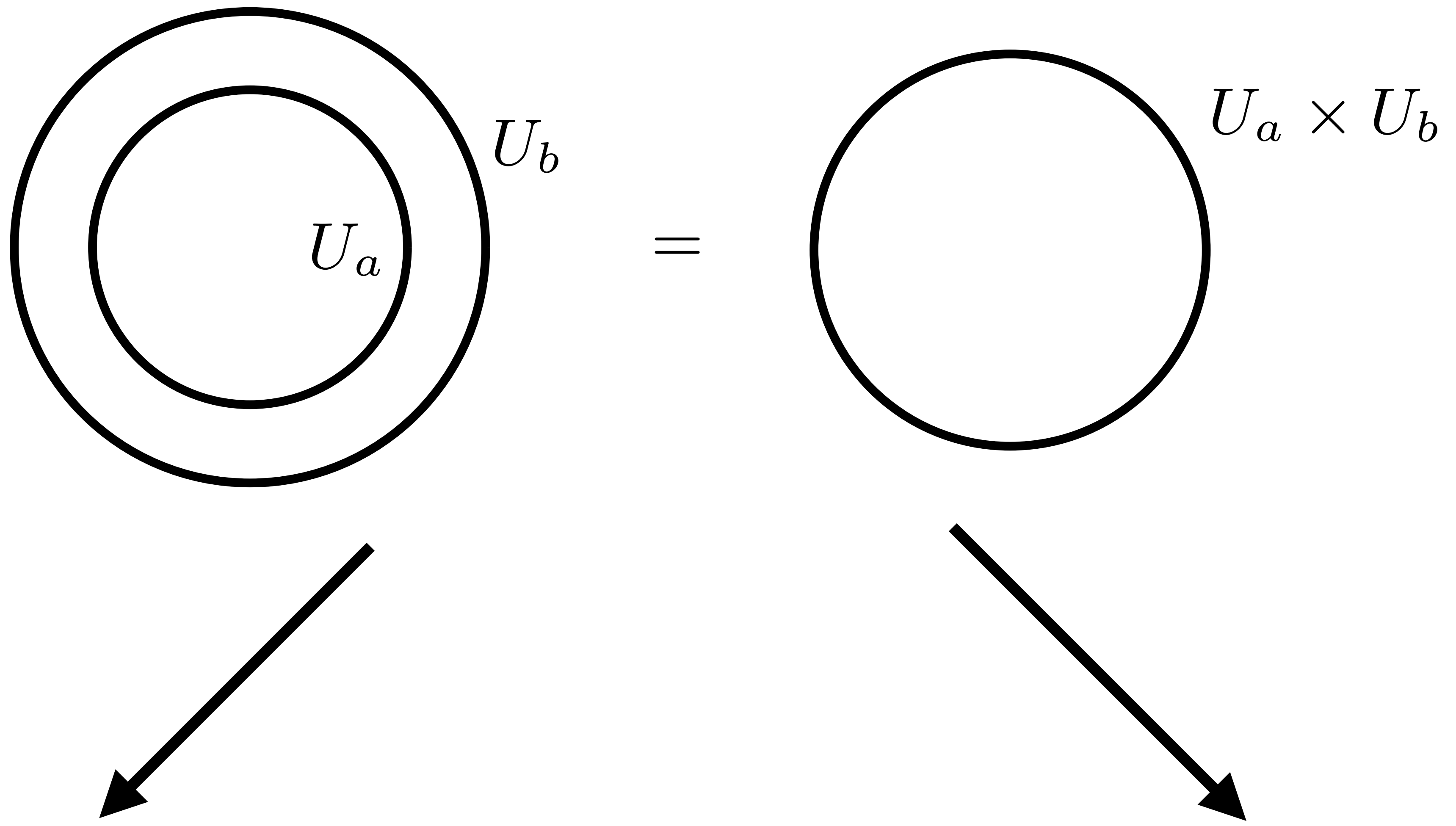


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$$U(M) = \exp\left(\int_M j\right)$$

(Invertible)

$$U_a \times U_b = 1$$

(Non-Invertible)

$$U_a \times U_b = \sum_c N_{ab}^c U_c$$

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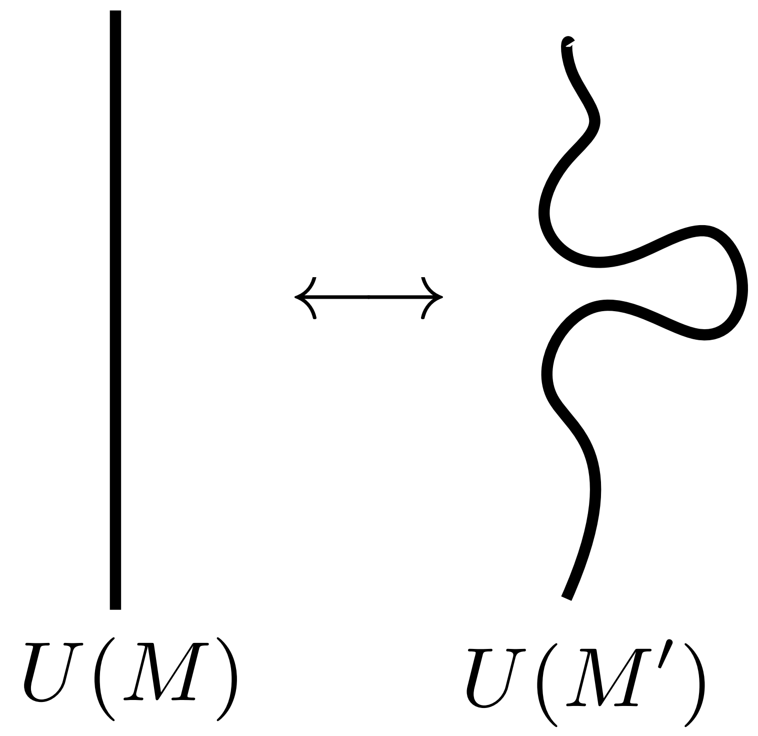


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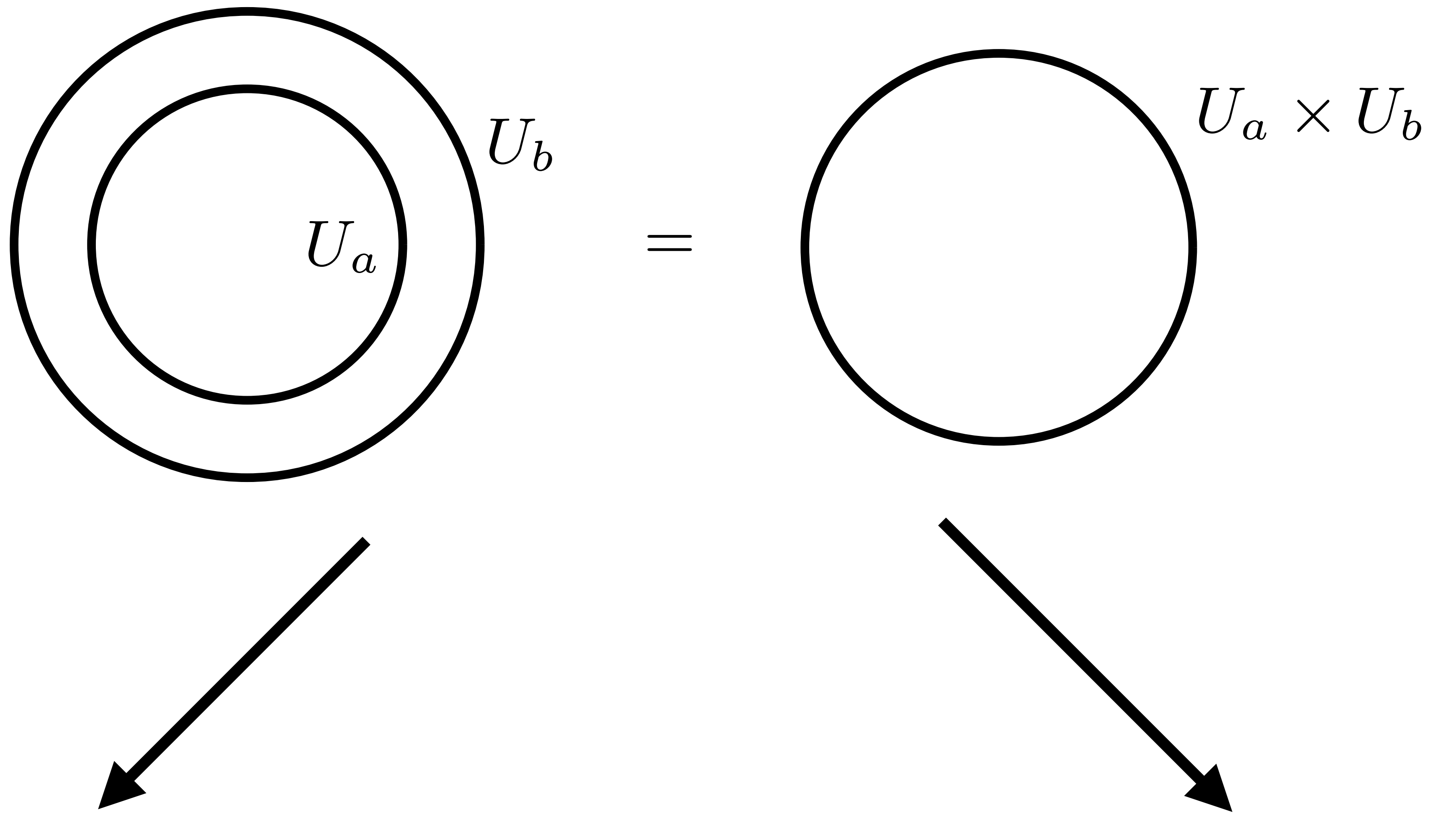


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Mathematically described by fusion category theory

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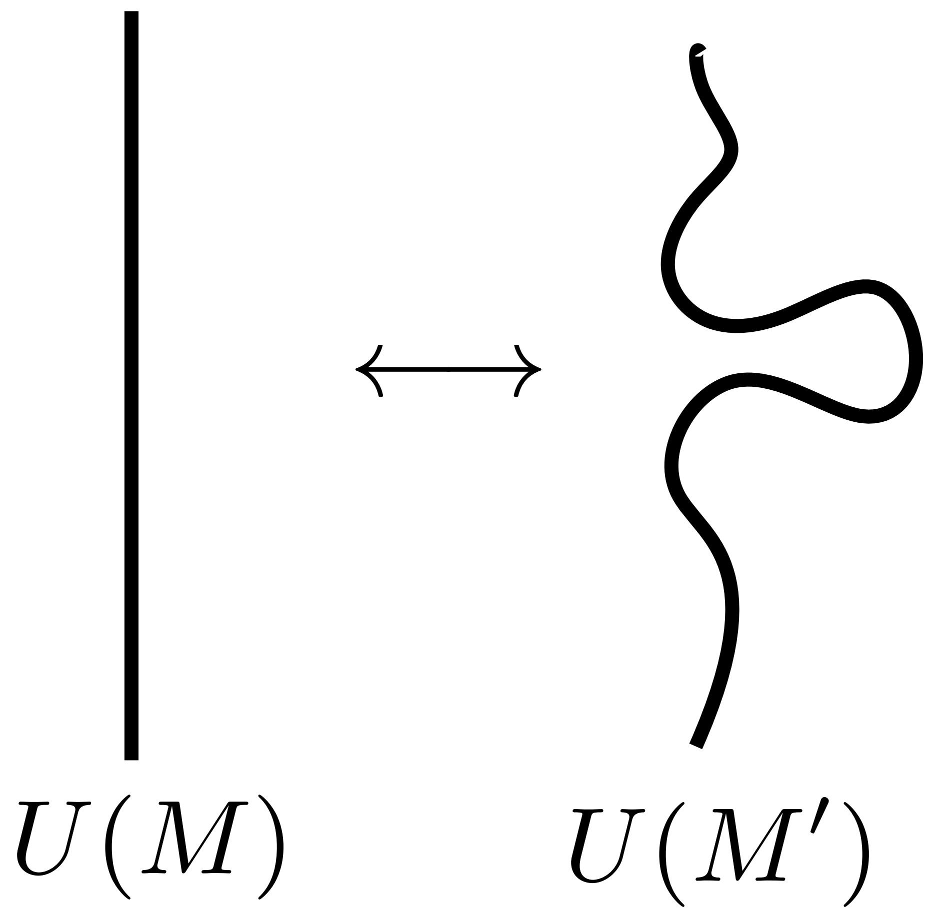


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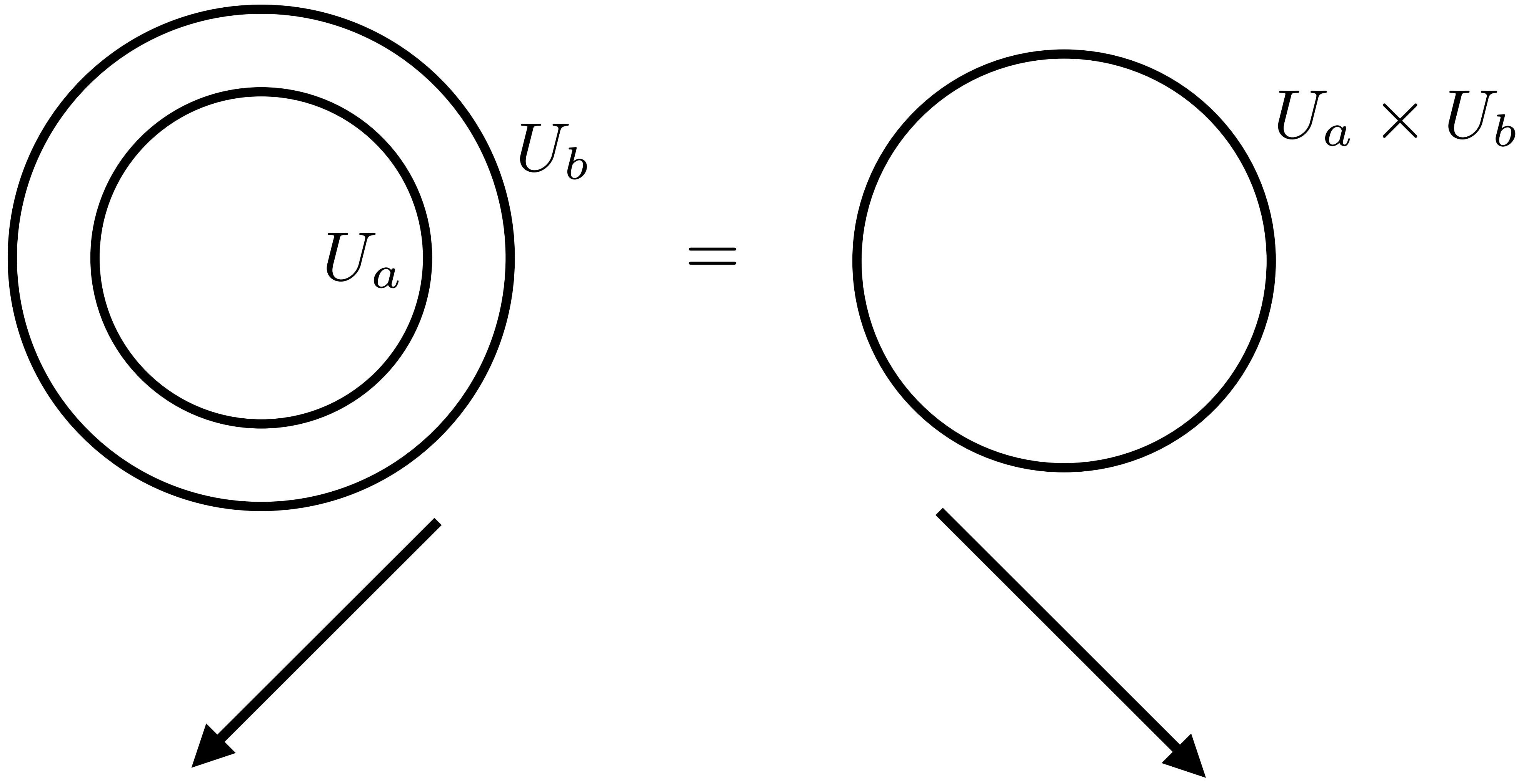


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- Spontaneous Symmetry Breaking.
- Imply energetic degeneracies.
- Can be anomalous (and thus constrain RG flows).
- May be gauged if non-anomalous

# *Simplest Example: Verlinde Line Operators in 2D RCFT*

[Verlinde Nucl. Phys. B300 (1988) 360–376].
[N. Drukker, D. Gaiotto, J. Gomis (1003.1112)].
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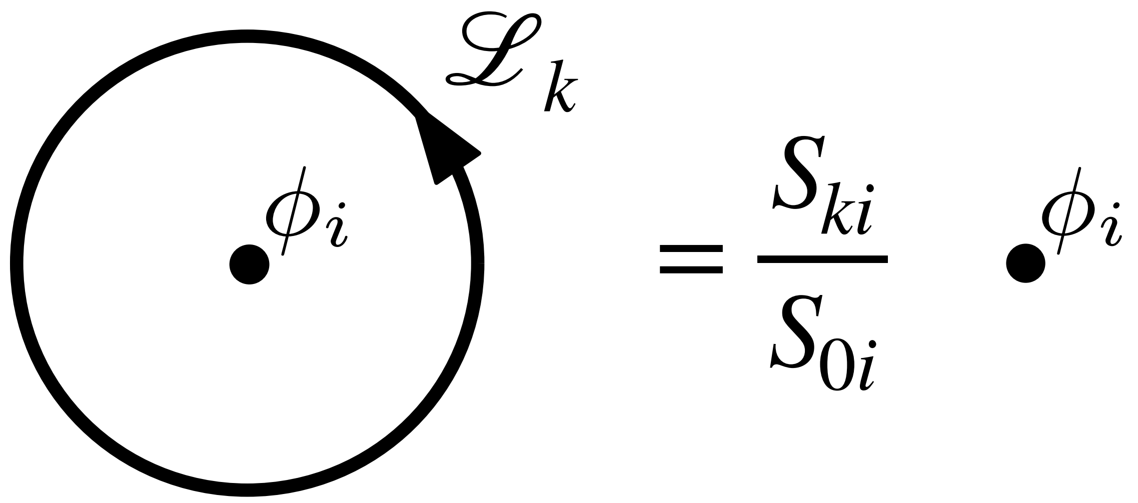
| Diagonal 2D RCFT $\mathscr{R}$: | Primary operators $\phi_i$ | Line operators $\mathscr{L}_i$ |
|---|---|---|
| $Z^{(\mathscr{R})}_{T^2}(\tau, \bar{\tau}) = \sum_i \chi_i(\tau)\,\bar{\chi}_i(\bar{\tau})$ | $\phi_i \times \phi_j = \sum_k N^k_{ij}\,\phi_k$ | $\mathscr{L}_i \times \mathscr{L}_j = \sum_k N^k_{ij}\,\mathscr{L}_k$ |

$$\mathscr{L}_k\,|\,\phi_i\rangle = \frac{S_{ki}}{S_{0i}}\,|\,\phi_i\rangle$$



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Primary operators $\phi_i$

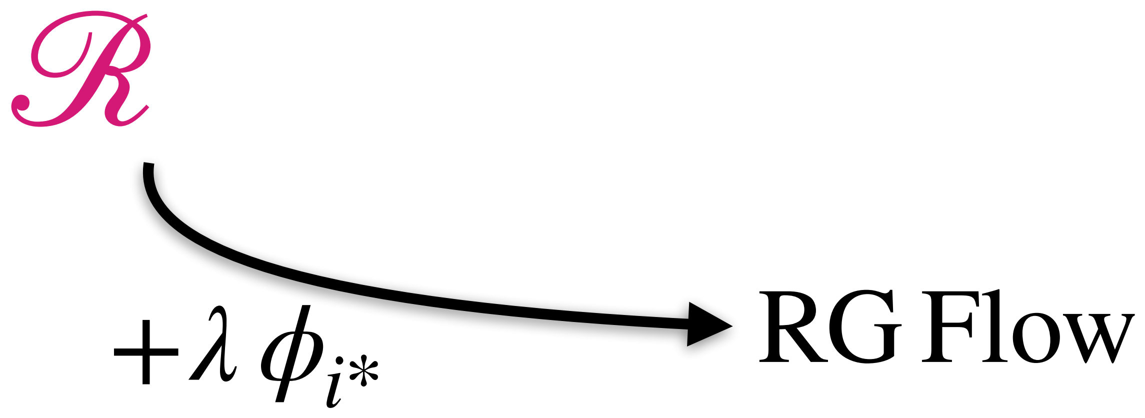
$$\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k$$

Line operators $\mathscr{L}_k$

$$\mathscr{L}_i \times \mathscr{L}_j = \sum_k N_{ij}^k \mathscr{L}_k$$

Trigger a Renormalization Group Flow by a relevant operator $\phi_{i*}$:

$$\mathscr{R} + \lambda\, \phi_{i*}$$

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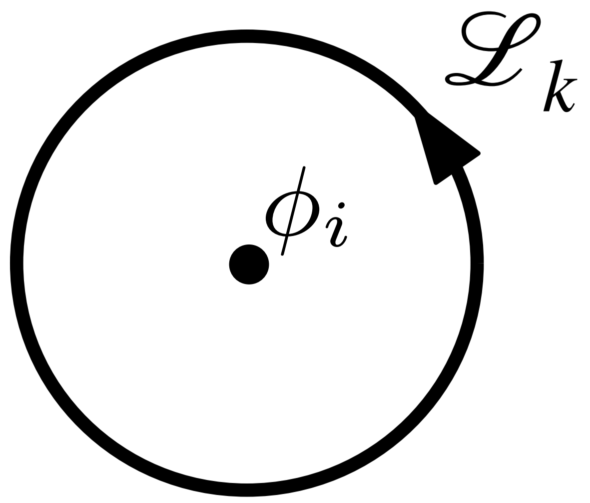


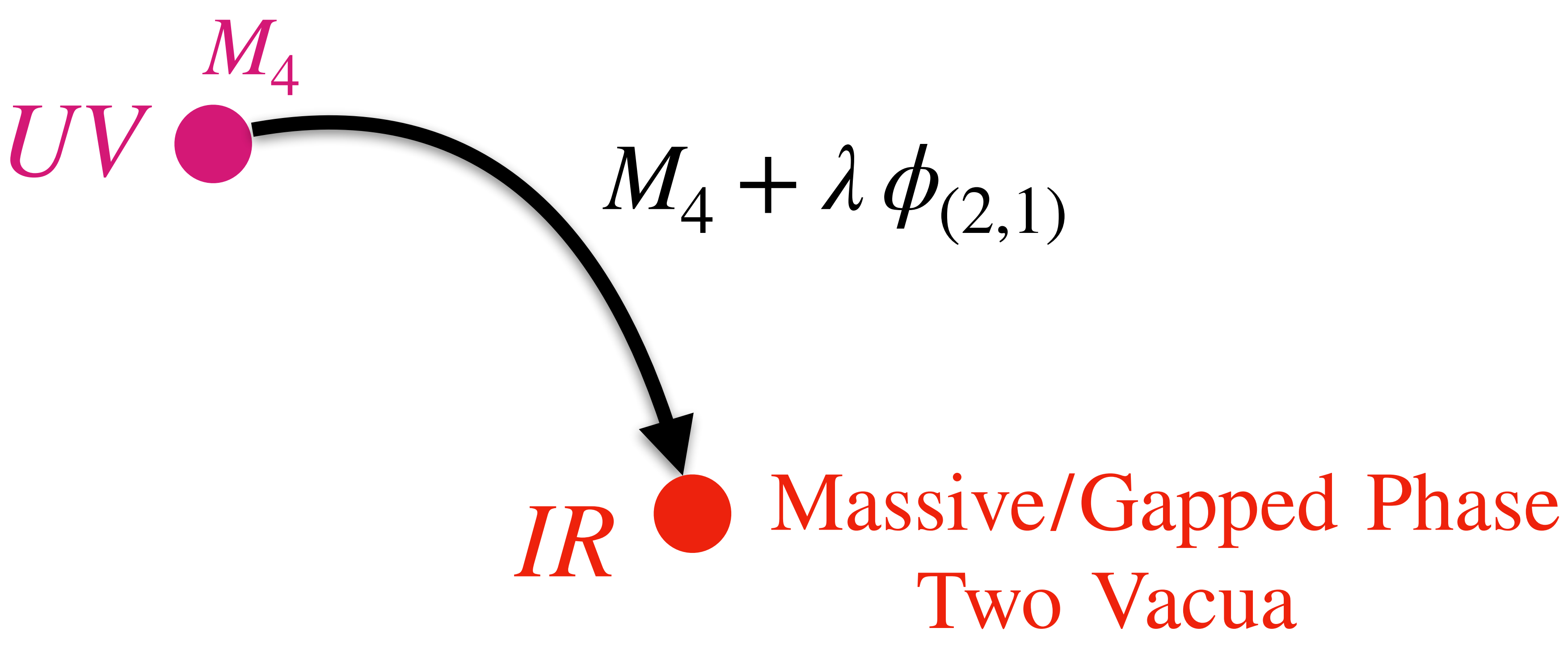

$$= \frac{S_{ki}}{S_{0i}} \quad \bullet \phi_i$$

Verlinde lines preserved throughout the RG flow

$$\mathscr{L}_k \text{ preserved} \Longleftrightarrow \frac{S_{k\,i*}}{S_{0\,i*}} = \frac{S_{i\,0}}{S_{0\,0}}$$

# *An integrable/pedagogical Example*

Minimal models are clearly under good technical control.
(Known spectra of primaries, modular S-matrix, etc…) [C-M. Chang, Y-H. Lin, S-H. Shao, Y. Wang, X. Yin. (1802.04445)]




| Tricritical Ising Model $M_4$ | | | |
|---|---|---|---|
| Kac label | Conformal Weight | Verlinde Line | Quantum Dimension |
| $(1,1)$ or $(3,4)$ | $h_{1,1} = 0$ | $1$ | $d_{1,1} = 1$ |
| $(1,2)$ or $(3,3)$ | $h_{1,2} = 1/10$ | $W \otimes \eta$ | $d_{1,2} = \frac{1+\sqrt{5}}{2}$ |
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| $(2,2)$ or $(2,3)$ | $h_{2,2} = 3/80$ | $W \otimes N$ | $d_{2,2} = \sqrt{2}(\frac{1+\sqrt{5}}{2})$ |
| $(2,4)$ or $(2,1)$ | $h_{2,1} = 7/16$ | $N$ | $d_{2,1} = \sqrt{2}$ |

At the UV CFT point:

$$\eta \times \eta = 1$$
$$\eta \times N = N \times \eta = N$$
$$N \times N = 1 + \eta$$
$$W \times W = 1 + W$$

Triggering the $\phi_{(2,1)}$ deformation

$+\lambda\, \phi_{(2,1)}$

$$W \times W = 1 + W$$

Preserved along the flow

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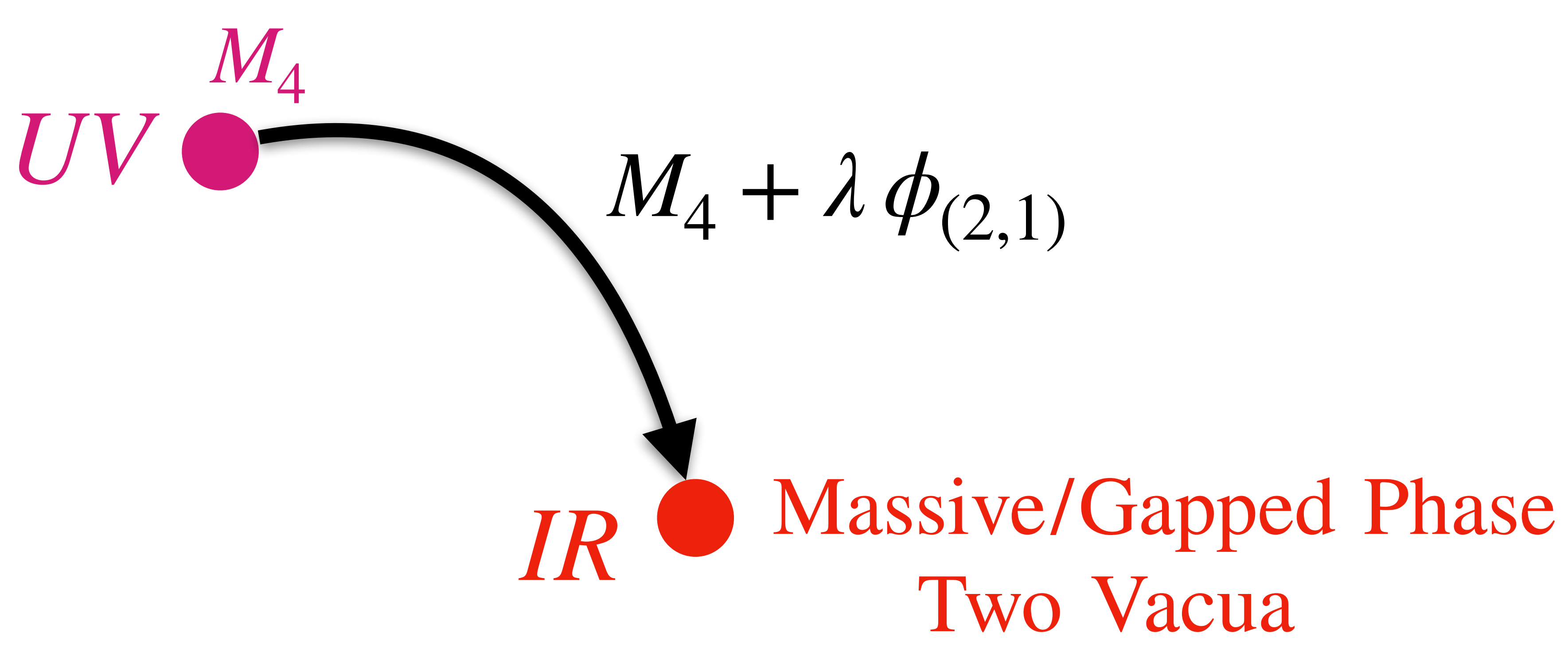


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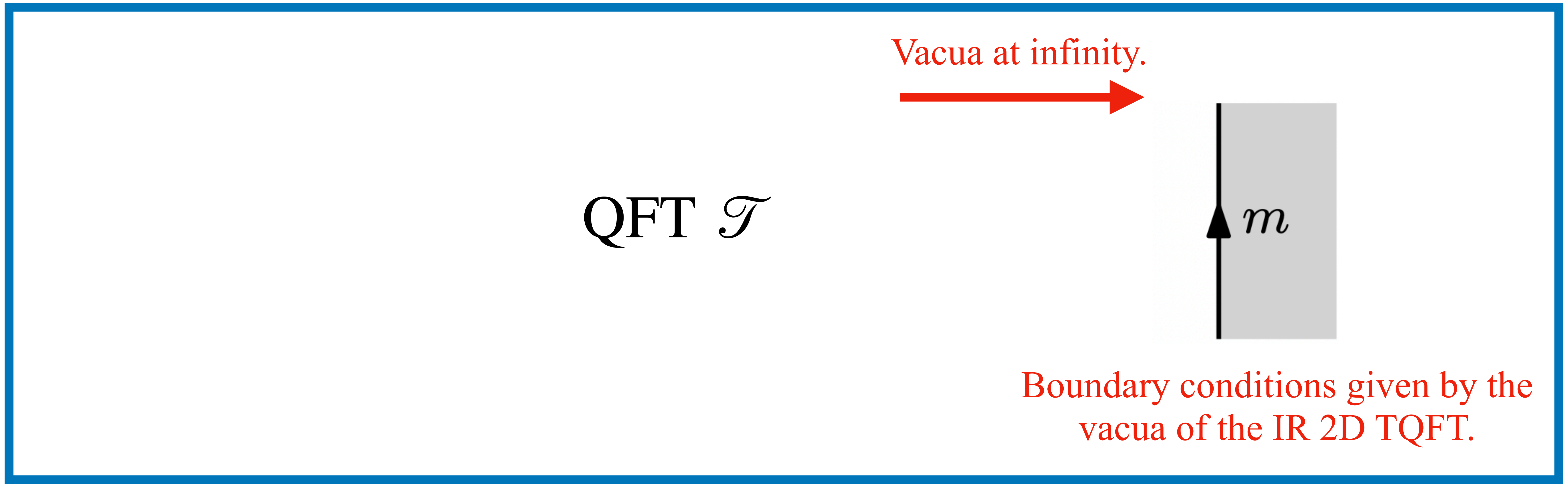
Preserved along the flow

Interestingly, this is a non-invertible symmetry!

# *Symmetry and Boundary Conditions*

Solitons: Need to quantize the theory over the real line. Impose boundary conditions at infinity.




Topologically gapped 2D QFT: IR is a 2D TQFT, consisting on many vacua (topological local operators), acted over by the (possibly non-invertible) symmetry of the QFT.

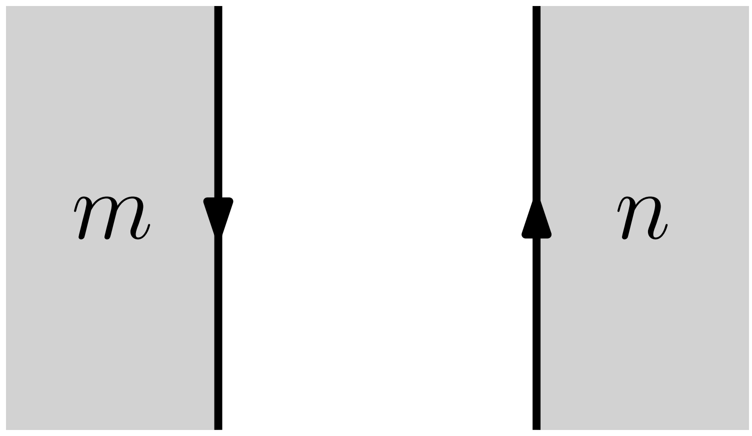
[G. Moore, G. Segal (0609042].

[T-C. Huang, Y-H. Lin, S. Seifnashri (2110.02958)].

[G. Moore. “ A few remarks on topological field theory ”]

# *Symmetry and Boundary Conditions*

The Hilbert space is decomposed into sectors labeled by the boundary conditions to the infinite left and to the infinite right




$: \ \mathcal{H}_{m,n}$

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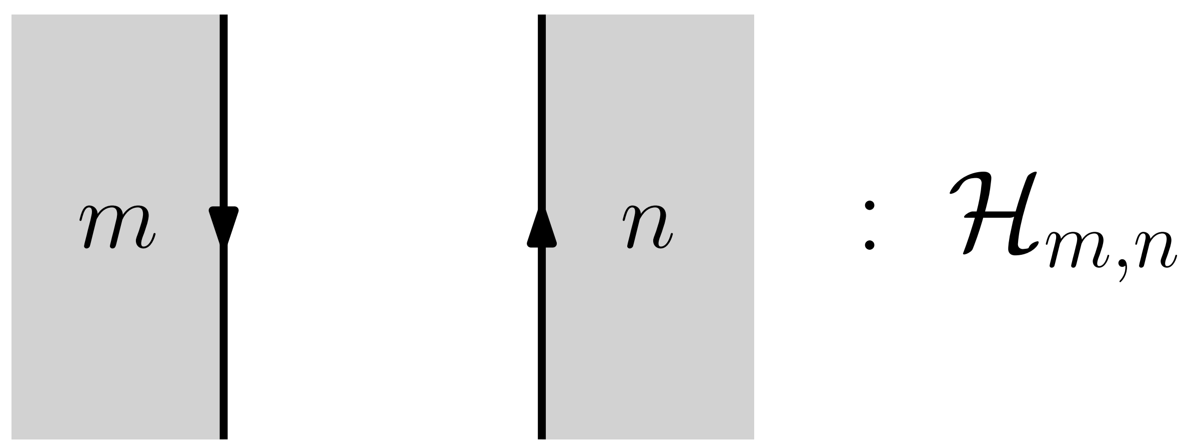


- If $|\psi_{m,n}\rangle \in \mathscr{H}_{m,n}$ with $m \neq n$: state is a soliton.

  $\longrightarrow$ "Interpolates between vacua $|\Omega_m\rangle$ at $-\infty$ and $|\Omega_n\rangle$ at $+\infty$"

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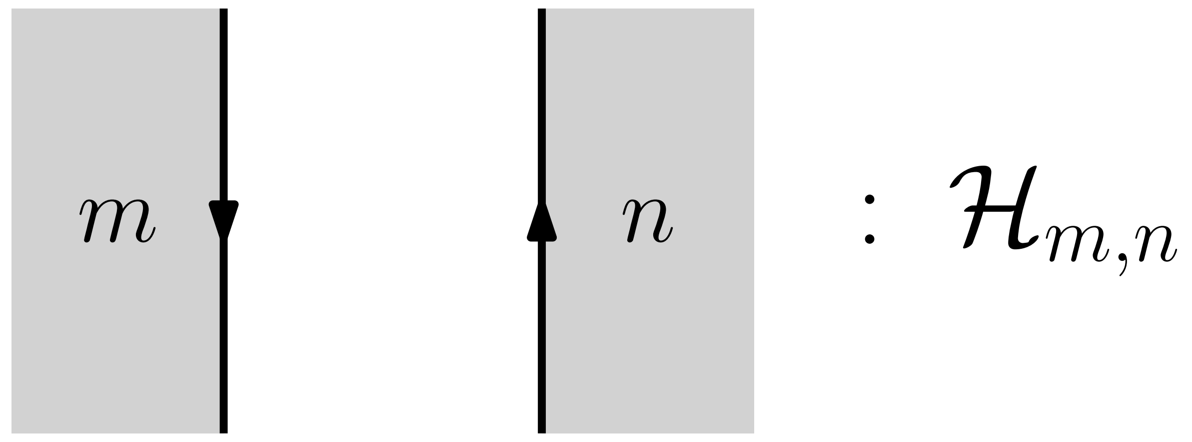


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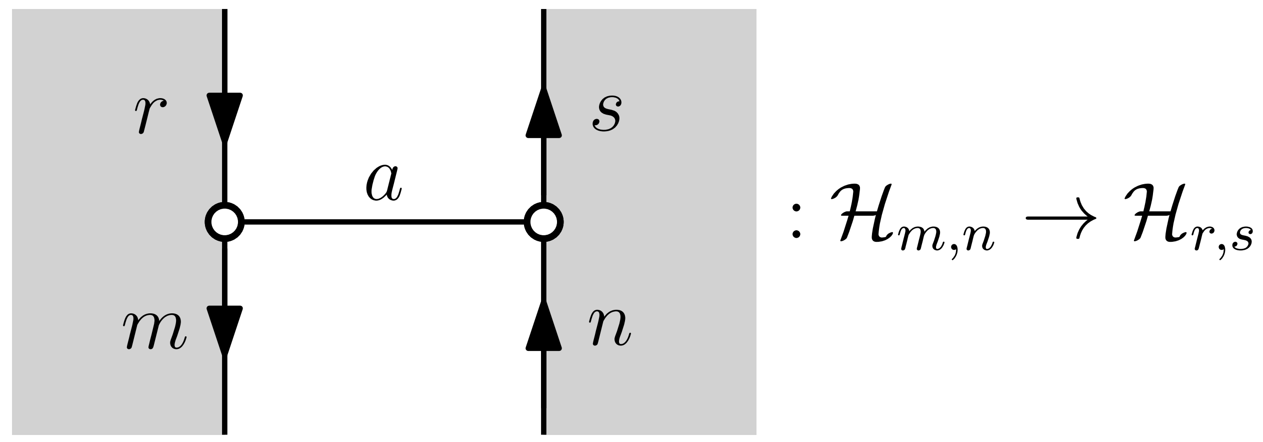
- Clearly, local operators preserve sector.




- Spontaneously broken symmetry: acts over the vacua. Sectors may be permuted/intertwined.

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Massive spectrum consists of a soliton-antisoliton pair connecting the two vacua, and a single particle state on one vacuum only. All the states have the same mass.

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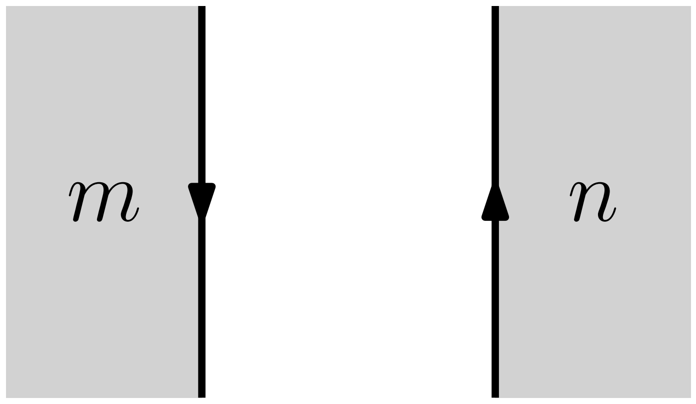


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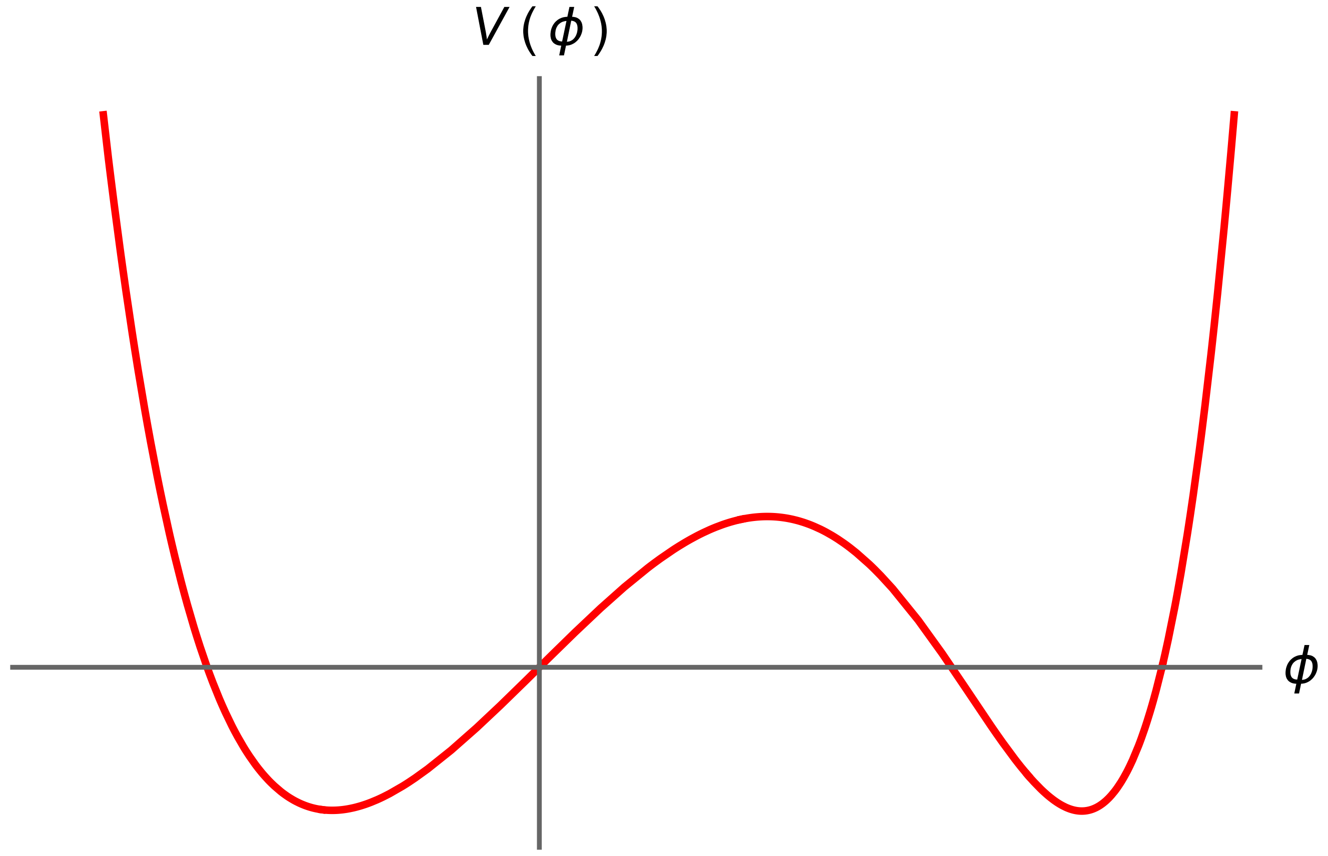
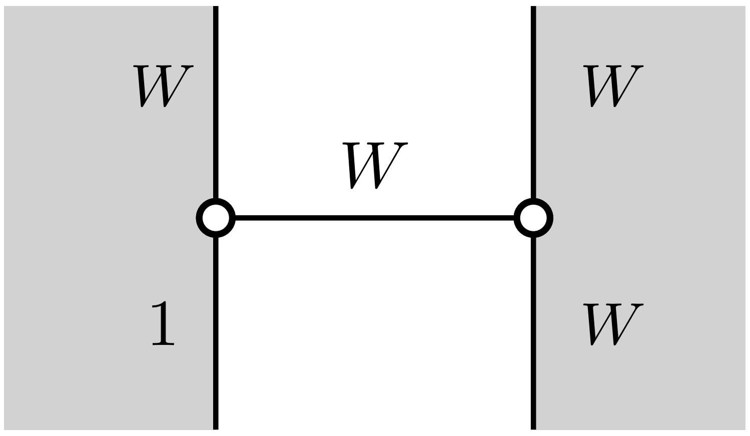
Symmetry spontaneously broken (gapped phase): $\mathscr{H}_{1,1} \oplus \mathscr{H}_{1,W} \oplus \mathscr{H}_{W,1} \oplus \mathscr{H}_{W,W}$




$$V(\phi) = \phi^6 - 10\lambda^3\phi^3 + 12\lambda^5\phi$$

e.g.



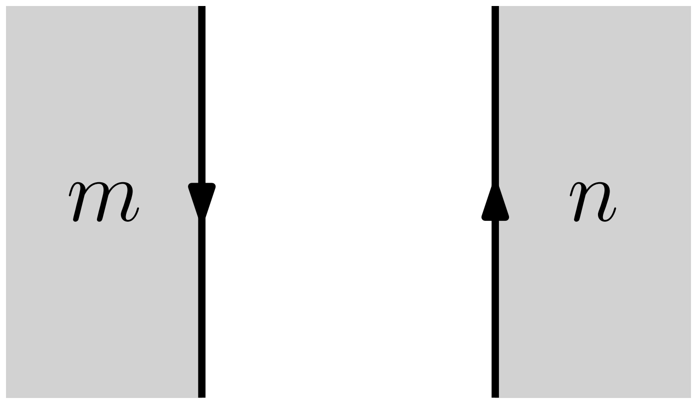

$: \mathscr{H}_{1,W} \to \mathscr{H}_{W,W}$

Multiplets of states relating particles and solitons in 2D via non-invertible symmetry [2403.08883]:

# *An integrable/pedagogical Example*

Massive spectrum consists of a soliton-antisoliton pair connecting the two vacua, and a single particle state on one vacuum only. All the states have the same mass.

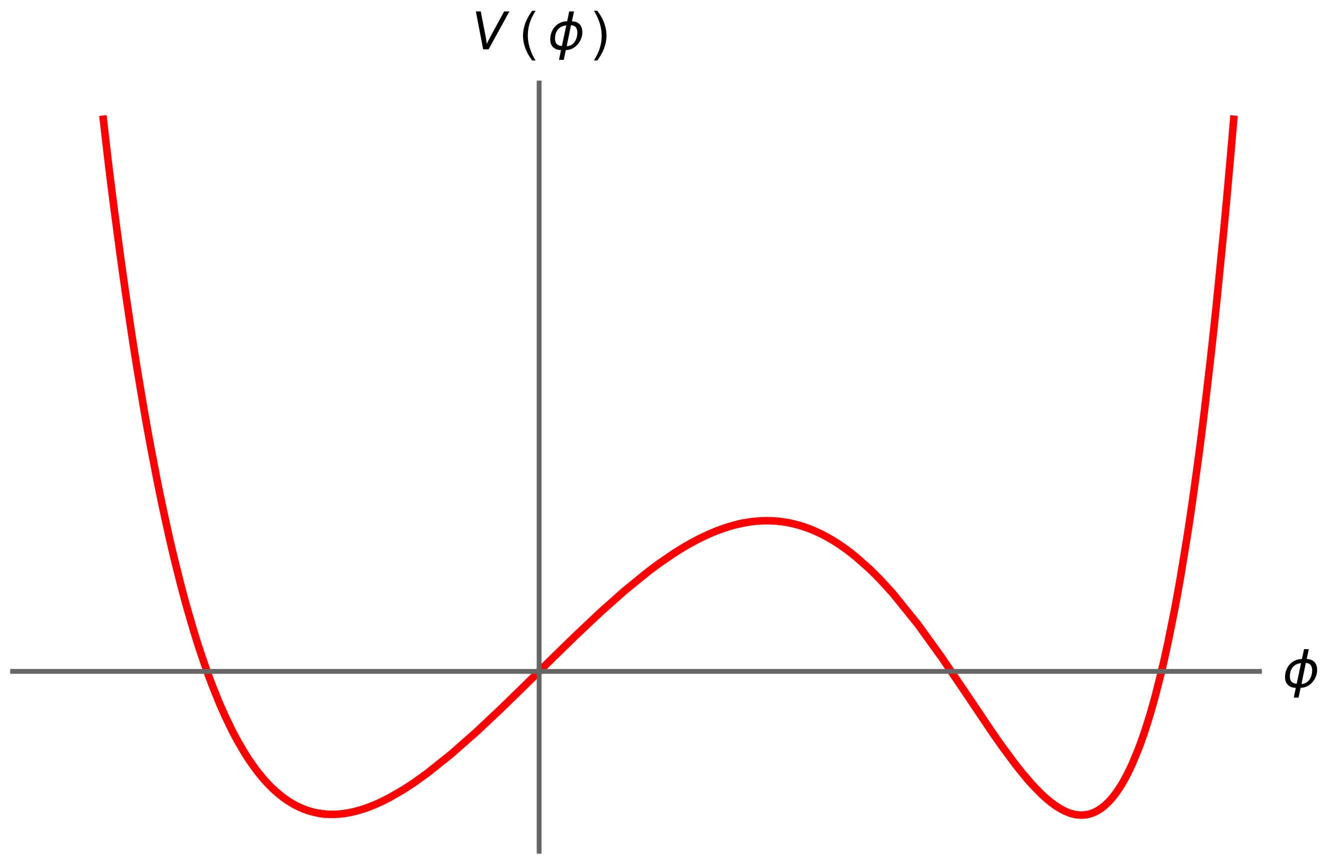
[C-M. Chang, Y-H. Lin, S-H. Shao, Y. Wang, X. Yin. (1802.04445)].
[Lassig-Mussardo-Cardy, Zamolodchikov]




$: \mathcal{H}_{m,n}$

$$W \times W = 1 + W$$

Symmetry spontaneously broken (gapped phase): $\mathscr{H}_{1,1} \oplus \mathscr{H}_{1,W} \oplus \mathscr{H}_{W,1} \oplus \mathscr{H}_{W,W}$

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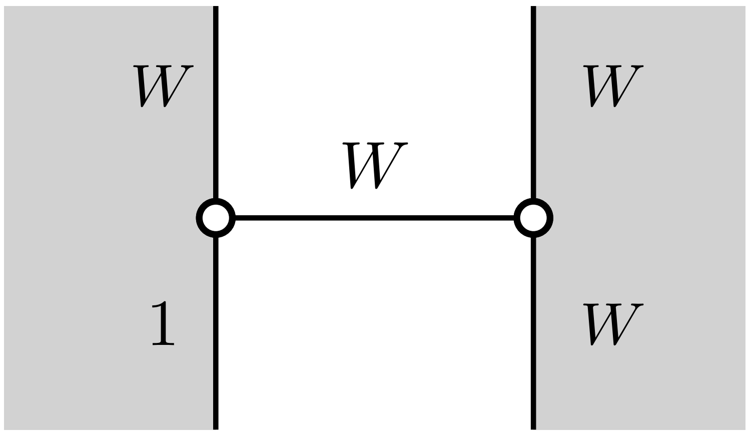


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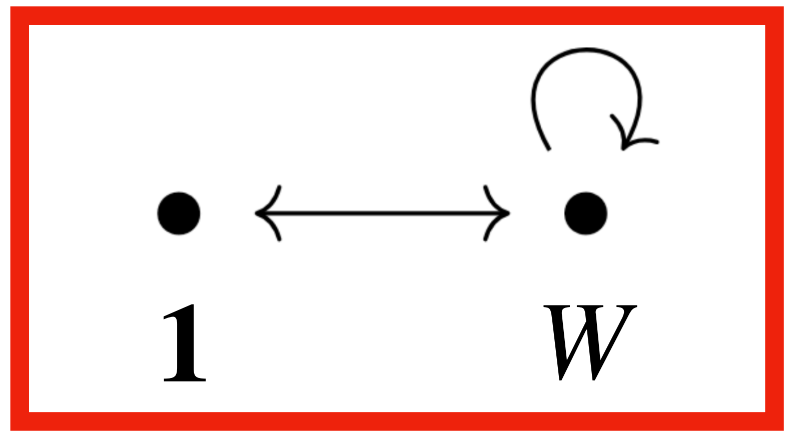


Recover the particle-soliton degeneracies of the spectrum from the non-invertible (Fibonacci) symmetry in the flow!

Multiplets of states relating particles and solitons in 2D via non-invertible symmetry [2403.08883]:

# *Another example: $M_4 \; - \; \lambda\,\phi_{(1,3)}$*

Lines $1, \eta$ and $N$ preserved by $\phi_{(1,3)}$ deformation.

$\mathbb{Z}_2$ Tambara-Yamagami (Ising) Soliton Degeneracy:

$$\eta \times \eta = 1 \qquad \eta \times N = N \times \eta = N$$
$$N \times N = 1 + \eta$$

Fully spontaneously broken.

| Tricritical Ising Model $M_4$ | | | |
|---|---|---|---|
| Kac label | Conformal Weight | Verlinde Line | Quantum Dimension |
| $(1,1)$ or $(3,4)$ | $h_{1,1} = 0$ | $1$ | $d_{1,1} = 1$ |
| $(1,2)$ or $(3,3)$ | $h_{1,2} = 1/10$ | $W \otimes \eta$ | $d_{1,2} = \frac{1+\sqrt{5}}{2}$ |
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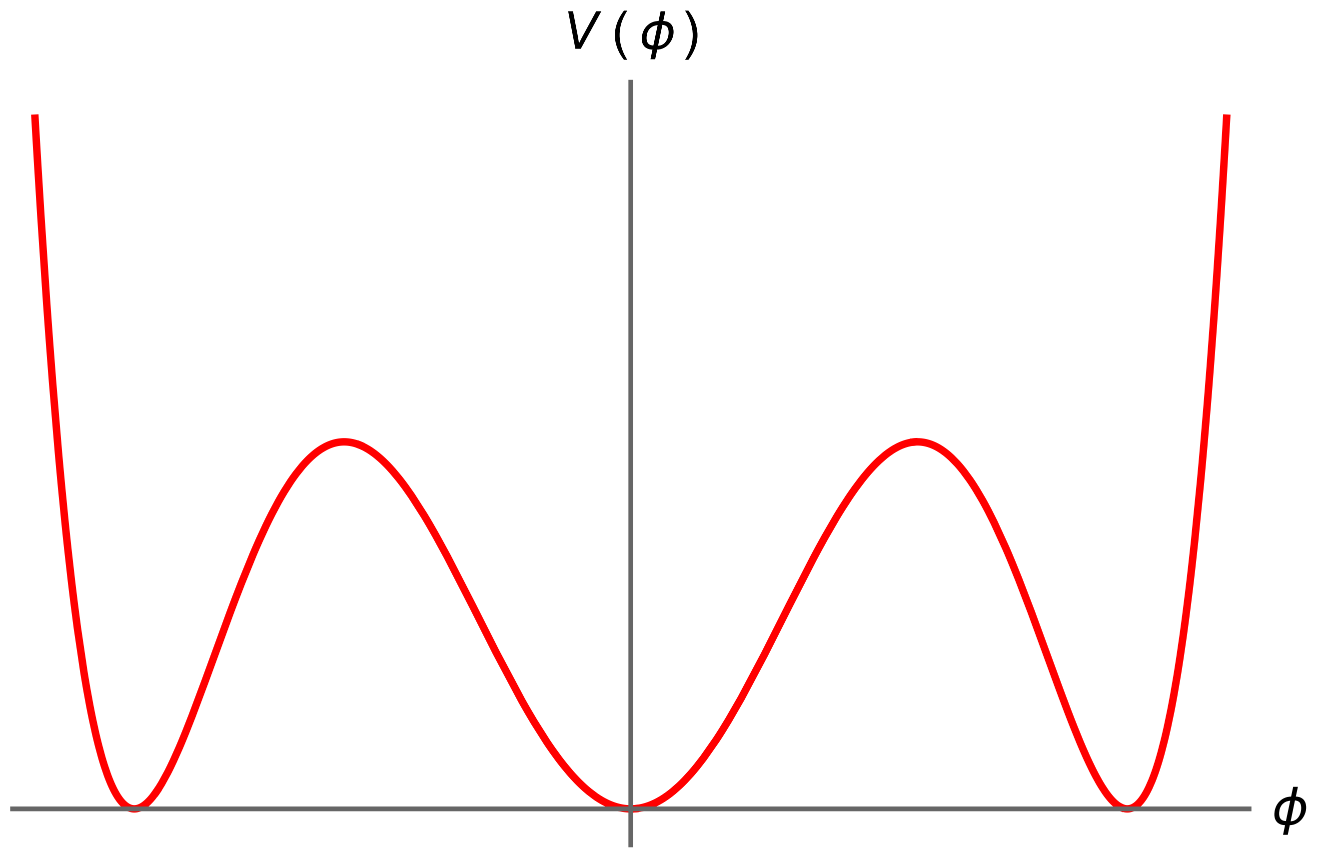
| $(2,4)$ or $(2,1)$ | $h_{2,1} = 7/16$ | $N$ | $d_{2,1} = \sqrt{2}$ |



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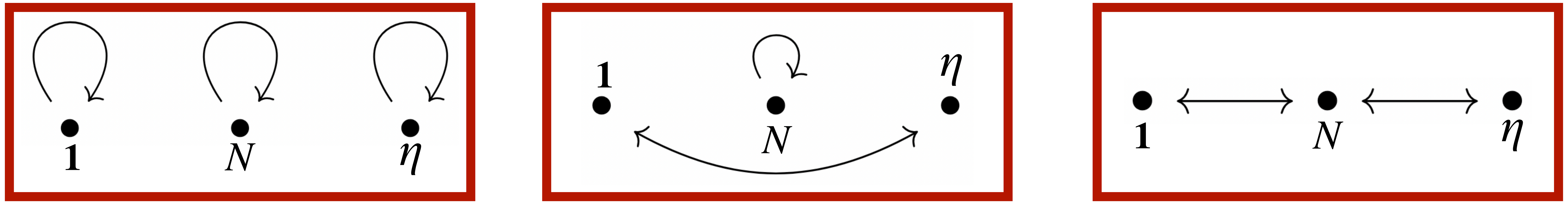
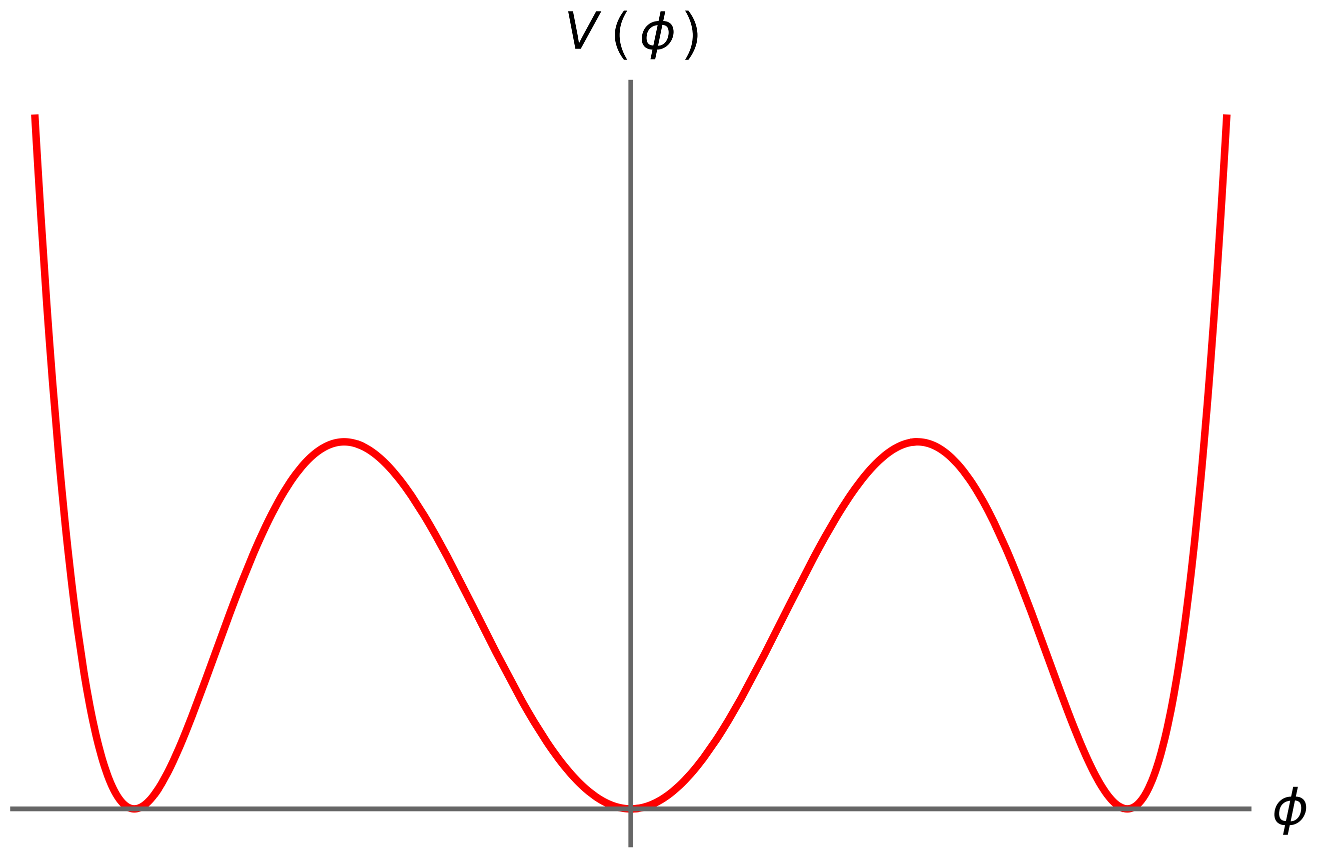
Allowed quiver diagrams (representations):



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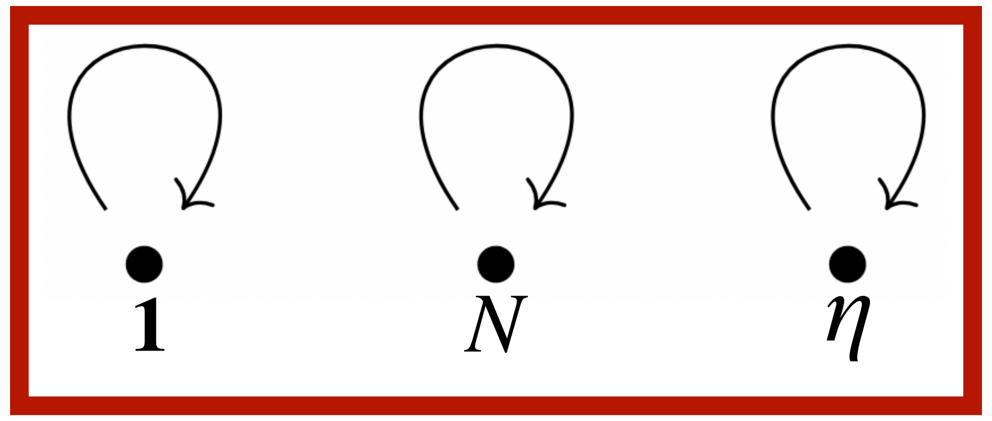


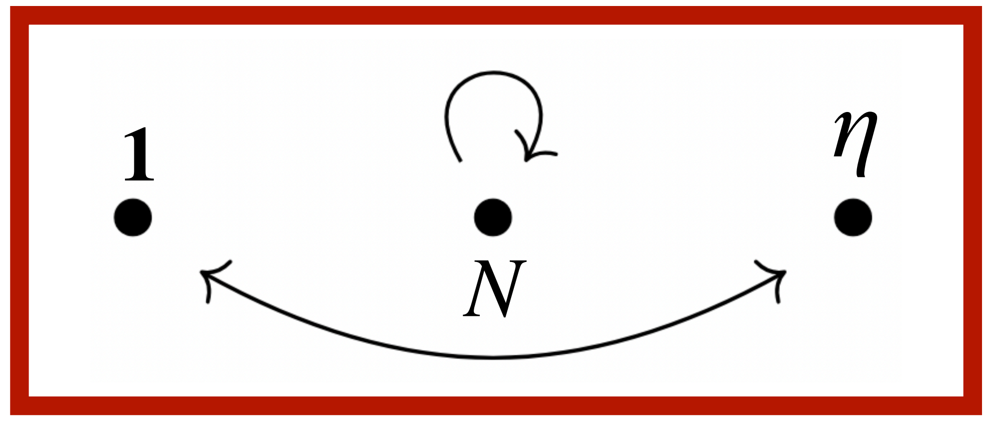


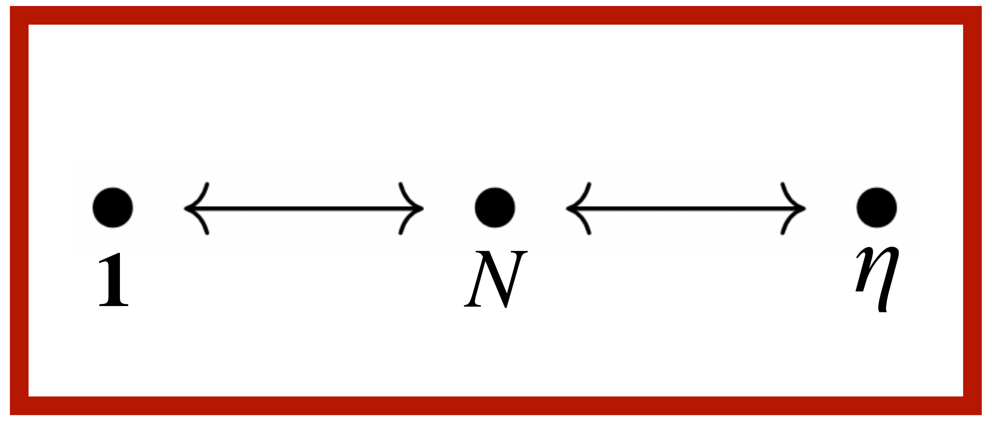


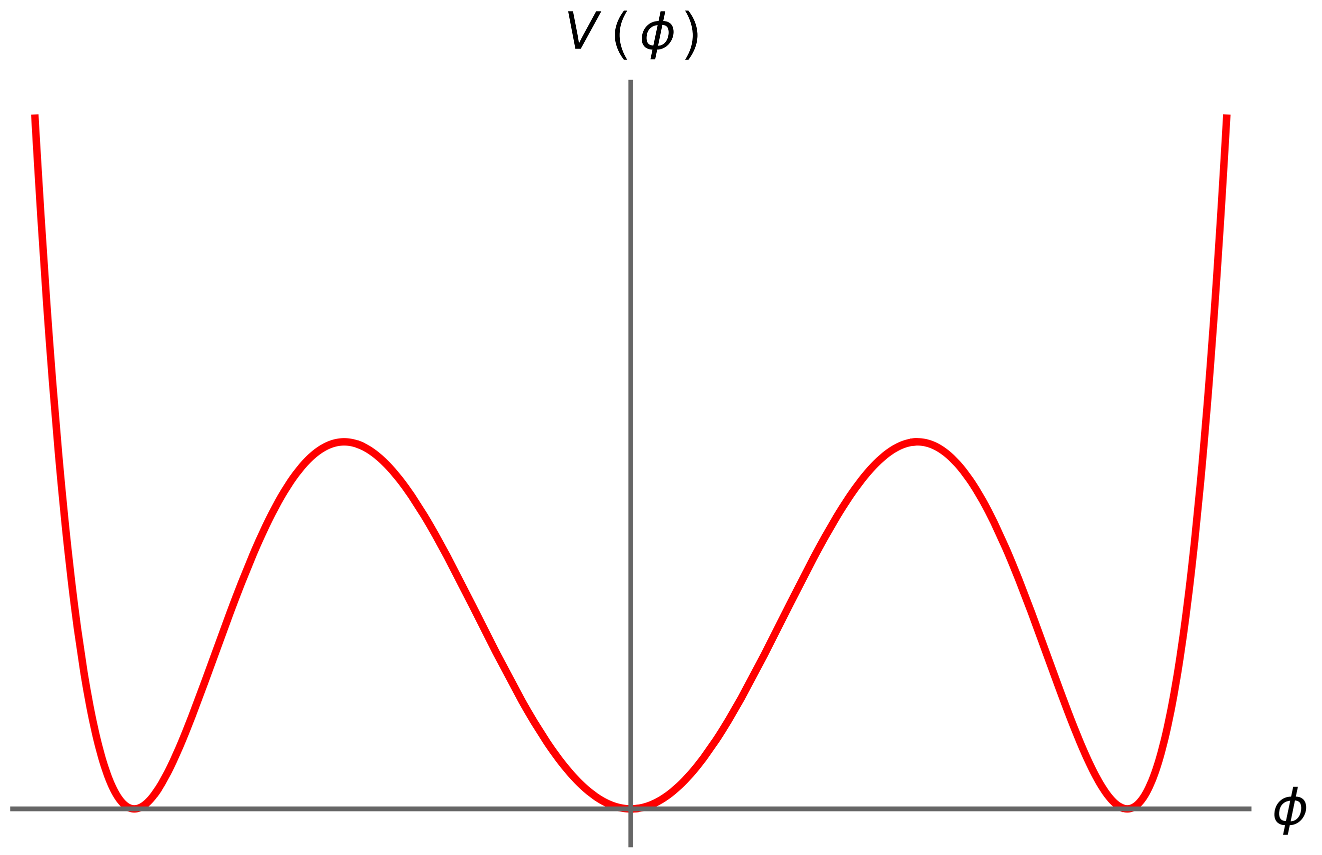


## *Generalization:* $M_n \;-\; \phi_{(1,3)}$

$2(n-2)$-fold soliton degeneracy

[Zamolodchikov. *Nucl.Phys.B* 358 (1991) 497-523]. [2403.08883]

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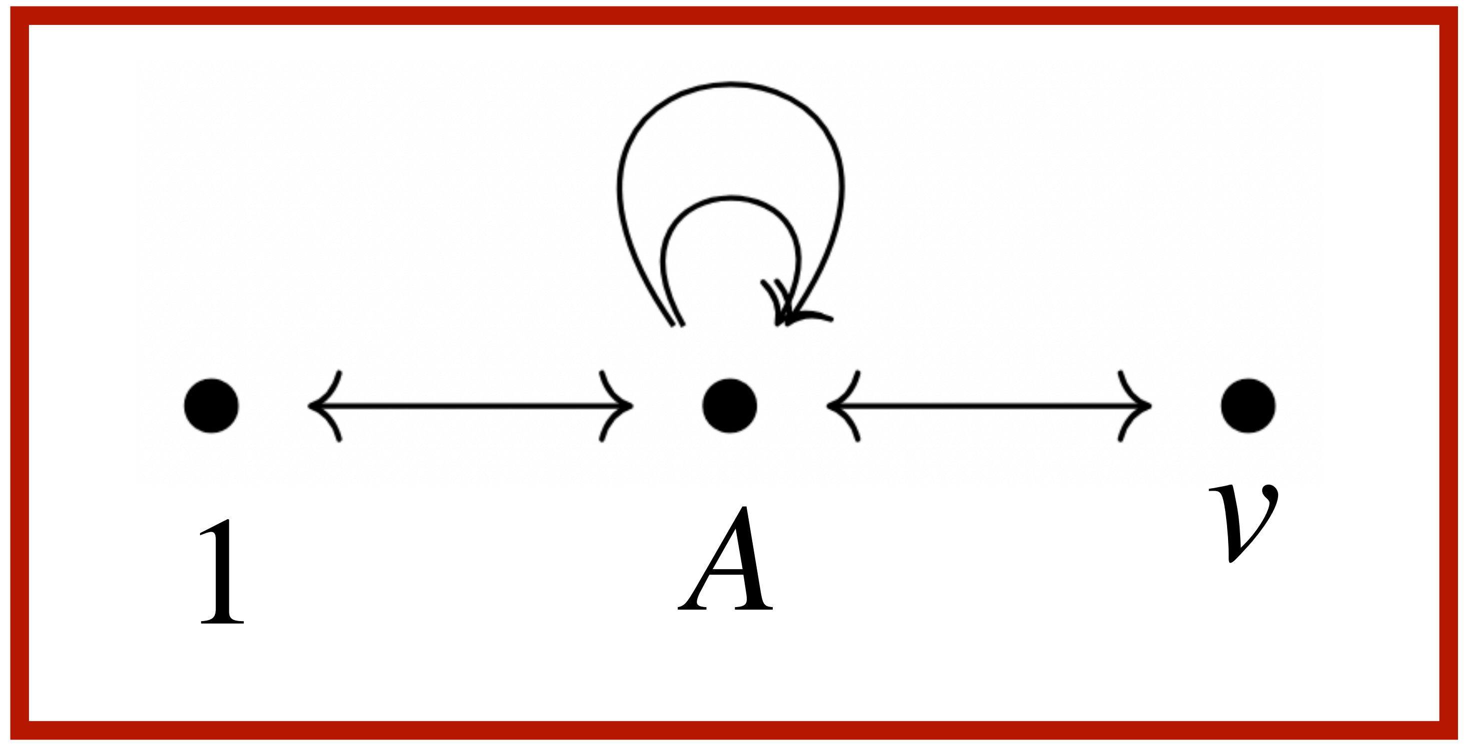
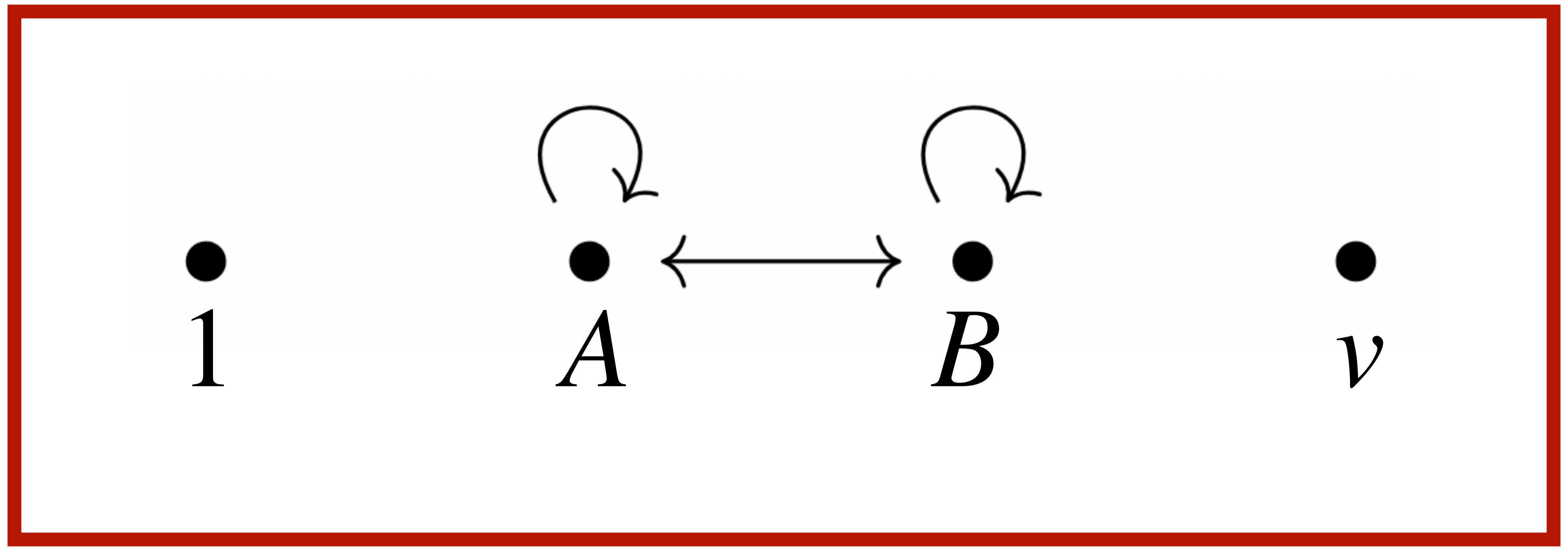



$PSU(4) + \psi_{\mathbf{15}}$



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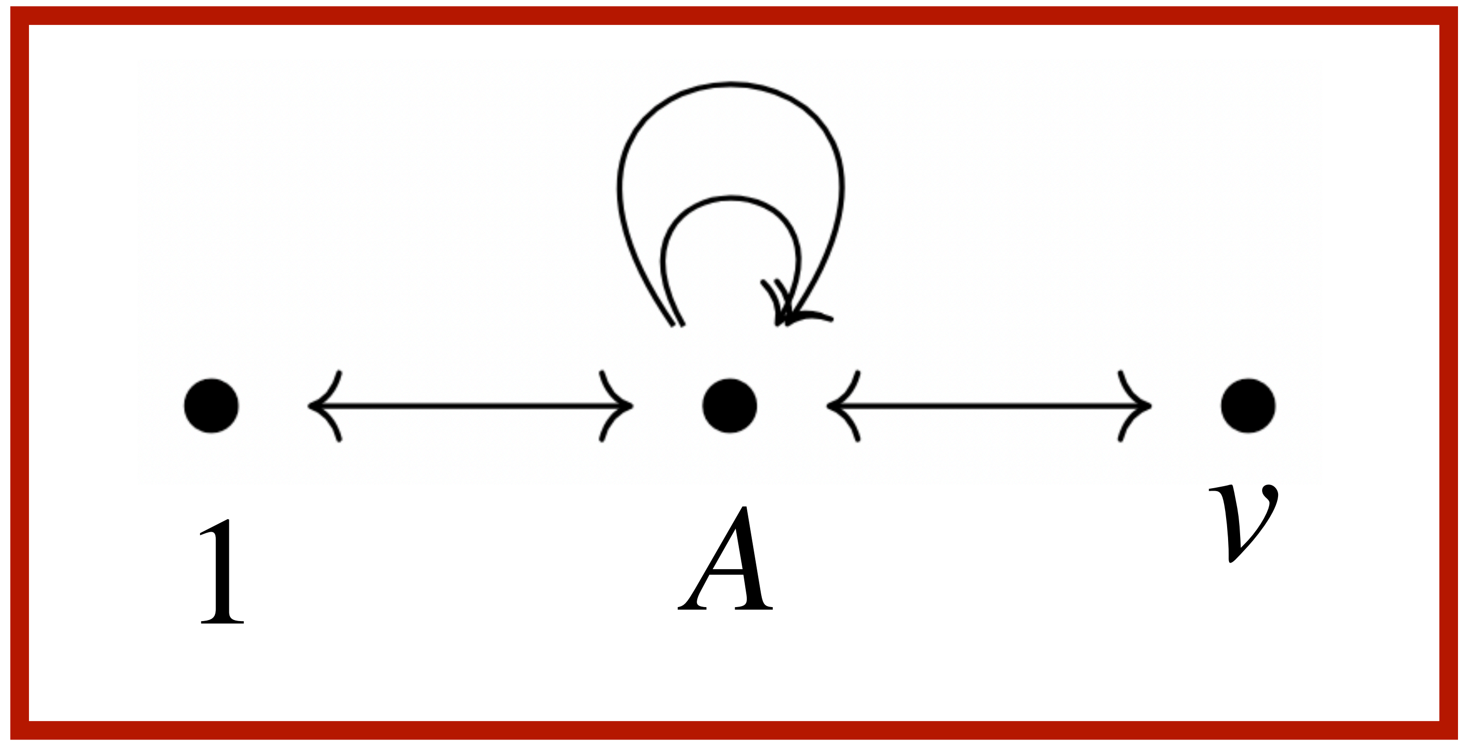


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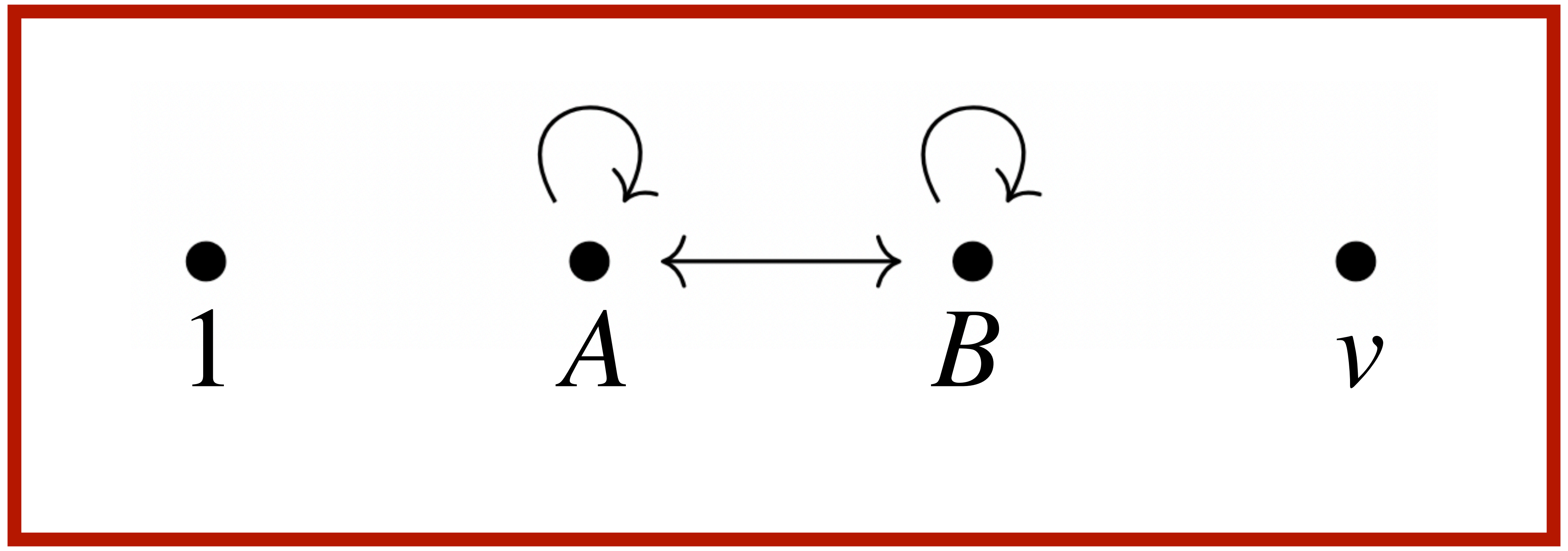


Non-simply-connected gauge groups to disregard one-form symmetry throughout the flow: finite excitations only

# *Setting up 2D QCD*

# *Setup for (massless) 2D QCD*

The theory we consider is QCD in two spacetime dimensions. The gauge group is $G$ and the matter fields are massless fermions transforming in some (irreducible) representation $\mathbf{R}$ of $G$. In other words:

$$S_f(g_{YM}) = \int d^2x \left[ -\frac{1}{4g_{YM}^2}\mathrm{Tr}(F^2) + \mathrm{Tr}(\psi^T\, iD\, \psi) \right]$$

Bosonize: Gauged WZW model with $\mathrm{Spin}(\dim(\mathbf{R}))_1$ matter content and $G$ gauge fields, plus kinetic term for gauge fields:

$$S_b(g_{YM}) = S_{WZW}(g_{YM}, A) - \frac{1}{4g_{YM}^2}\int_\Sigma d^2x\, \mathrm{Tr}(F^2)$$

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In principle, no loss of information: fermionization/bosonization invertible.

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When is a 2D QCD theory gapped?

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Constrains on Particle-Soliton states $\Longrightarrow$ Focus on gapped 2D QCD theories.

A series of results illuminate how to describe the low-energy regime:

[Z. Komargodski, K. Ohmori, K. Roumpedakis, S. Seifnashri. (2008.07567)]
[D. Delmastro, J. Gomis, M. Yu. (2108.02202)]

Gap criterion: The theory is gapped if and only if the corresponding coset has a vanishing central charge:

$$\frac{\mathrm{Spin}(\dim(\mathbf{R}))_1}{G_{I(\mathbf{R})}} \qquad \text{with} \qquad c_{\frac{\mathrm{Spin}(\dim(\mathbf{R}))_1}{G_{I(\mathbf{R})}}} = c_{\mathrm{Spin}(\dim(\mathbf{R}))_1} - c_{G_{I(\mathbf{R})}} = 0$$

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Wish to study the action of non-invertible symmetries over vacua:

Topological Coset $\dfrac{\mathrm{Spin}(\dim(\mathbf{R}))_1}{G_{I(\mathbf{R})}}$

$$\lim_{g_{YM}\to\infty} S_b(g_{YM}) = S_{gWZW}(A)$$

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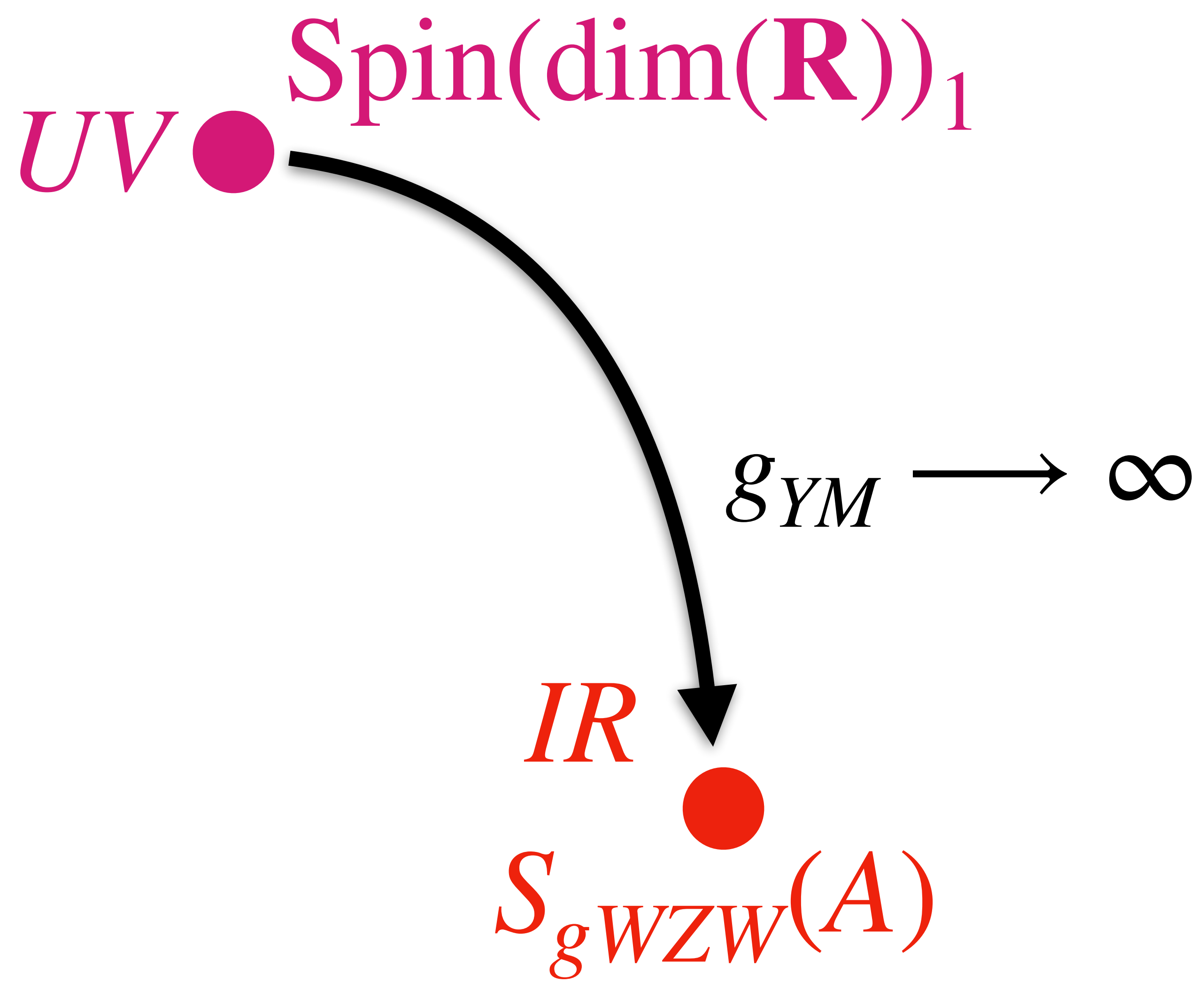



In the UV: Free Fermions (Bosonized).

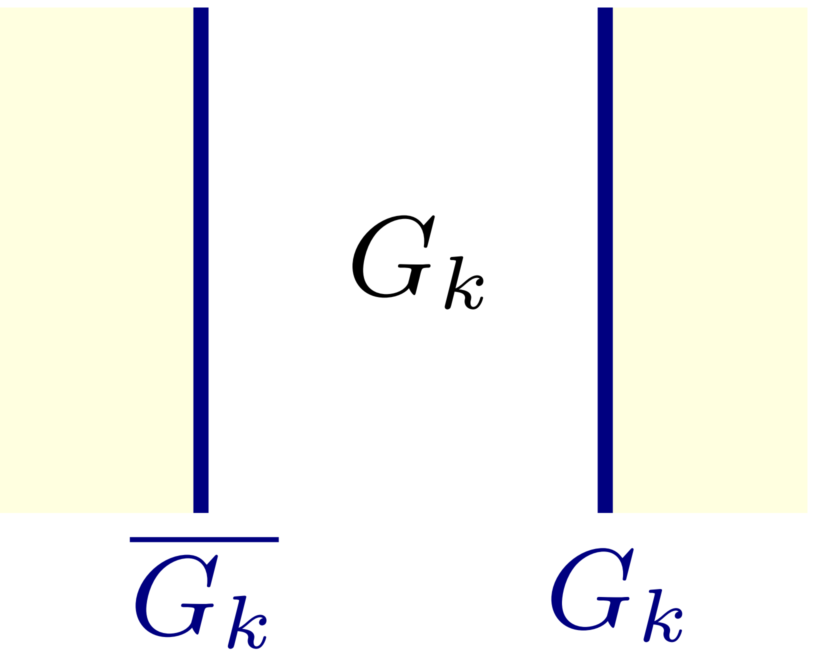
In the IR: Gauged WZW Model.

# *2D Theories from 3D TQFTs*

A fruitful way of describing cosets makes use of an appropriate 3D TQFT construction.

Recall the relationship between $G_k$ Chern-Simons theory and $G_k$ WZW theory.

[S. Elitzur, G. Moore, A. Schwimmer, N. Seiberg. (*Nucl.Phys.B* 326 (1989) 108-134)]

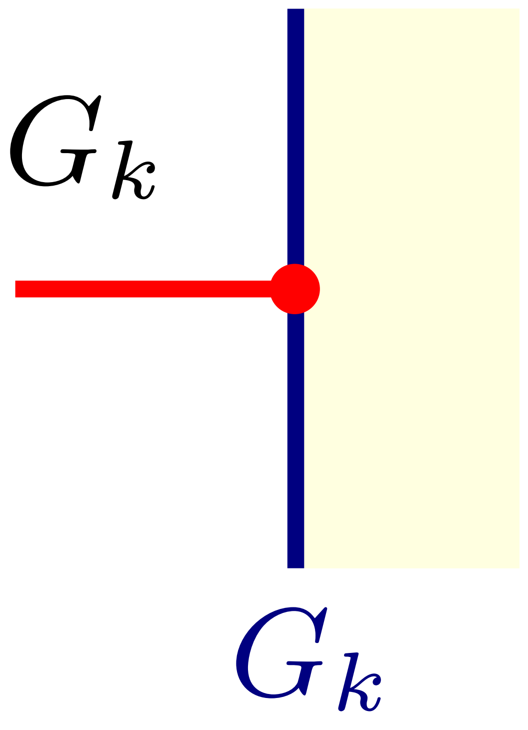



$$S = \frac{k}{4\pi}\int_Y \mathrm{Tr}(A\, dA + \frac{2}{3}A^3) \qquad A_0\Big|_{\partial Y} = 0$$

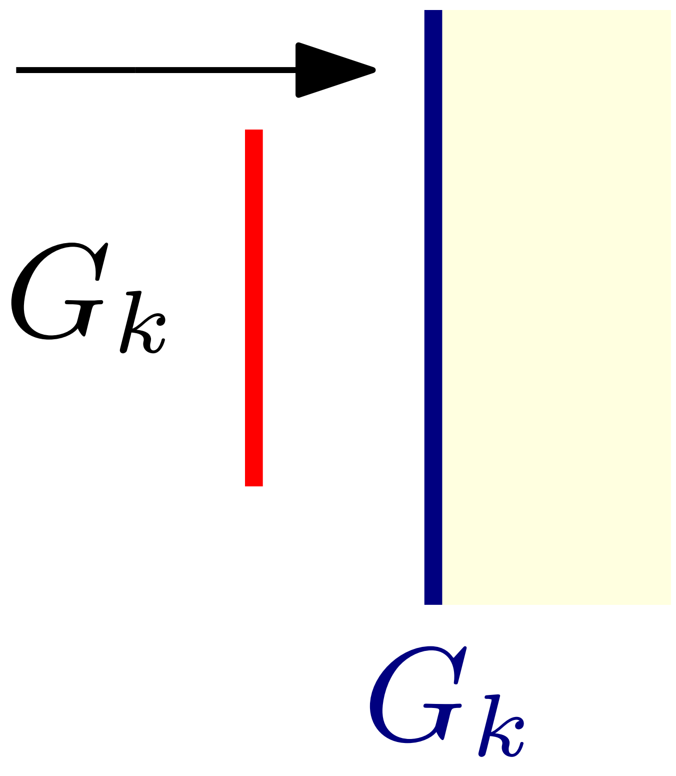
$$\frac{k}{4\pi}\int_{\partial Y} \mathrm{Tr}(U^{-1}\partial_\phi U\, U^{-1}\partial_t U) + \frac{k}{12}\int_Y \mathrm{Tr}(U^{-1}dU)^3$$

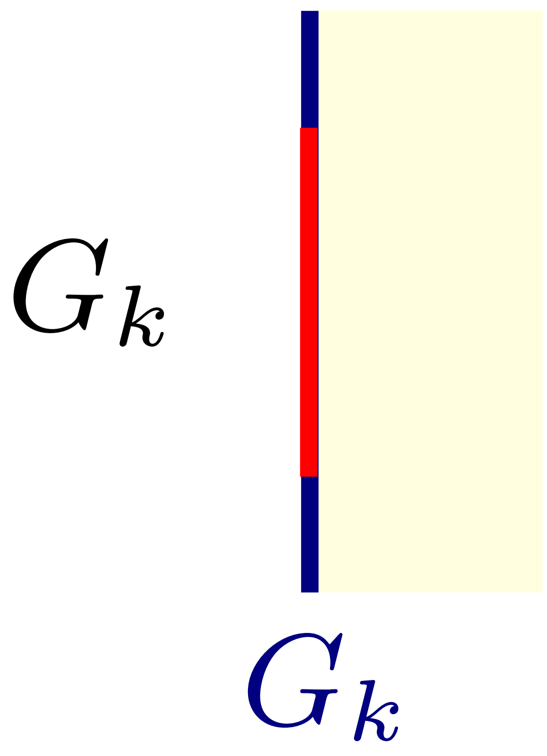
The data of the 2D theory is easily reconstructed from that of the bulk TQFT:

Local operators:




Verlinde lines:



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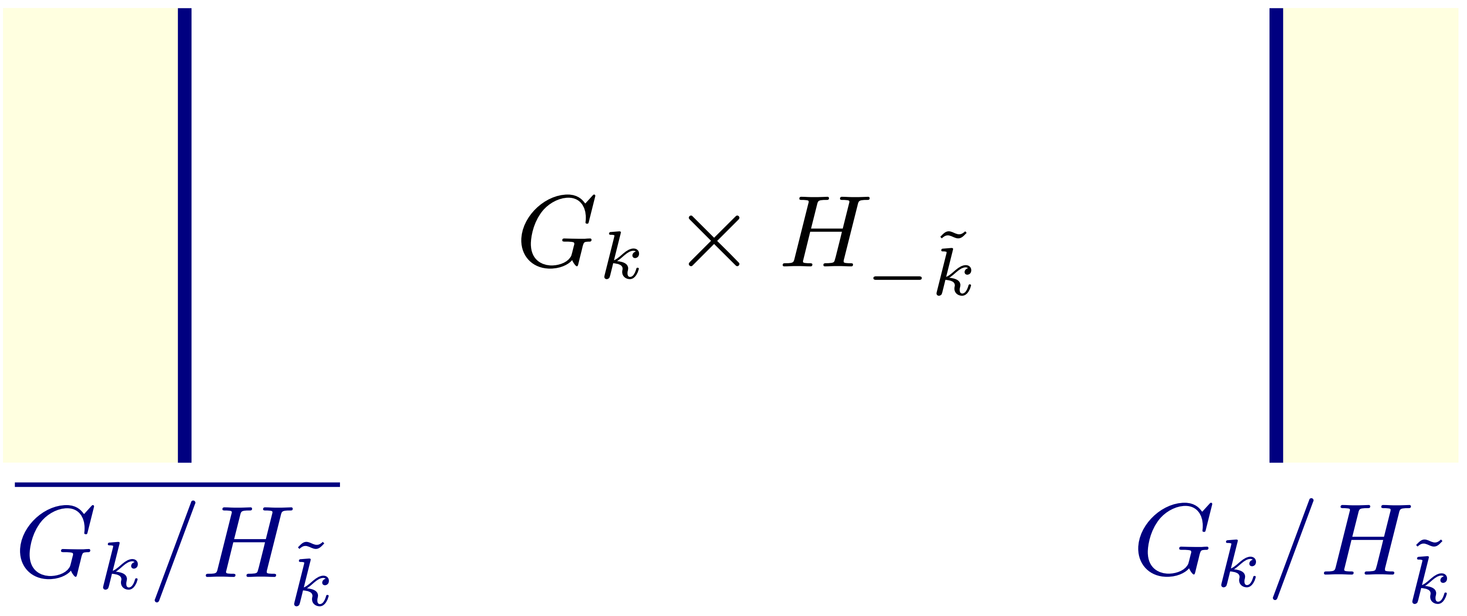


New boundary condition describing the embedding of gauge groups at the boundary:

$$H_{\tilde{k}} \hookrightarrow G_k$$

The 2D CFT obtained at the boundary corresponds to the gauged WZW model $G_k/H_{\tilde{k}}$ .

More precisely, Moore & Seiberg instructs us to construct the Chern-Simons theory

$$(G_k \times H_{-\tilde{k}})/\mathbb{Z}^{(1)}_{G\cap H}$$

- The algebraic data of the coset CFT $G_k/H_{\tilde{k}}$ may then be obtained following the "three-step gauging rule".
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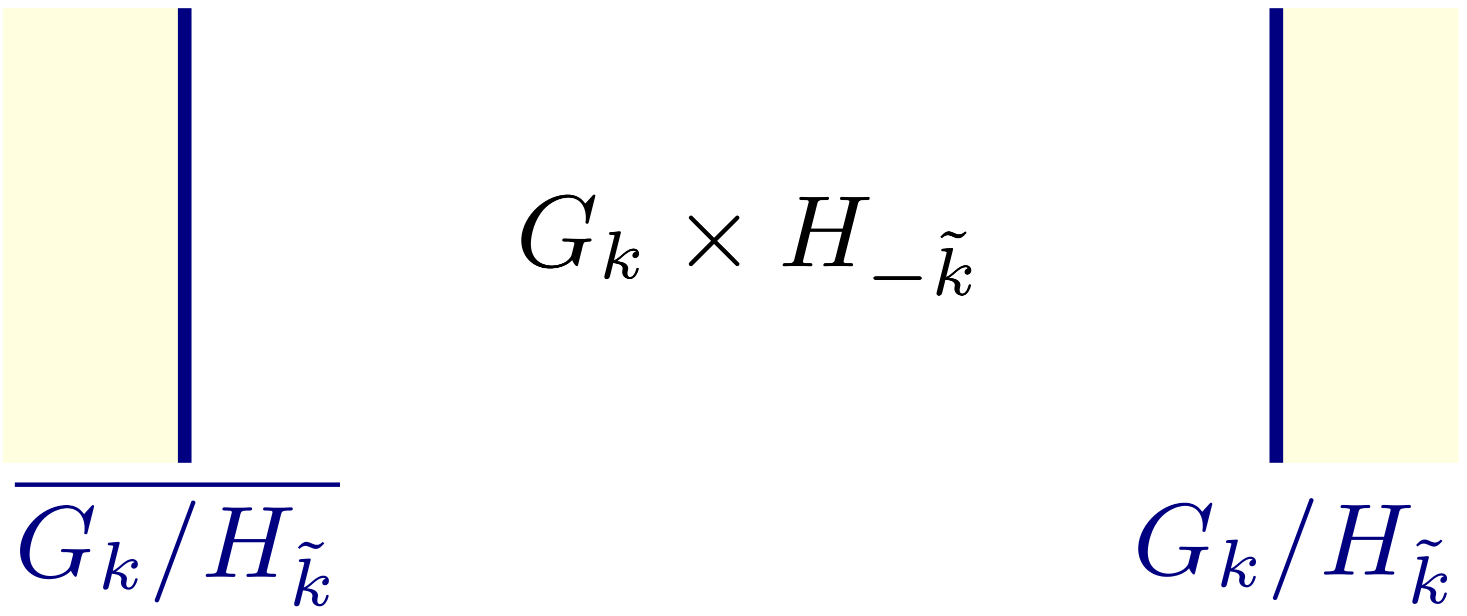


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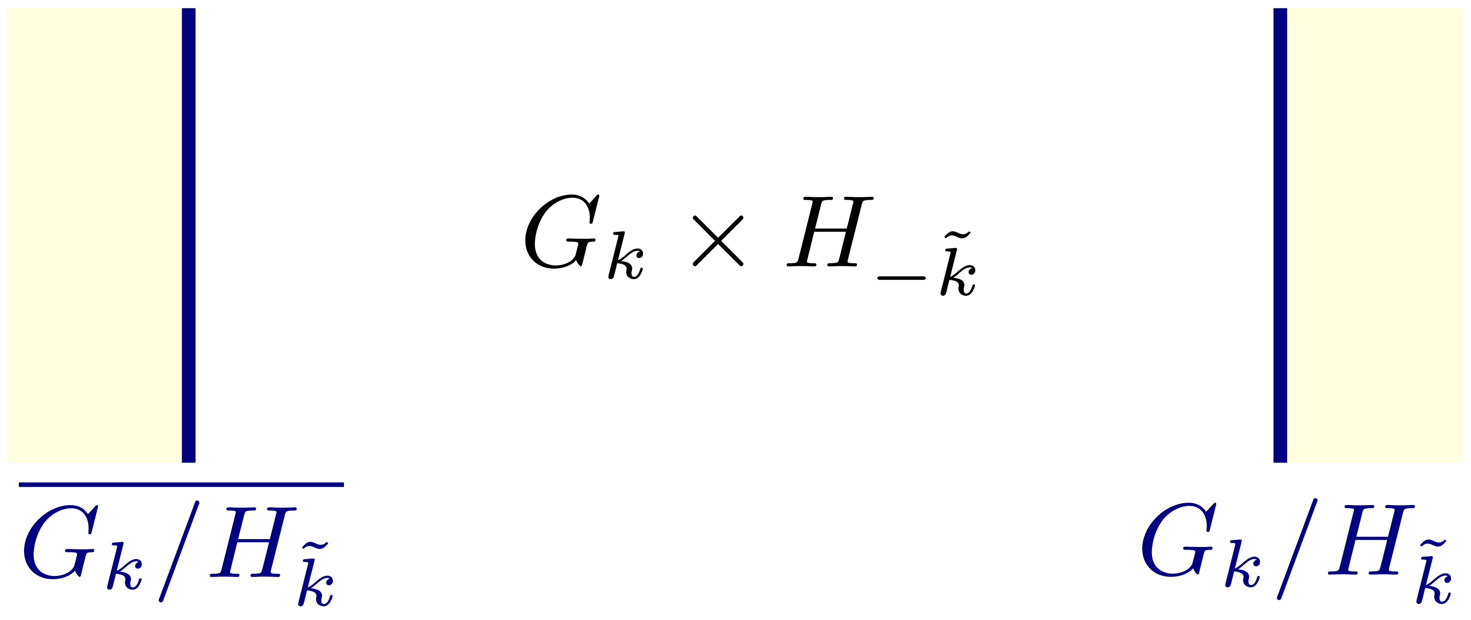


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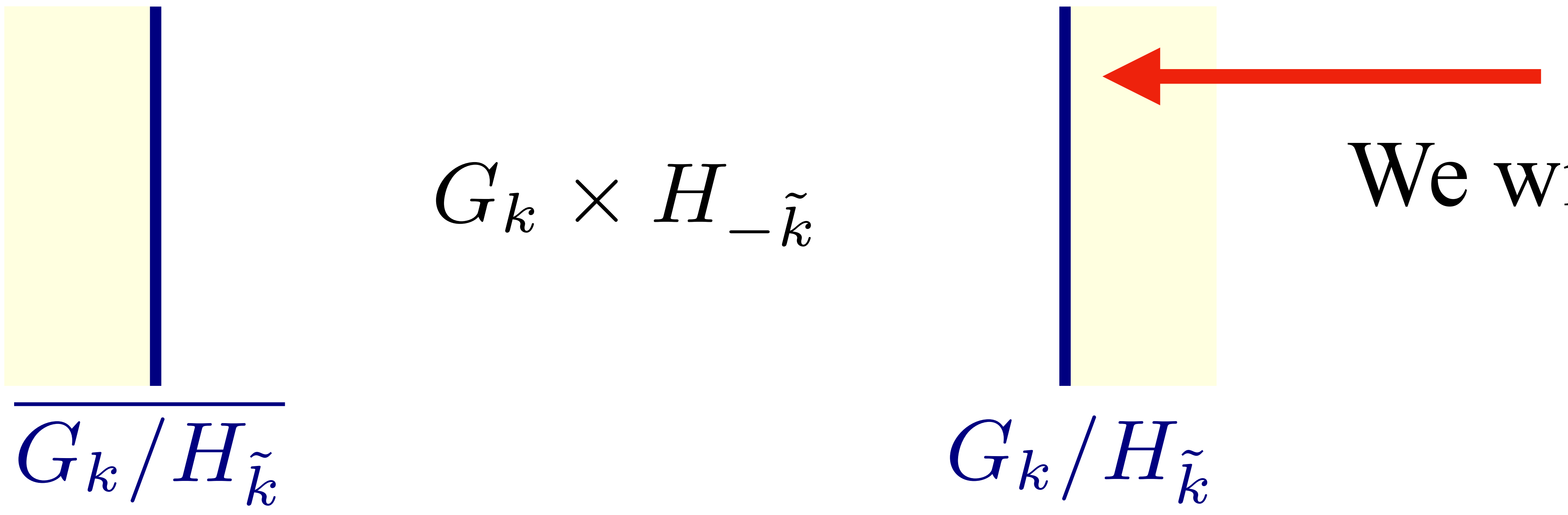


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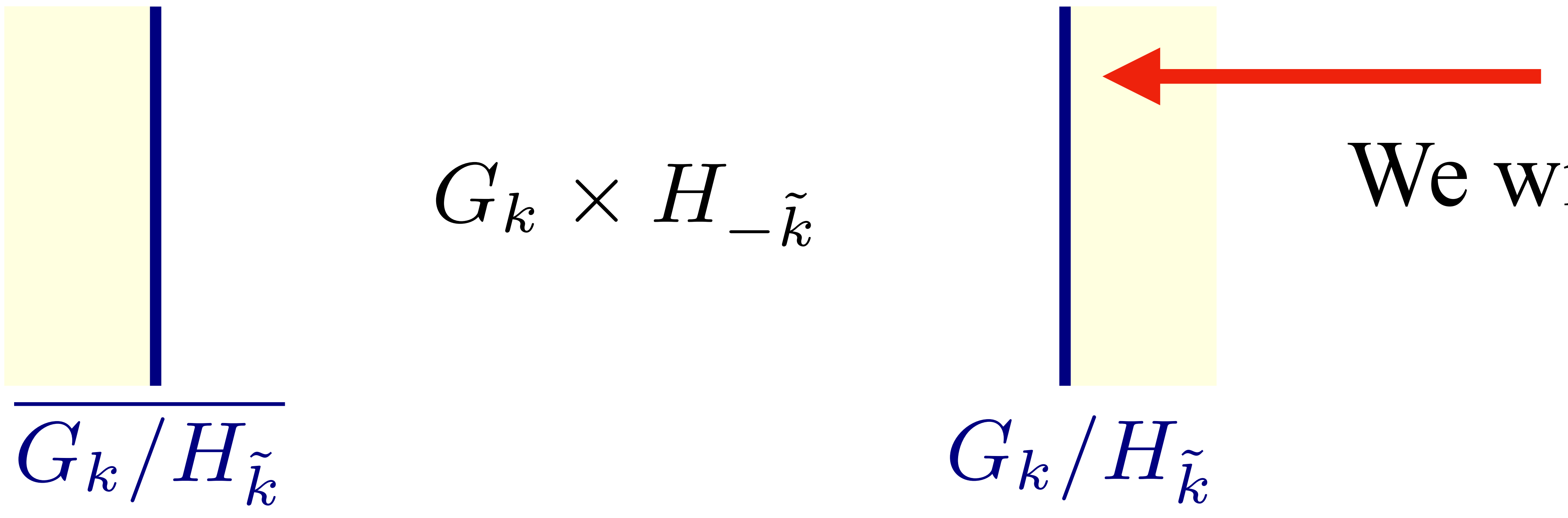


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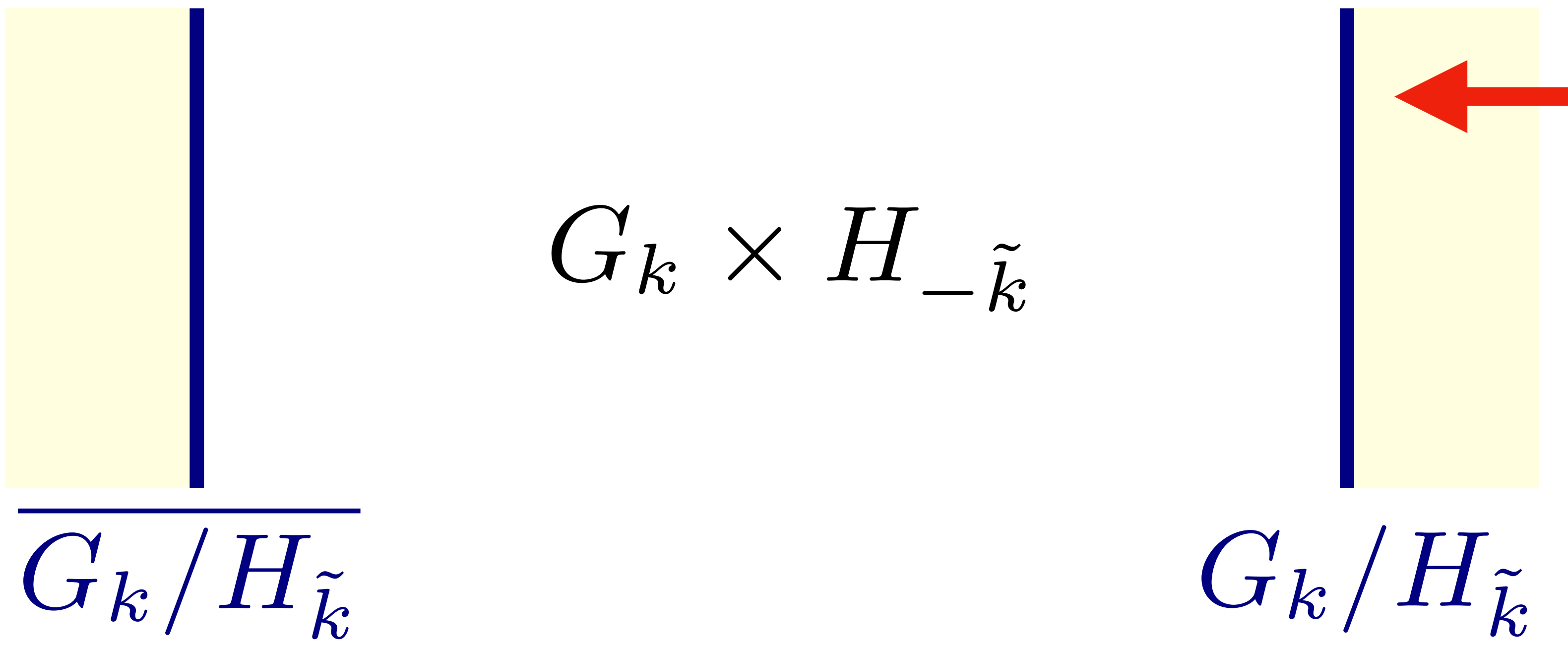


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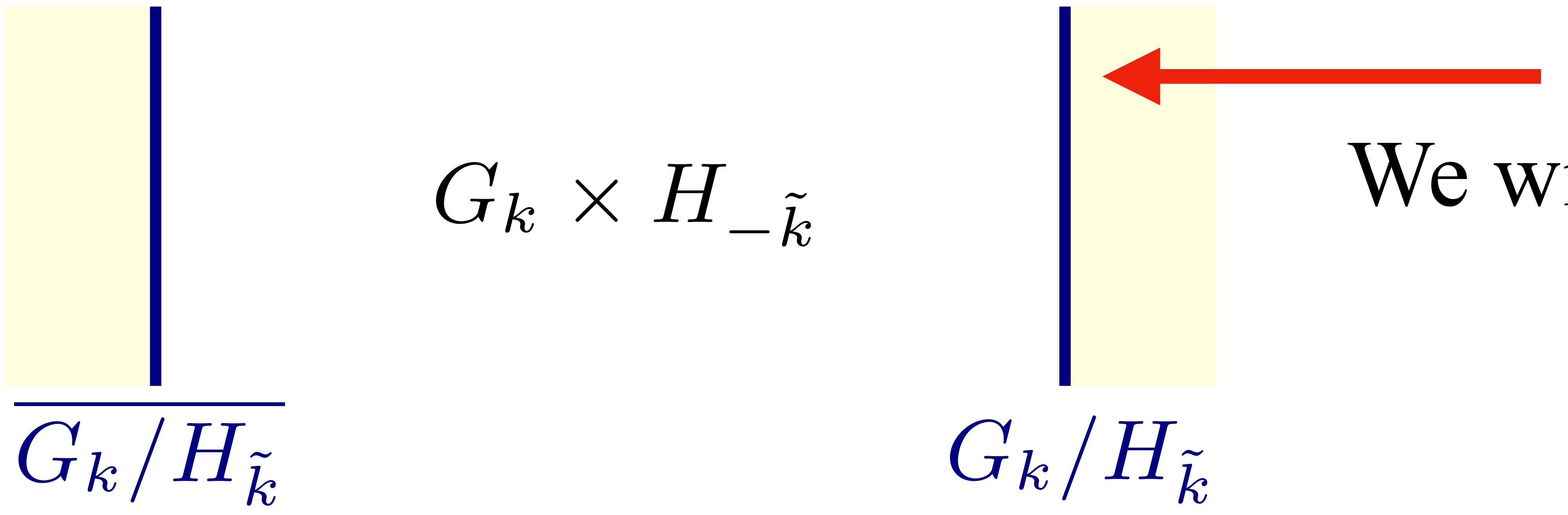


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Common "Moore-Seiberg framework" for gapless cosets and topological cosets

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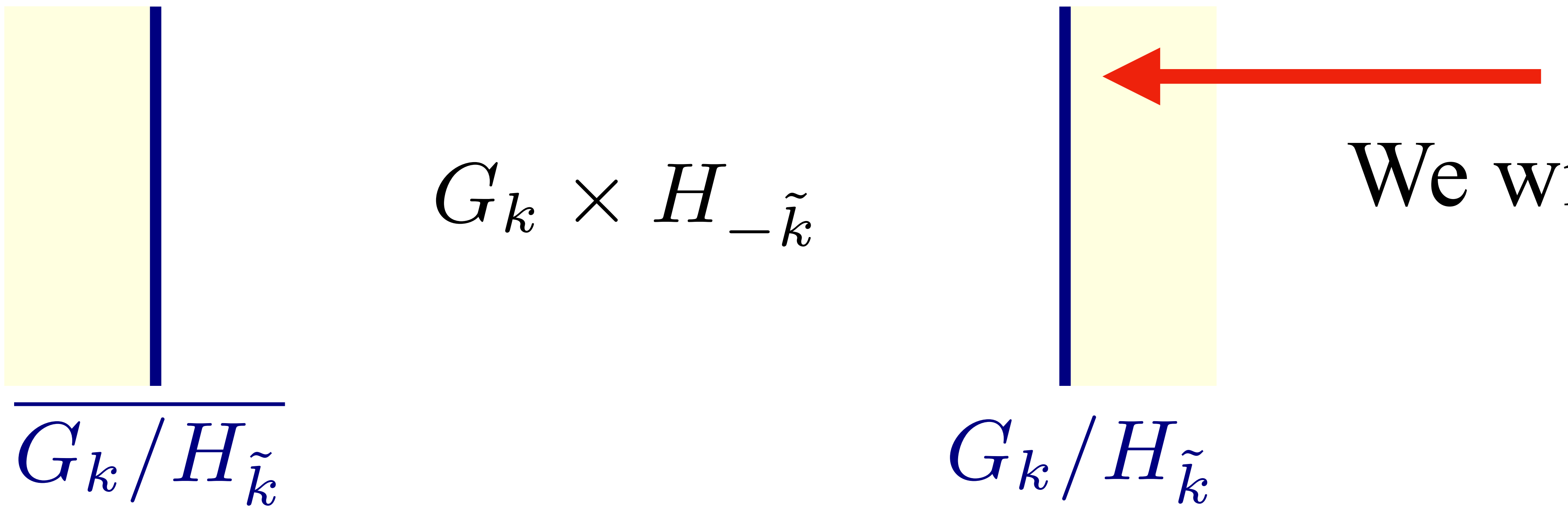



We wish to describe this coset in the topological case.

Description of the IR of 2D QCD!

Observation: lift the original proposal of Moore and Seiberg to allow for *topological boundary conditions* of the bulk 3D TQFT.

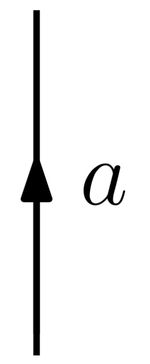
Common "Moore-Seiberg framework" for gapless cosets and topological cosets

Important observation: Topological cosets are always associated to a topological boundary condition of the corresponding bulk Chern-Simons theories.
[A. Davydov, M. Müger, D. Nikshych, V. Ostrik. (1009.2117)]
[YZ. Huang, A. Kirillov Jr., J. Lepowsky. (1406.3420)]

# *3D TQFT Description of Topological Cosets*

# *The algebraic theory of anyons*

**Main important point:** 3D TQFTs are theories consisting solely of topological line operators: **anyons**.

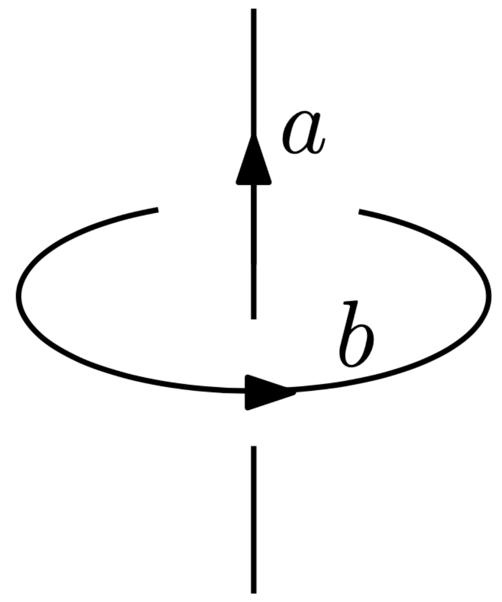


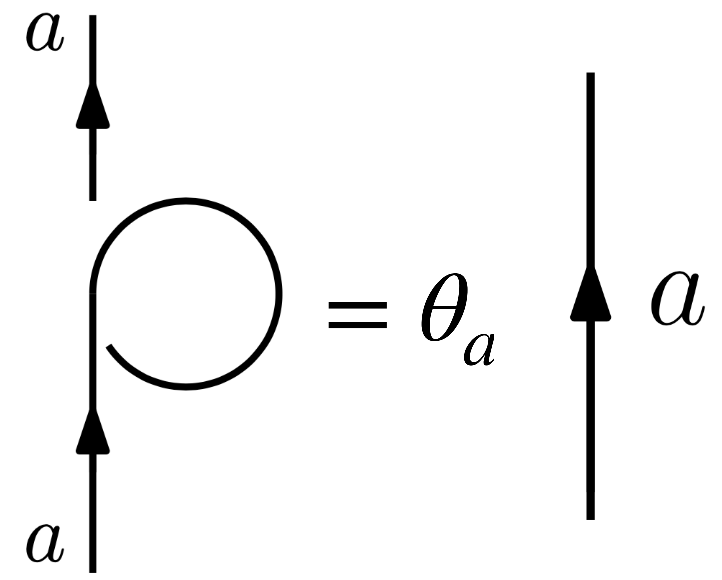

Mathematically, these line operators are described by the formalism of modular tensor categories.

In practice: a finite set of lines fulfilling a finite set of data.

In principle, any correlator can be computed from this finite set of data.

- The set of lines
- Fusion rules
- Topological Spin/Conformal weight
- Modular S Matrix

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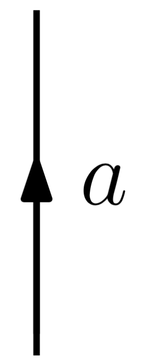


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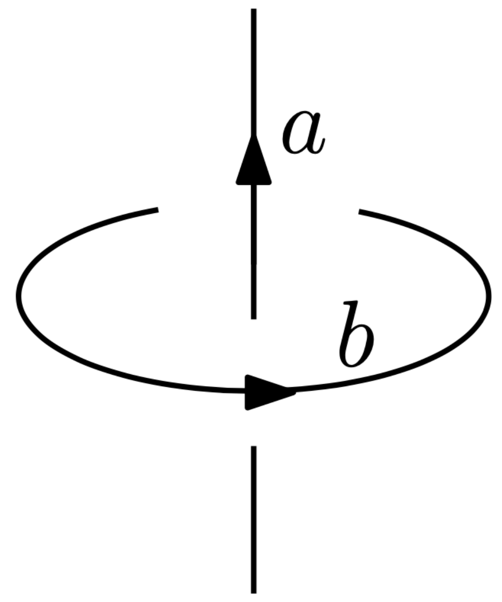


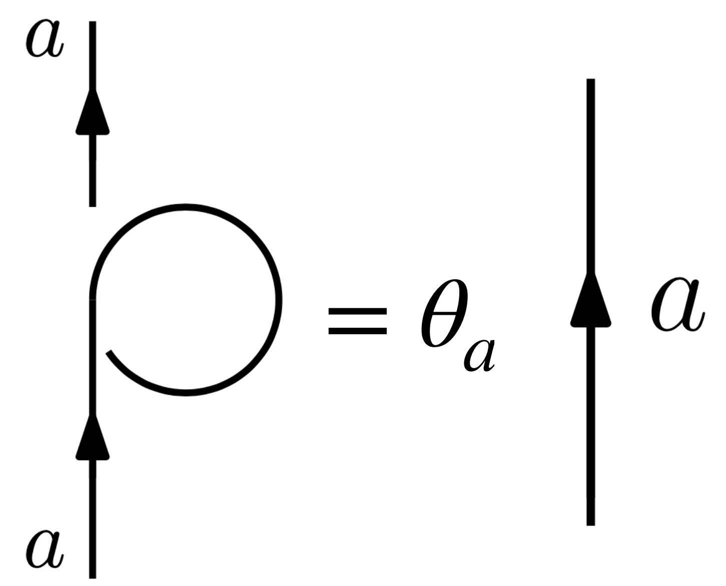


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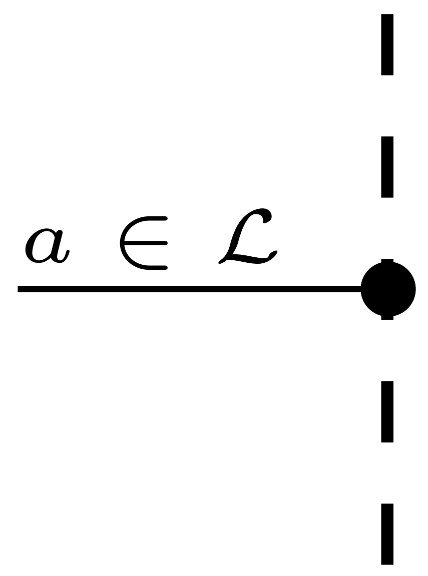
→ In the current application we will be mostly working with $G_k$ Chern-Simons theories, in which case the above properties are mostly inherited from the corresponding WZW theory.

# *Topological Boundary Conditions in 3D TQFTs*

- Topological boundary conditions of a 3D TQFT described by a linear combination of anyons in the theory called *Lagrangian Algebra:*

$$\mathscr{L} = \bigoplus_a n_a a, \quad n_a \in \mathbb{N}$$

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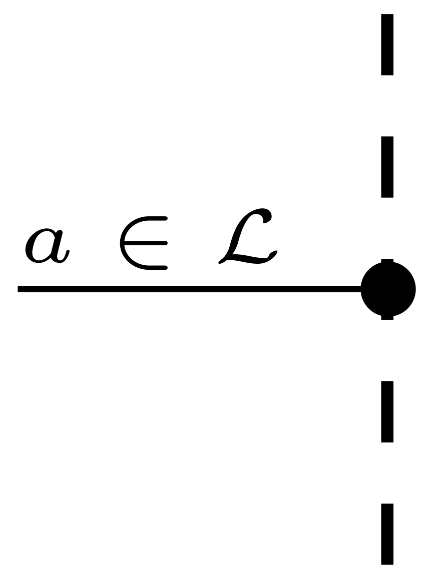


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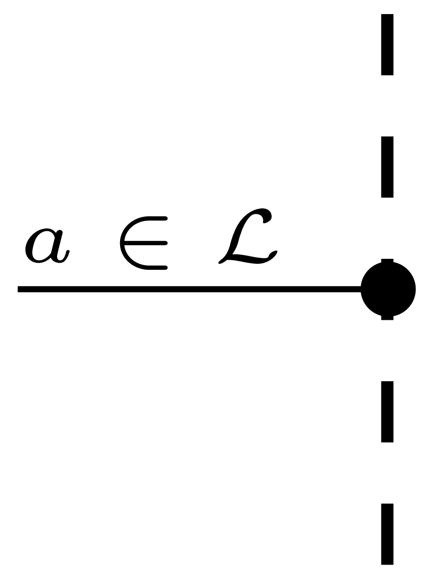


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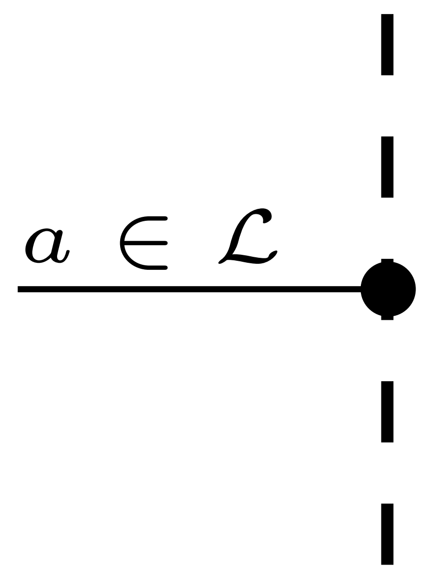


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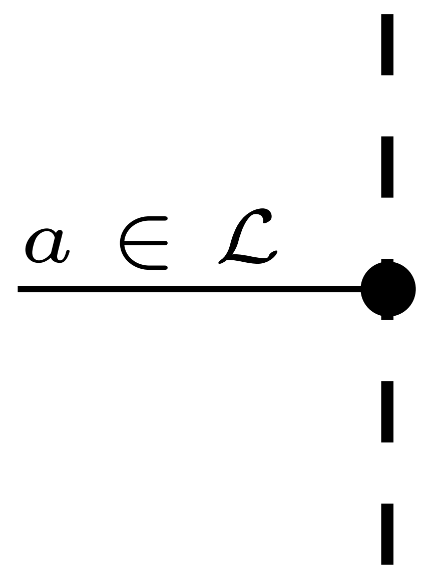


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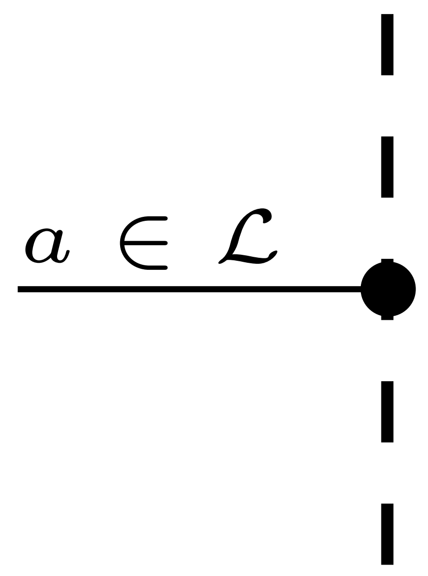


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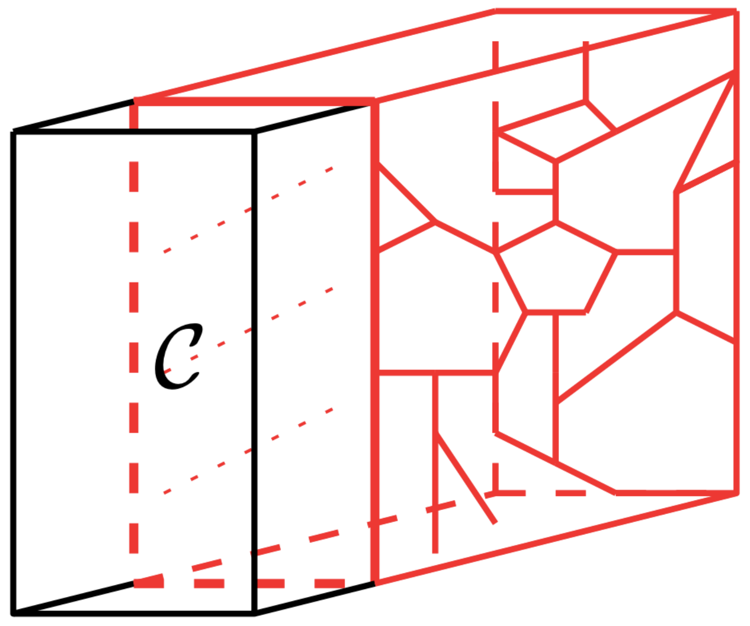
Gauging of generalized symmetry $\mathscr{F}$: Insert a mesh of $\mathscr{F}$ throughout the spacetime region where the original theory $\mathscr{C}$ is defined. [1412.5148].




"Anyon condensation" in condensed matter.

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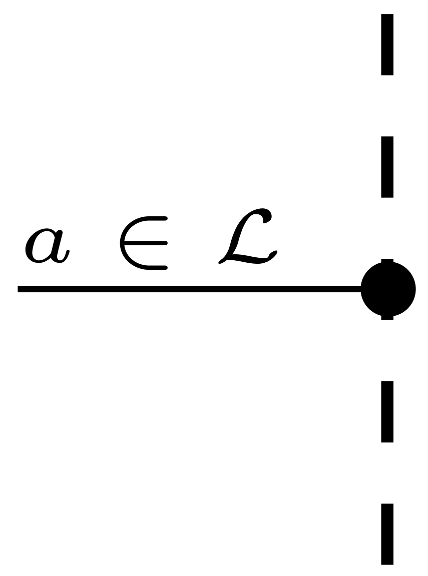


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**How do we characterize this associated fusion category $\mathscr{F}$ at the boundary?**

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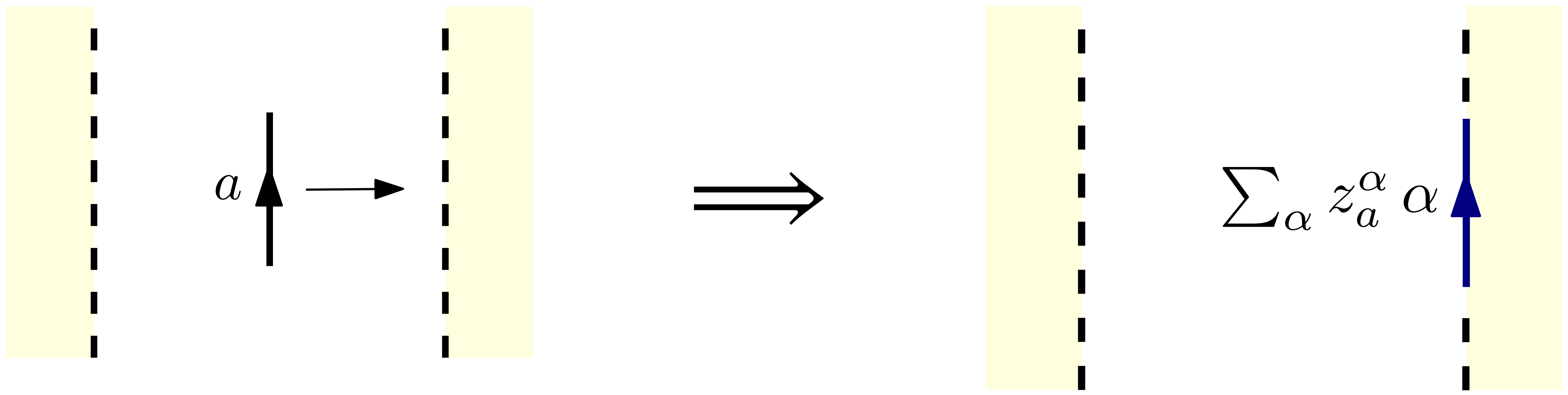



- Simple bulk anyons $a$ become generically at boundary via "splitting rule":

$$a = \sum_\alpha z_a^\alpha\, \alpha, \quad z_a^\alpha \in \mathbb{N}$$

- How the bulk anyons split depend on the Lagrangian algebra.

$$a = \sum_\alpha z_a^\alpha\, \alpha \Longrightarrow \bar{a} = \sum_\alpha z_a^\alpha\, \bar{\alpha}\,.$$

$$a = \sum_\alpha z_a^\alpha\, \alpha \Longrightarrow d_a = \sum_b z_a^\alpha\, d_\alpha\,.$$

$$a \otimes b = \bigoplus_c N_{a,b}^c\, c \Longrightarrow \left(\sum_\alpha z_a^\alpha\, \alpha\right) \times \left(\sum_\beta z_b^\beta\, \beta\right) = \sum_{c,\gamma} N_{a,b}^c\, z_c^\gamma\, \gamma\,.$$

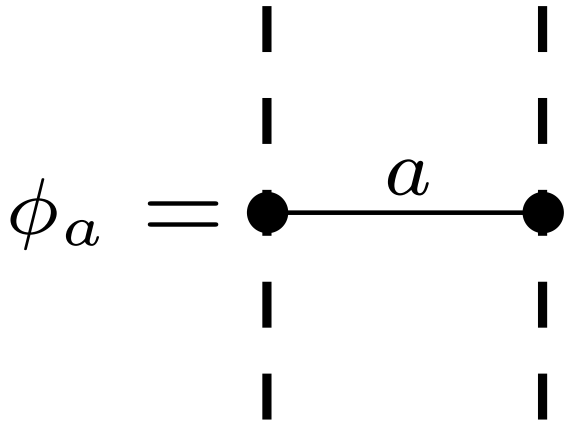
Simple anyons in the Lagrangian algebra $\mathscr{L} = \oplus_a a$ always have a component of the identity line of the boundary theory: $a \to \mathbf{1} + \cdots$

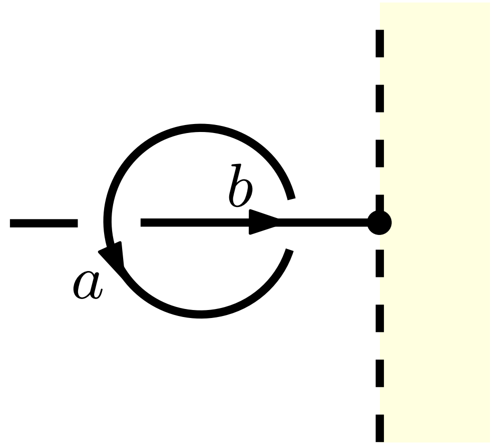
↓

Find fusion ring of the 2D boundary by these splitting rules

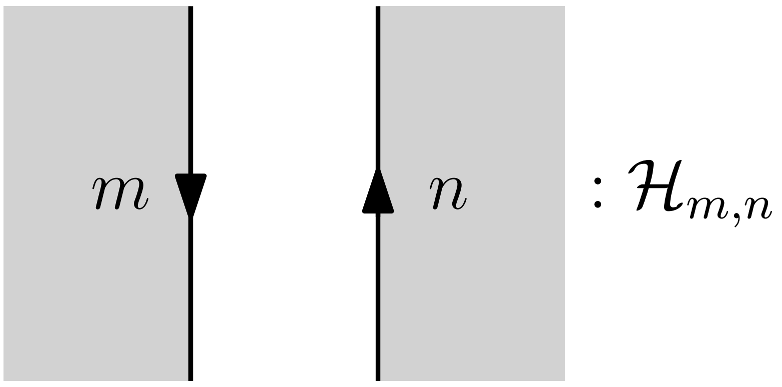
# Topological Boundary Conditions in 3D TQFTs

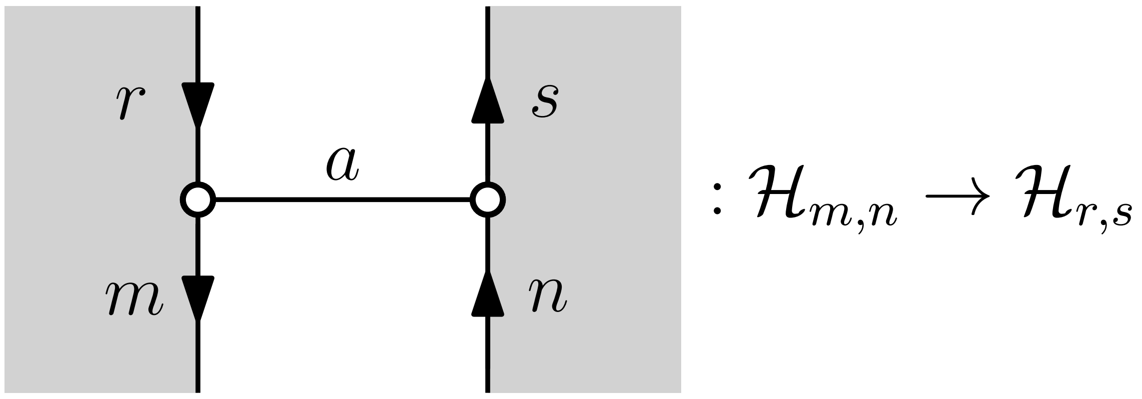
Reconstruct local operators of the 2D IR theory from the 3D bulk TQFT:




Once we know how the fusion category acts over the boundary conditions, we can ask how Hilbert spaces defined by different boundary conditions are related by the action of the non-invertible symmetry:




General Story
[C. Córdova, K. Ohmori, N. Holfester. (2408.11045)]

For our purposes, the degeneracies can be encoded in quiver diagrams:

- For each vacuum/topological local operator $m$, write a node in the quiver.
- Irreducible representations are labeled by the lines $a$ of the fusion category. Write $N^n_{ma}$ directed arrows from $m$ to $n$.

[S. Cecotti, C. Vafa. (9211097)]

# *Applications to Gapped 2D QCD*

# *Strategy for 2D QCD*

- Start with a gapped 2D QCD theory. This is, a QCD theory such that $c_{\mathrm{Spin}(\dim(\mathbf{R}))_1} - c_{G_{I(\mathbf{R})}} = 0.$

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In order to deal with finite excitations only, we can project by gauging any exact abelian one-form symmetry. This changes the bulk TQFT as

$$Z(\mathcal{F}) = \frac{\mathrm{Spin}(\dim(\mathbf{R}))_1 \times G_{-I(\mathbf{R})}}{\mathbb{Z}^{(1)}_{\mathrm{Spin}(\dim(\mathbf{R}))\cap G}}$$

Effectively, in the 2D theory this quotient shifts the global form of the gauge group $G$ to a non-simply connected version thereof.

# *Characterizing the vacua of 2D QCD*

Flow of local operators in 2D QCD
[D. Delmastro, J. Gomis. (2211.09036)]

Characterize the vacua of the theory by examining the expectation value of local operators in each vacua (order parameters):

$$\phi_a^{(UV)} \xrightarrow{g_{YM} \longrightarrow \infty} \phi_a^{(IR)}$$

$$\langle \phi_a \rangle_i = \langle \mathbf{v}_i | \phi_a | \mathbf{v}_i \rangle$$

Matrix of condensates
$$B_{ai} = \langle \mathbf{v}_i | \phi_a | \mathbf{v}_i \rangle$$

***SO(3)*** $+ \boldsymbol{\psi_5}$

$$\mathrm{B} = \begin{pmatrix} 1 & 1 & 1 \\ 2+\sqrt{3} & 2+\sqrt{3} & -1 \\ 3+\sqrt{3} & -3-\sqrt{3} & 0 \end{pmatrix}$$

***PSU(5)*** $+ \boldsymbol{\psi_{15}}$

$$\mathrm{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3+2\sqrt{2} & 3+2\sqrt{2} & -1 & -1 \\ 2+\sqrt{2} & -2-\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 2+\sqrt{2} & -2-\sqrt{2} & \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

# *Particle-Soliton Degeneracy in SO(3) + $\psi_5$*

## Concrete example: $SO(3)$ gauge theories with fermions in the $\mathbf{5}$

Coset TQFT: $\dfrac{Spin(5)_1 \times SU(2)_{-10}}{\mathbb{Z}_2^{(1)}}$

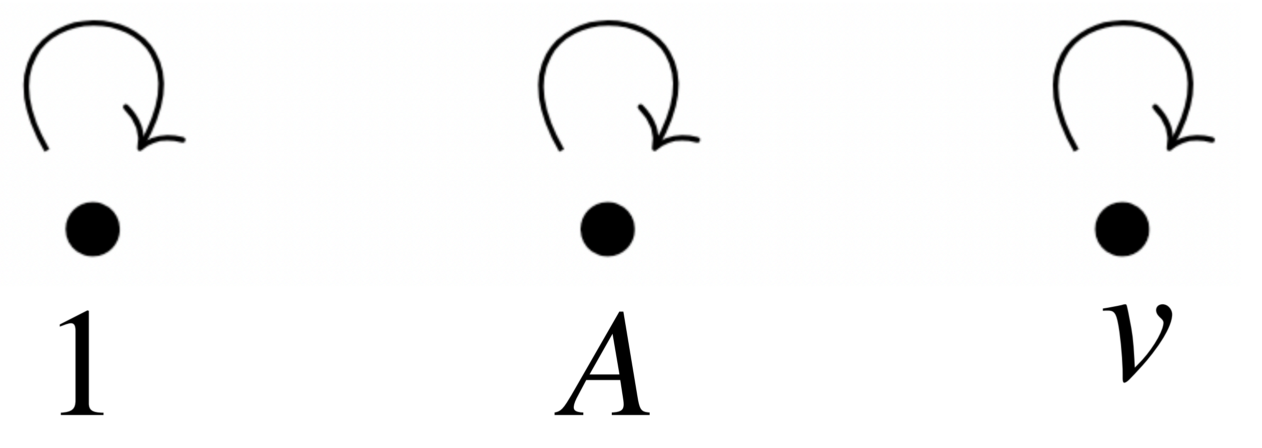
Fusion Category:
Three lines $\{1, v, A\}$ with fusion rules:

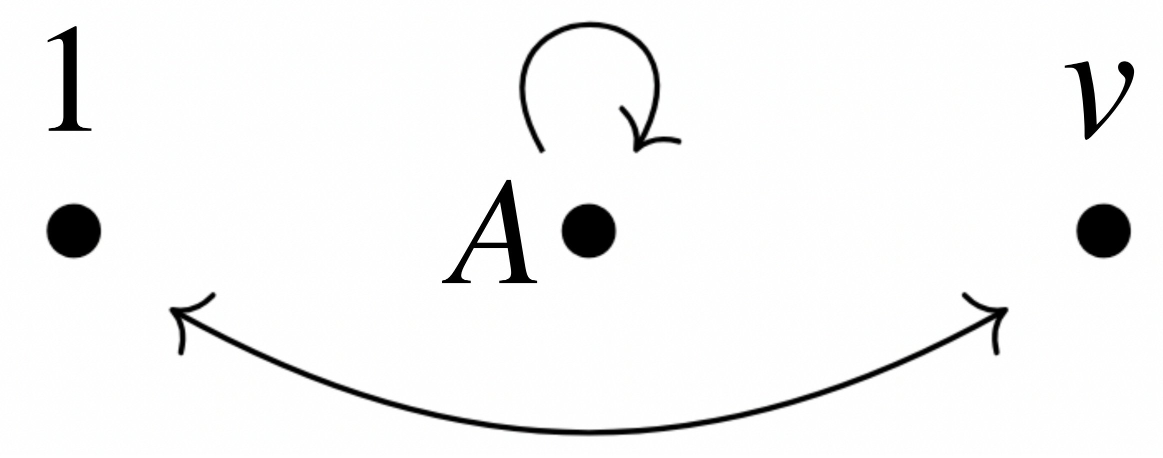
$v \times v = 1$

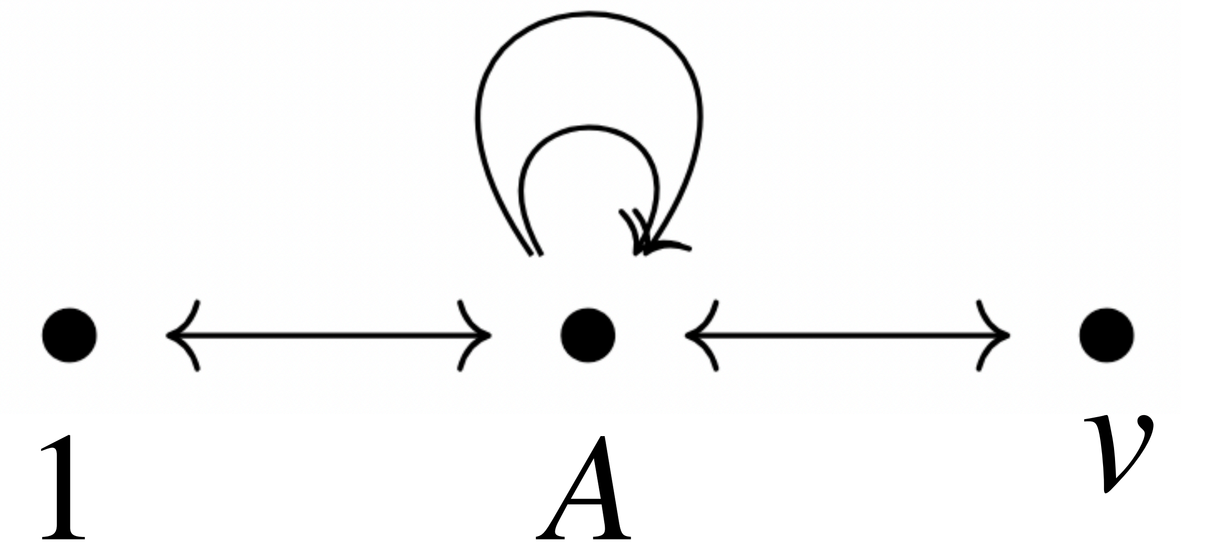
$v \times A = A \times v = A$

$A \times A = 1 + v + 2A$

Three vacua with three irreducible multiplets.



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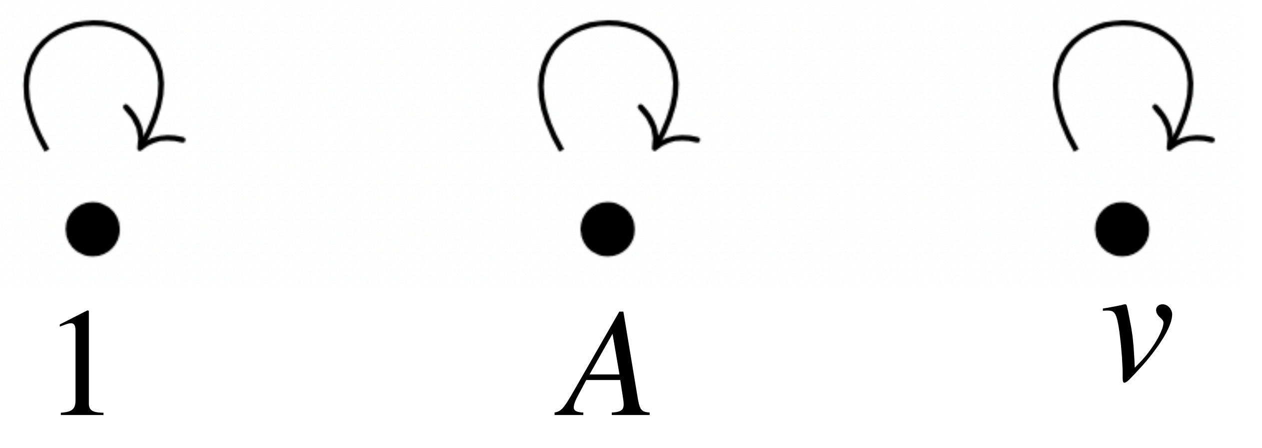
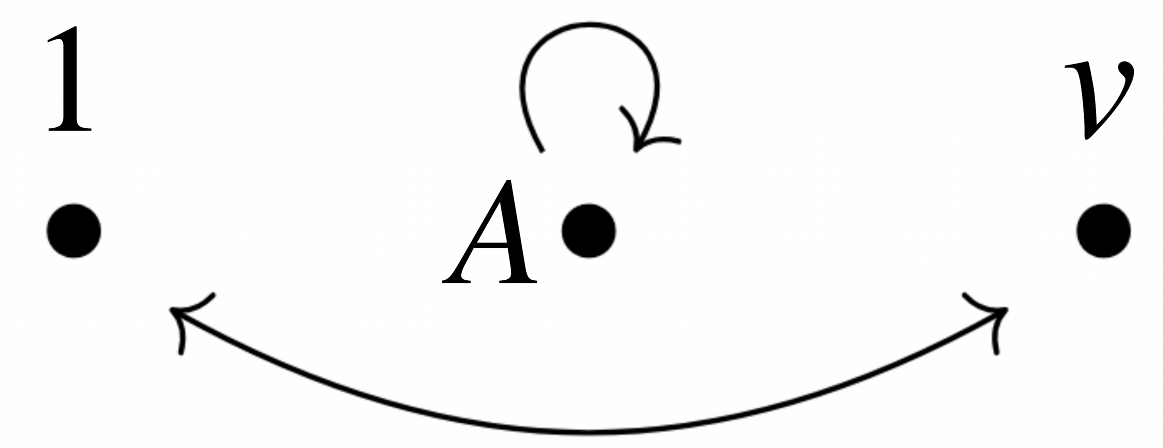
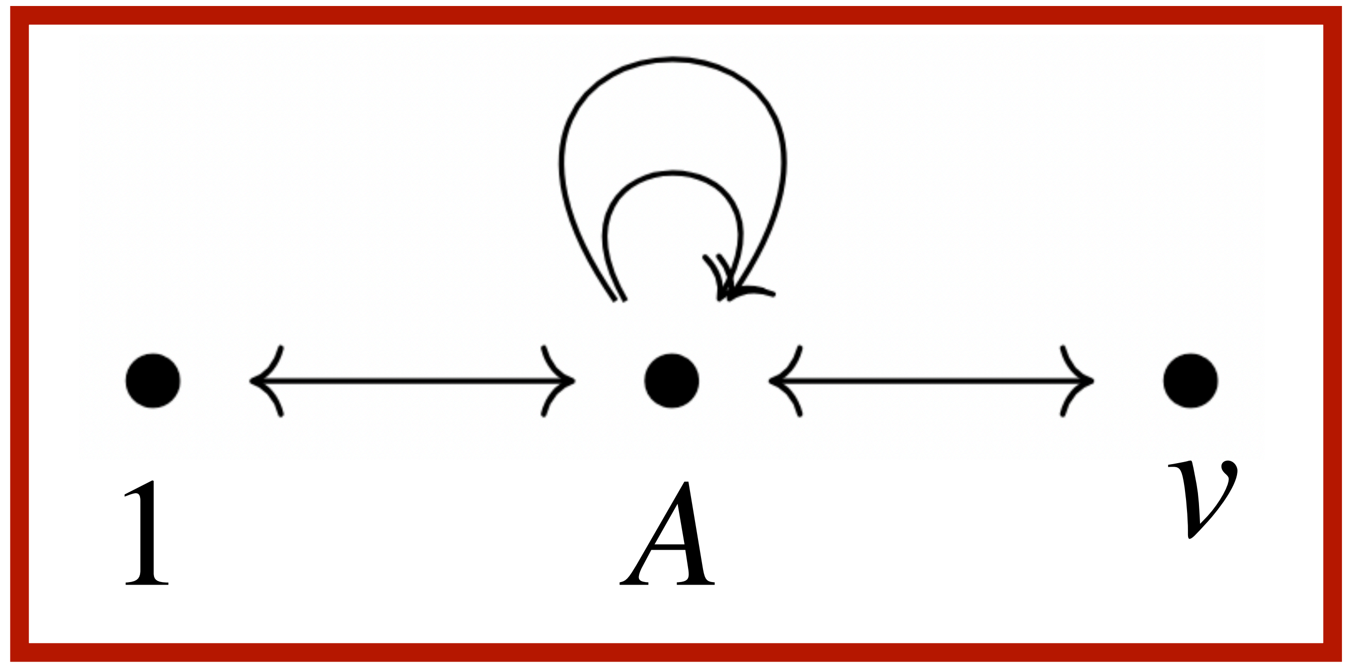
Three vacua with three irreducible multiplets.
Non-empty Hilbert space: States in the connected quiver must exist!



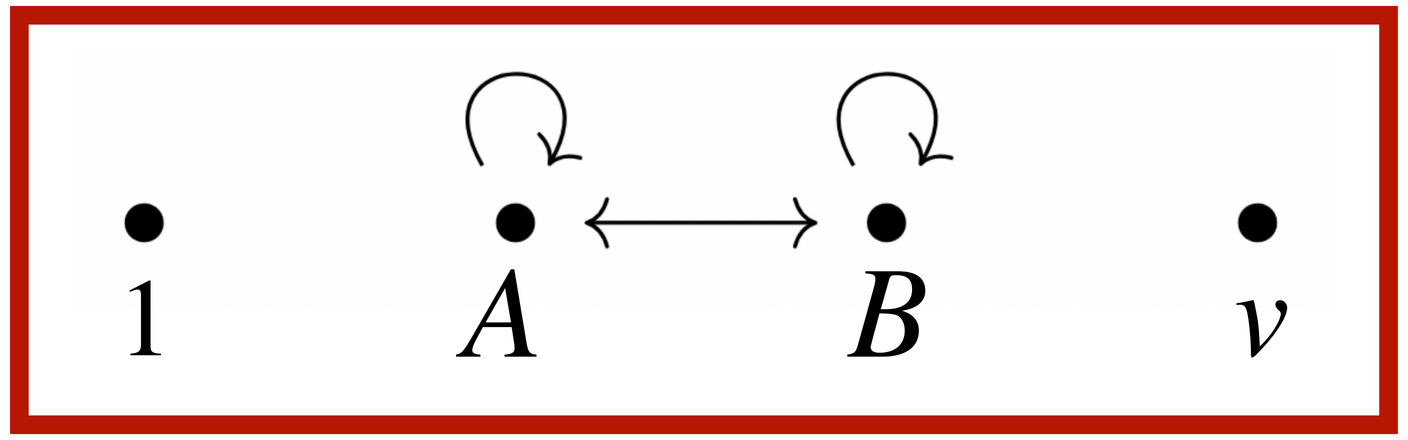
# *Particle-Soliton Degeneracy in PSU(4)* + $\psi_{\mathbf{15}}$

## Another example: *PSU*(4) gauge theories with fermions in the **15**

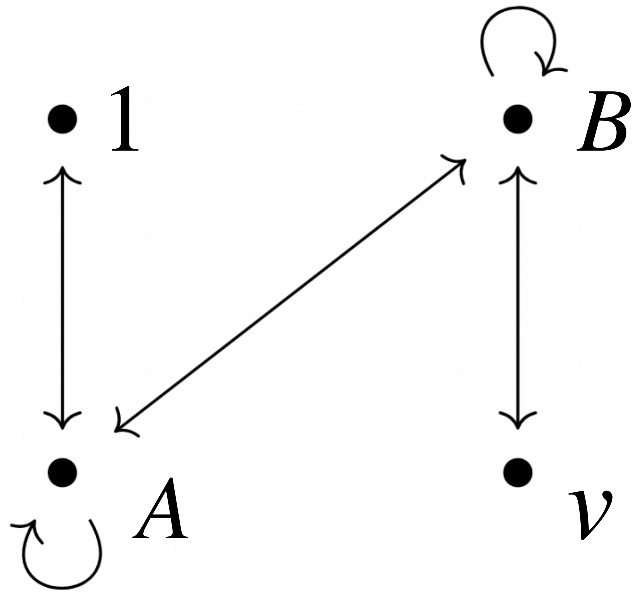
$$\text{Coset TQFT: } \frac{Spin(15)_1 \times SU(4)_{-4}}{\mathbb{Z}_4^{(1)}}$$

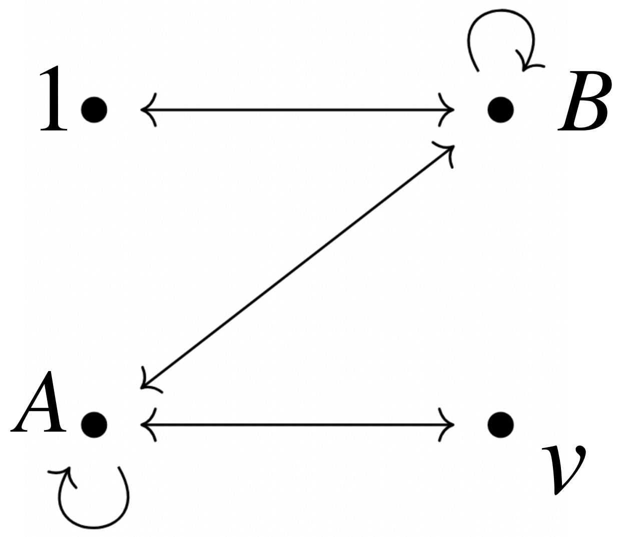
- Fusion Category: Four lines $\{1, v, A, B\}$ with fusion rules:

| $\times$ | $v$ | $A$ | $B$ |
|---|---|---|---|
| $v$ | $1$ | $B$ | $A$ |
| $A$ | $B$ | $1+A+B$ | $v+A+B$ |
| $B$ | $A$ | $v+A+B$ | $1+A+B$ |




Excitations are necessarily members of the multiplets:



# *Summary of Results and Future Directions*

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- Used non-invertible to constrain spectra of gapped 2D QCD theories.
- In particular, non-invertible symmetry implies novel mass degeneracies, e.g.  relating the mass of particles and solitons.
- We have seen the importance of appropriately taking into account topological sectors in 2D QFT to understand finite energy physics.
- Seen the practical use of anyon condensation ("gauging of non-invertible one-form symmetry") to derive specific fusion categories in 2D QFTs.

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- Analysis in higher dimensions. Domain Walls? Particle-Monopole Degeneracy?

# *Summary of Results and Future Directions*

- Extend analysis to other 2D gauge theories with non-invertible symmetry.
- Consequences over spectrum in the gapless case?
- Other applications of 2D topological cosets?
- Particle-Soliton Degeneracy in 2D QCD by direct calculation (e.g. lattice)?
- Infinite families of gapped 2D QCD theories (e.g. adjoint)?
- Constrain dynamics of 2D QCD with non-invertible symmetries.
  [C. Copetti, L. Cordova, S. Komatsu. (2403.04835)]
- Analysis in higher dimensions. Domain Walls? Particle-Monopole Degeneracy?

*Thanks for your attention! Questions?*