Generalized symmetries and gauge theory multiverses

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Recently there's been a lot of interest in "generalized symmetries" of QFT.

Today I will outline some basics aspects of those symmetries.

We'll see that sometimes, local theories with generalized symmetries
are equivalent to
disjoint unions of other local theories, known as "universes" in this context,
which gives rise to a notion of multiverses in gauge theories.

This is called **decomposition**, and explaining this will be the goal of this talk.

Let's begin with a quick review of (ordinary) symmetries in physics.

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Classical physics:

Recall a group defines a symmetry of a theory if the action S is invariant.

This leads to Noether's theorem, conserved currents.

Quantum mechanics:

Given a group G,

we can represent elements $g \in G$ by unitary operators $A(g) = \exp(iT(g))$,

such that $\quad A(g)A(h) = A(gh)$.

We say this is a symmetry if this commutes with the Hamiltonian, in the sense

$$A(g)\, H\, A(g)^{-1} \;=\; H \qquad\qquad \left[T(g), H\right] \;=\; 0$$

A simple common example: gauge transformations in electromagnetism

Here, physically, if A is the gauge field of electromagnetism,

then $A \sim A + d\alpha$

for α any function,

because both define the same electric fields $\overrightarrow{E}$ and magnetic fields $\overrightarrow{B}$.

The α describes an infinitesimal action of the group $U(1)$,

and since we're identifying fields related by that action,

we say that we *gauged* the symmetry.

Now, how can this be generalized?

One way is to generalize the groups appearing to `higher' groups.
A higher group is much like a group, except that some axioms are weakened.

Example: associativity

In a group, we require $g_1(g_2g_3) \; = \; (g_1g_2)g_3$

In a higher group, we instead merely require the existence of isomorphisms

$$\psi(g_1, g_2, g_3) : \; (g_1g_2)g_3 \xrightarrow{\sim} g_1(g_2g_3)$$

such that

$$\begin{tikzcd}
 & (g_1g_2)(g_3g_4) \arrow[rrd, "\psi(1,2,34)"] & & \\
((g_1g_2)g_3)g_4 \arrow[ru, "\psi(12,3,4)"] \arrow[d, "\psi(1,2,3)"'] & & & g_1(g_2(g_3g_4)) \\
(g_1(g_2g_3))g_4 \arrow[rr, "\psi(1,23,4)"'] & & g_1((g_2g_3)g_4) \arrow[ru, "\psi(2,3,4)"'] &
\end{tikzcd}$$

commutes.

Example: B fields

The B field is a two-form tensor potential $B = B_{\mu\nu} dx^\mu \wedge dx^\nu$ with a gauge invariance:

$$B \ \sim \ B + d\Lambda$$

where Λ is a $U(1)$ gauge field.

Here, the gauge transformation itself admits gauge transformations:

a gauge transformation by Λ
is equivalent to
a gauge transformation by $\Lambda + d\alpha$

As a result, gauge transformations can merely hope to be isomorphic to one another, so associativity only holds up to isomorphism.

(Fields with this & related towers of gauge transformations for gauge transformations are common in string theory.)

These structures may seem obscure,
but they've been known in various circles for a long time.

A few examples:

- In math, higher groups have been studied since at least the early '70s
- *B* fields and more general tensor field potentials arise in sugrav
- The String 2-group has been known in elliptic genus circles for many decades
- Two-dimensional gauge theories with trivially-acting subgroups

(Pantev, ES '06)

Recently, `generalized symmetries' have become very popular:
a 2014 paper of Gaiotto et al now has 1065 cites according to Inspire (4-28-24)....

I'll specialize to 2d gauge theory examples next,
as they'll provide prototypical examples of decomposition.

- Two-dimensional gauge theories with trivially-acting subgroups (Pantev, ES '06)

Example: Consider a theory of electromagnetism (a $U(1)$ gauge theory)
in which all of the matter fields (electrons, ...) have charges that are multiples of k
so that $\mathbb{Z}_k \subset U(1)$ acts trivially.

Technical point:

why is that different from a theory in which everything has charges that are multiples of 1 ?

Can't I just rescale all the charges ?

Answer: Perturbatively yes, but nonpert'ly that's only one option. (Pantev, ES '06)

Another option: Add heavy charge ± 1 fields, with masses above cutoff scale.

This certainly distinguishes cases.

In 2d, at low energies, their presence can be detected via θ angle periodicity.

Upshot: the difference is nonperturbative; are identical perturbatively.

- Two-dimensional gauge theories with trivially-acting subgroups (Pantev, ES '06)

Example: Consider a theory of electromagnetism (a $U(1)$ gauge theory)
in which all of the matter fields (electrons, ...) have charges that are multiples of k
so that $\mathbb{Z}_k \subset U(1)$ acts trivially.

This theory has a (generalized) symmetry,
that interchanges the bundles / instantons of the $U(1)$ gauge theory:

$$(U(1)\text{ bundle}) \mapsto (U(1)\text{ bundle}) \otimes (\mathbb{Z}_k\text{ bundle})$$

for *any* $\mathbb{Z}_k$ bundle

Formally: $F \mapsto F + \tilde{F}$

Because the subgroup $\mathbb{Z}_k \subset U(1)$ acts trivially on all matter,
the action S weighting these contributions is the same under the replacement above.

So, this is some kind of symmetry,
interchanging the nonperturbative contributions....

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Formally: $F \mapsto F + \tilde{F}$

Because the subgroup $\mathbb{Z}_k \subset U(1)$ acts trivially on all matter,
the action S weighting these contributions is the same under the replacement above.

An action, *not* of an element of $\mathbb{Z}_k$, but rather a $\mathbb{Z}_k$ *bundle,*
that interchanges nonperturbative contributions to the QFT.

So, it's a symmetry,
and because the symmetry parameters themselves have gauge transformations,
associativity etc only hold up to isomorphism.

This is a generalized symmetry, denoted $B\mathbb{Z}_k$ or $\mathbb{Z}_k^{(1)}$

So far, we've seen that a gauge theory with a trivially-acting subgroup has a generalized symmetry, that interchanges instanton sectors.

Let's try to characterize such symmetries more precisely….

One way to think about these symmetries is in terms of operators.

Noether's theorem:

Consider an ordinary global symmetry.

Under an infinitesimal symmetry transformation parametrized by α,

$$S \mapsto S + \int (d\alpha) \wedge j$$

(Hodge dual of typical description)

where j is a $(d-1)$-form (Hodge dual of Noether current),

which obeys $dj = 0$ (conservation law).

Formally, we can associate an operator

$$U_\alpha = \exp\left(\int_{M_{d-1}} j \right)$$

supported along a submfld M_{d-1} of dim $d-1$.

It's invariant under deformations of M_{d-1}, b/c of conservation law $dj = 0$.

"topological operator"

That picture can be generalized. Consider a symmetry parametrized by a p-form α.

$$S \mapsto S + \int_M (d\alpha) \wedge j$$

where j is a $(d-p-1)$-form, obeying $dj = 0$ (conservation law).

We can associate an operator

$$U_\alpha = \exp\left(\int_{M_{d-p-1}} j \right)$$

supported along a submanifold M_{d-p-1} of dim $d-p-1$.

It's invariant under deformations of M_{d-p-1}, b/c of conservation law $dj = 0$.

We call this a *p-form symmetry*.

Ordinary symmetries are 0-form symmetries.

Gauge theory with trivially-acting subgroup has a 1-form symmetry (BK).

These generalized symmetries can sometimes have exotic effects.....

In $d > 1$ spacetime dimensions,

if a local quantum field theory has a global $(d-1)$-form symmetry,
it is equivalent to a disjoint union of other local QFT's,
known in this context as `universes.'

We call this **decomposition**.

(2d: Hellerman et al '06, ...;
d>2: Tanizaki-Unsal '19, Cherman-Jacobson '20, ...)

When this happens, we say the QFT `decomposes.'
Decomposition of the QFT can be applied to give insight
into its properties, which I will explore in this talk.

In $d > 1$ spacetime dimensions,
if a local quantum field theory has a global $(d-1)$-form symmetry,
it is equivalent to a disjoint union of other local QFT's,
known in this context as `universes.'

Prototypical example:

A two-dimensional G-gauge theory
with trivially-acting central subgroup $K \subset G$
is equivalent to
a disjoint union of $|K|$ copies of (G/K) gauge theories,
each with a possibly different (discrete) theta angle.

$$G\text{-gauge theory} = \coprod_{|K|} (G/K\text{-gauge theory})_\theta$$

(the *universes* of the decomposition)

Why is the existence of decomposition surprising?

To explain, let me distinguish a *sum* of QFTs from a *product* of QFTs.

Consider two QFTs with path integrals: $Z(T_1) = \int [D\phi_1] \exp(-S_1)$, $Z(T_2) = \int [D\phi_2] \exp(-S_2)$

In a product of QFTs, we *multiply* partition functions:

$$Z(T_1 \otimes T_2) = Z(T_1)\, Z(T_2) = \int [D\phi_1][D\phi_2] \exp(-S_1 - S_2)$$

There always exists a local action for a product. Here, it's $S_1 + S_2$

In a sum of QFTs, we *add* partition functions: (connected spacetime)

$$Z(T_1 \coprod T_2) = Z(T_1) + Z(T_2) = \int [D\phi_1] \exp(-S_1) + \int [D\phi_2] \exp(-S_2)$$

Ordinarily, no way to write this in the form $\int [D\phi_1][D\phi_2] \exp(-S)$ for some S: $\log(x+y) \neq (\log x) + (\log y)$

But that's exactly what happens in decomposition!

What does it mean for one local QFT to be a sum of other local QFTs?

(Hellerman et al '06)

1) Existence of projection operators

The theory contains topological operators Π_i such that

$$\Pi_i \Pi_j = \delta_{i,j} \Pi_j \qquad \sum_i \Pi_i = 1 \qquad [\Pi_i, \mathcal{O}] = 0$$

Operators Π_i simultaneously diagonalizable; state space = $\mathcal{H} = \oplus_i \mathcal{H}_i$

In the language of extended objects / defects from earlier,
a $p = (d-1)$-form symmetry in d dimensions has operators supported along
submanifolds of dimension $d - p - 1$, which here $= d - (d-1) - 1 = 0$.
These are the projectors Π_i above.

In the case of gauge theories w/ triv' acting subgroups, because the action is trivial,
the operators commute with everything — hence diagonalize the state space.

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Operators Π_i simultaneously diagonalizable; state space = $\mathcal{H} = \oplus_i \mathcal{H}_i$

Correlation functions:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \Pi_i \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle (\Pi_i \mathcal{O}_1) \cdots (\Pi_i \mathcal{O}_m) \rangle = \sum_i \langle \tilde{\mathcal{O}}_1 \cdots \tilde{\mathcal{O}}_m \rangle_i$$

2) Partition functions decompose

$$Z = \sum_{\text{states}} \exp(-\beta H) = \sum_i \sum \exp(-\beta H_i) = \sum_i Z_i$$

(on a connected spacetime)

There are many examples of decomposition !

Finite gauge theories in 2d (orbifolds):

(T Pantev, ES '05; D Robbins, ES, T Vandermeulen '21)

Common thread: a subgroup of the gauge group acts trivially.

Example: If $K \subset \text{center}(\Gamma) \subset \Gamma$ acts trivially, then $[X/\Gamma] = \coprod_{\text{irreps } K} [X/(\Gamma/K)]_{\hat{\omega}}$

Gauge theories:

- 2d $U(1)$ gauge theory with nonmin' charges = sum of $U(1)$ theories w/ min charges (Hellerman et al '06)

  Ex: charge p Schwinger model

- 2d G gauge theory w/ center-invt matter = sum of $G/Z(G)$ theories w/ discrete theta (ES '14)

  Ex: $SU(2)$ theory (w/ center-invt matter) = $SO(3)_+ \coprod SO(3)_-$ (w/ same matter)

- 2d pure G Yang-Mills = sum of trivial QFTs indexed by irreps of G (Nguyen, Tanizaki, Unsal '21) (U(1): Cherman, Jacobson '20)

  Ex: pure $SU(2) = \coprod_{\text{irreps } SU(2)}$ (sigma model on pt)

There are also higher-dimensional examples....

There are many examples of decomposition !

More examples :

- 3d Chern-Simons theory with gauged noneffectively-acting 1-form symmetry (Pantev, ES '22)

  = disjoint union of ordinary Chern-Simons theories

  (On the boundary, this reduces to 2d decomposition.)

- 3d orbifold by finite noneffectively-acting 2-group (Pantev, Robbins, ES, Vandermeulen '22; Perez-Lona, ES '23)

  = disjoint union of ordinary 3d orbifolds

  (Example: Yetter model vs union of Dijkgraaf-Witten theories)

- 4d Yang-Mills w/ restriction to instantons of deg' divisible by k (Tanizaki, Unsal '19)

  = disjoint union of ordinary 4d Yang-Mills w/ different θ angles

More examples ….

There are many examples of decomposition !

More examples :

TFTs: 2d unitary TFTs w/ semisimple local operator algebras decompose to invertibles

Examples:

(Implicit in Durhuus, Jonsson '93; Moore, Segal '06)

(Also: Komargodski et al '20, Huang et al 2110.02958)

- 2d abelian BF theory at level k = disjoint union of k invertibles (sigma models on pts)

(Hellerman, ES, 1012.5999)

- 2d G/G model at level k = disjoint union of invertible theories as many as integrable reps of the Kac-Moody algebra

(Komargodski et al 2008.07567)

- 2d Dijkgraaf-Witten = sum of invertible theories, as many as irreps (In fact, is a special case of finite gauge theories already mentioned.)

Sigma models on gerbes = disjoint union of sigma models on spaces w/ B fields

Solves tech issue w/ cluster decomposition.

(T Pantev, ES '05)

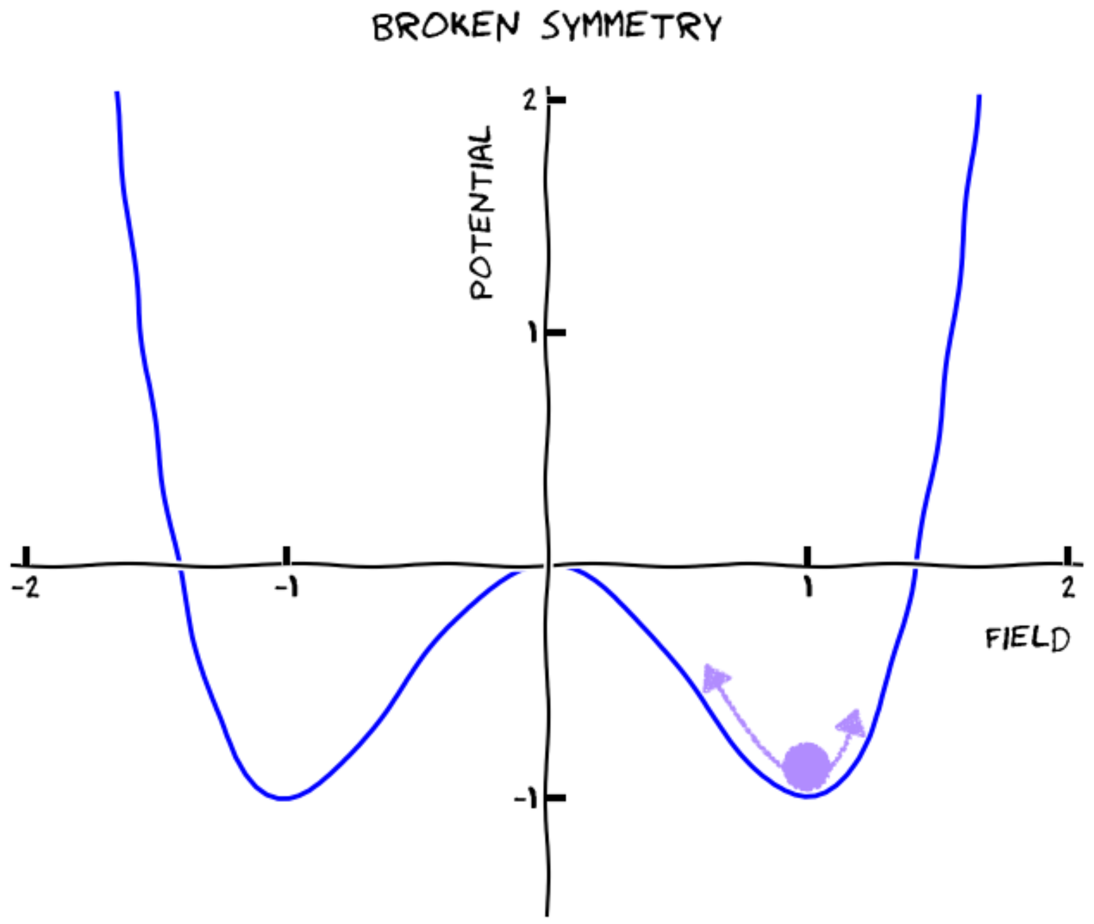
Decomposition $\neq$ spontaneous symmetry breaking

SSB:

Superselection sectors:

- separated by dynamical domain walls
- only genuinely disjoint in IR
- only one overall QFT

Prototype:

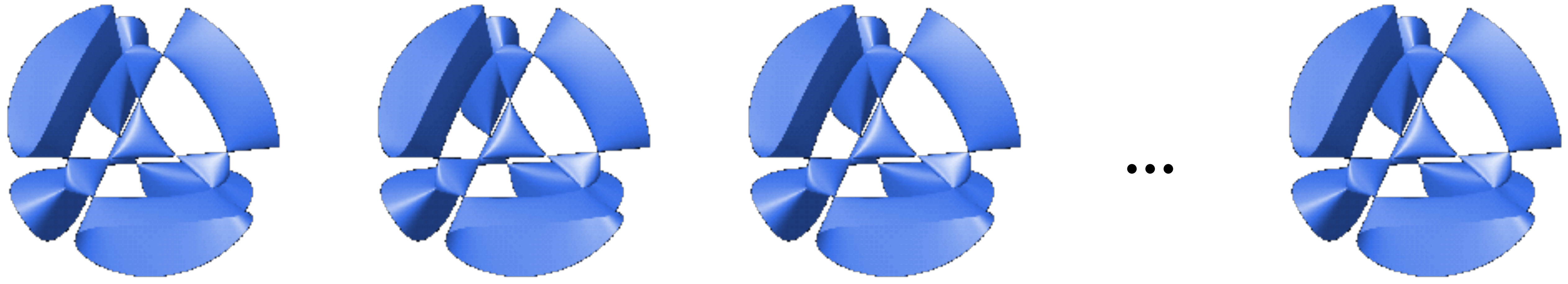


Decomposition:

Universes:

- separated by *non*dynamical domain walls
- disjoint at *all* energy scales
- *multiple* different QFTs present

Prototype:



(see e.g. Tanizaki-Unsal 1912.01033)

Since 2005, decomposition has been checked in many examples in many ways. Examples:

- GLSM's: mirrors, quantum cohomology rings (Coulomb branch) (T Pantev, ES '05; Gu et al '18-'20)
- Orbifolds: partition f'ns, massless spectra, elliptic genera (T Pantev, ES '05; Robbins et al '21)
- Open strings, K theory (Hellerman et al hep-th/0606034)
- Susy gauge theories w/ localization (ES 1404.3986)
- Nonsusy pure Yang-Mills ala Migdal (ES '14; Nguyen, Tanizaki, Unsal '21)
- Adjoint QCD_2 (Komargodski et al '20)
- Numerical checks (lattice gauge thy) (Honda et al '21)
- Versions in d-dim'l theories w/ (d-1)-form symmetries (Tanizaki, Unsal, '19; Cherman, Jacobson '20)

This list is incomplete; apologies to those not listed.

Applications include:

- Sigma models with target stacks & gerbes (T Pantev, ES '05)
- Predictions for Gromov-Witten theory (checked by H-H Tseng, Y Jiang, E Andreini, etc starting '08)
- Nonperturbative constructions of geometries in GLSMs (Caldararu et al 0709.3855, Hori '11, ..., Romo et al '21)
- Elliptic genera (Eager et al '20)
- Anomalies in orbifolds (Robbins et al '21)

So far:

In d spacetime dimensions,
a theory decomposes when it has a global $(d-1)$-form symmetry.

This has been checked in many ways since 2005,
and there are lots of examples of decomposition in practice.

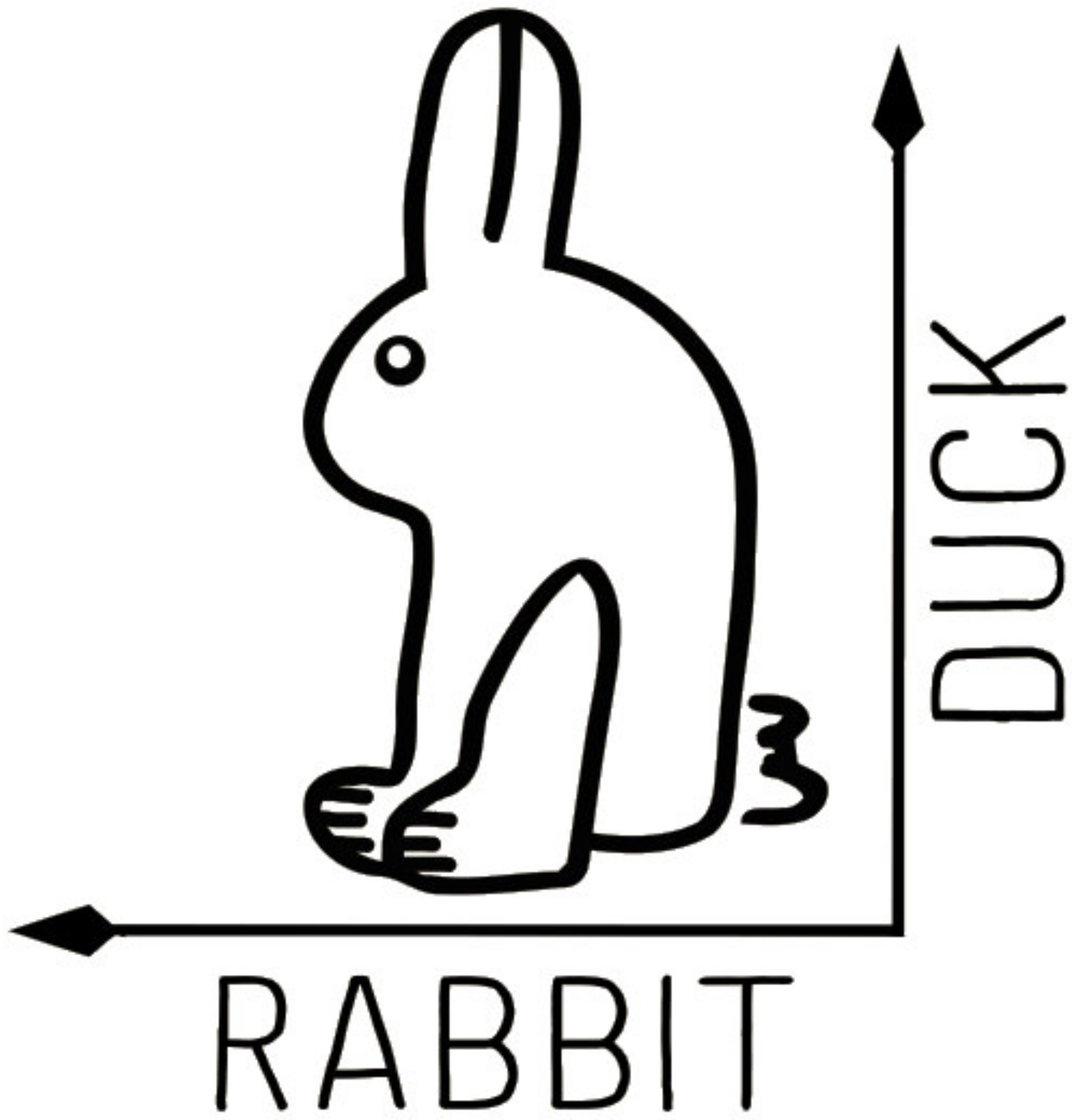
Next, I'll focus on one particular family of examples:
2d gauge theories with trivially-acting subgroups

Decomposition in 2d gauge theories

(Hellerman et al '06)

S'pose have G-gauge theory, G semisimple, with finite central $K \subset G$ acting trivially.
As discussed previously, has 1-form symmetry (specifically, BK).

So far, this sounds like just one QFT.



However, I'll outline how, from another perspective,
QFTs of this form are also each
a disjoint union of other QFTs;
they "decompose."

Decomposition in 2d gauge theories

(Hellerman et al '06)

S'pose have G-gauge theory, G semisimple, with finite central $K \subset G$ acting trivially.

As discussed previously, has 1-form symmetry (specifically, BK).

Claim this theory decomposes.

Where are the projection operators?

Math understanding:

Briefly, the projection operators (twist fields, Gukov-Witten) correspond to elements of the center of the group algebra $\mathbb{C}[K]$.

Existence of those projectors (idempotents), forming a basis for the center, is ultimately a consequence of Wedderburn's theorem.

Universes $\longleftrightarrow$ Irreducible representations of K

Partition functions & relation of decomp' to restrictions on instantons....

Decomposition in 2d gauge theories

(Hellerman et al '06)

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As discussed previously, has 1-form symmetry (specifically, BK).

Statement of decomposition (in this example):

$$\mathrm{QFT}(G-\text{gauge theory}) = \coprod_{\text{char's } \hat{K}} \mathrm{QFT}\left(G/K-\text{gauge theory w/ discrete theta angles}\right)$$

Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where $\pm$ denote discrete theta angles (w_2)

Perturbatively, the $SU(2)$, $SO(3)_\pm$ theories are identical

— differences are all nonperturbative.

Decomposition in 2d gauge theories

(Hellerman et al '06)

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where $\pm$ denote discrete theta angles (w_2)

$SU(2)$ instantons (bundles) $\subset$ $SO(3)$ instantons (bundles)

The discrete theta angles weight the non-$SU(2)$ $SO(3)$ instantons so as to cancel out of the partition function of the disjoint union.

Summing over the $SO(3)$ theories projects out some instantons, giving the $SU(2)$ theory.

Decomposition in 2d gauge theories

(Hellerman et al '06)

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$$\mathrm{QFT}(G-\text{gauge theory}) \;=\; \coprod_{\text{char's } \hat{K}} \mathrm{QFT}\left(G/K-\text{gauge theory w/ discrete theta angles}\right)$$

Formally, the partition function of the disjoint union can be written

$$Z \;=\; \underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} \;=\; \int [DA] \exp(-S) \overbrace{\left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)}^{\text{projection operator}}$$

where we have moved the summation inside the integral.

This is an interference effect between universes: **multiverse interference**

Decomposition in 2d gauge theories

(Hellerman et al '06)

projection operator

$$Z = \underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \left(\overline{\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right]} \right)$$

Decomposition in 2d gauge theories

(Hellerman et al ’06)

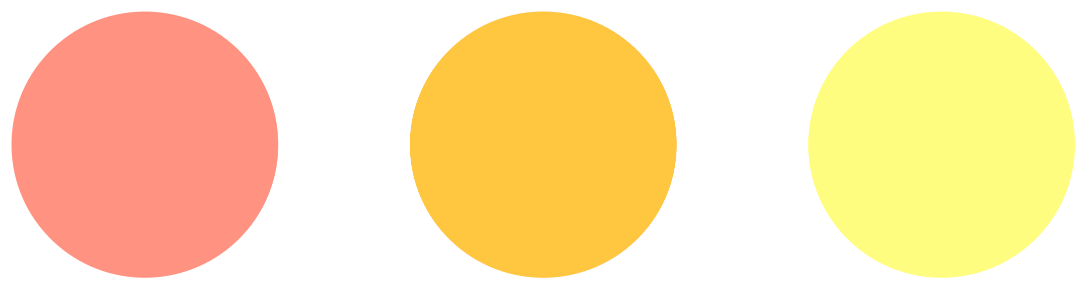
One effect is a projection on nonperturbative sectors:

$$\underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \left(\overbrace{\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right]}^{\text{projection operator}} \right)$$

Disjoint union of several QFTs / universes = `One’ QFT with a restriction on nonperturbative sectors = `multiverse interference’

Schematically,
two theories combine to form a distinct third:

universe ($SO(3)_+$)



universe ($SO(3)_-$)

multiverse interference effect ($SU(2)$)

Before going on, let's quickly check these claims for pure $SU(2)$ Yang-Mills in 2d.

The partition function Z, on a Riemann surface of genus g, is

(Migdal, Rusakov)

$$Z(SU(2)) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SU(2) reps

$$Z(SO(3)_+) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SO(3) reps

(Tachikawa '13)

$$Z(SO(3)_-) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SU(2) reps

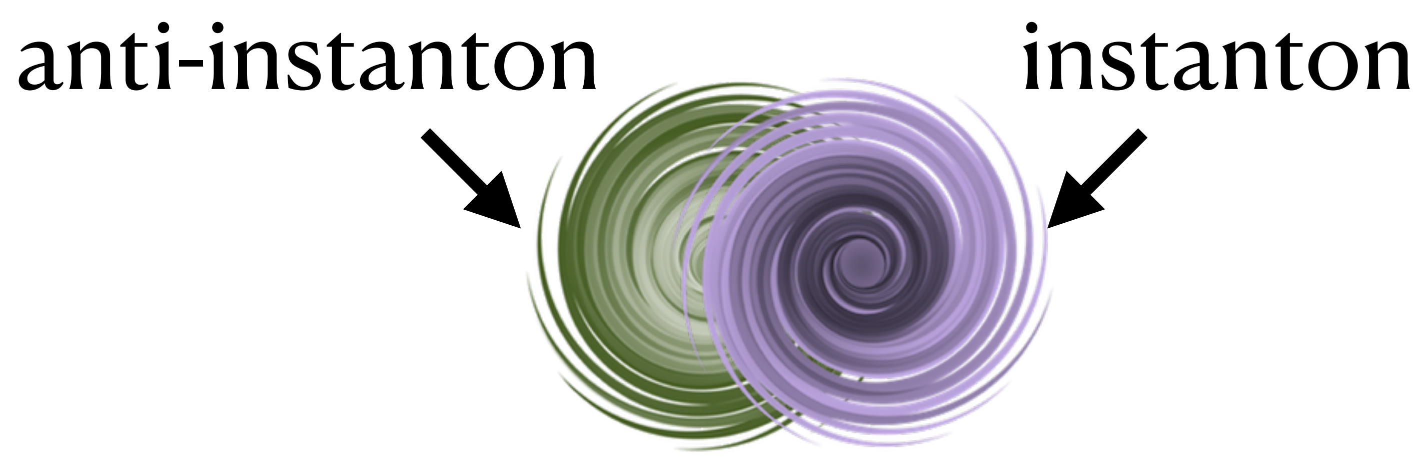
that are not SO(3) reps

Result: $Z(SU(2)) = Z(SO(3)_+) + Z(SO(3)_-)$ as expected.

Suppose we try to require that the total instanton number always vanish in our QFT.

Start with a field configuration with no net instantons.

Now, move them far away from one another:



Nonzero instanton number here!

Total instanton number : 0

Nonzero instanton number here!

If physics is local ("cluster decomposition"),
then in those widely-separated regions, the theories have instantons.
So, even if we start with no net instantons,
cluster decomposition implies we get instantons!

Cluster decomposition:

For this reason, Steven Weinberg taught us:

All local quantum field theories must sum over all instantons, so as to preserve cluster decomposition.

Loophole: Disjoint unions of QFTs also violate cluster decomposition (ex: multiple dimension zero operators), but in principle are straightforward to deal with.

So, if a theory with a restriction on instantons is also a disjoint union, of theories which are well-behaved, then all is OK.

Recap:

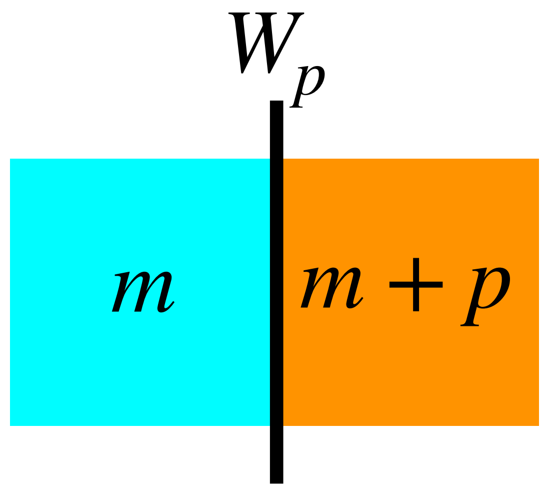
So far we have discussed, in a simple set of examples,
the form of decomposition,
and how it explains restrictions on instantons —
— as a multiverse interference effect.

What if one has a Wilson line that is charged under the trivially-acting $K \subset G$?

Such Wilson lines are defects linking different universes.

Here's an easy example in a different context:

Ex: 2d abelian BF theory at level k



Projectors: $\Pi_m = \frac{1}{k} \sum_{n=0}^{k-1} \xi^{nm} \mathcal{O}_n \qquad \xi = \exp(2\pi i/k)$

Clock-shift commutation relations: $\mathcal{O}_p W_q = \xi^{pq} W_q \mathcal{O}_p \qquad \Leftrightarrow \qquad \Pi_m W_p = W_p \Pi_{m+p \mod k}$

Decomposition has been checked in many ways,
including, for example, gauge duals & mirrors.

In such a dual, the nonperturbative physics of the original theory becomes perturbative in the dual theory, and so one can see decomposition perturbatively.

Example: susy $\mathbb{CP}^N$ model

The susy $\mathbb{CP}^N$ model is a 2d susy $U(1)$ gauge theory, with $N+1$ (chiral super)fields each of charge $+1$.

Semiclassically, the Higgs moduli space is $\mathbb{CP}^N$, thus the name.

The mirror to this theory is a susy Landau-Ginzburg model with superpotential

$$W = \exp(-Y_1) + \exp(-Y_2) + \cdots + \exp(-Y_{N-1}) + q \exp(+Y_1 + Y_2 + \cdots + Y_{N-1})$$

The mirror encodes the nonperturbative physics of the original theory (eg instantons) as classical / perturbative physics in the mirror.

Decomposition?

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$$W \;=\; \exp(-Y_1) + \exp(-Y_2) + \cdots + \exp(-Y_{N-1}) \;+\; q \exp(+Y_1 + Y_2 + \cdots + Y_{N-1})$$

Example: gerby susy $\mathbb{CP}^N$ model

Consider a 2d susy $U(1)$ gauge theory with $N+1$ chiral superfields of charge $k > 1$.
Has $B\mathbb{Z}_k$ one-form symmetry, decomposes into k copies of $\mathbb{CP}^N$ model.

Mirror was computed using methods of (Hori, Vafa '00) in (Pantev, ES, '06); result:

$$W \;=\; \exp(-Y_1) + \exp(-Y_2) + \cdots + \exp(-Y_{N-1}) \;+\; q\Upsilon \exp(+Y_1 + Y_2 + \cdots + Y_{N-1})$$

where Υ is a $\mathbb{Z}_k$-valued field.

Path integral sum over values of Υ = disjoint union, perturbatively.

Example: gerby susy $\mathbb{CP}^N$ model

Consider a 2d susy $U(1)$ gauge theory with $N+1$ chiral superfields of charge $k > 1$.

Has $B\mathbb{Z}_k$ one-form symmetry, decomposes into k copies of $\mathbb{CP}^N$ model.

Mirror was computed using methods of (Hori, Vafa '00) in (Pantev, ES, '06); result:

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where Υ is a $\mathbb{Z}_k$-valued field.

Path integral sum over values of Υ = disjoint union, perturbatively.

In passing:

Ordinarily I describe decomposition in terms of universes with variable θ angles or B fields — complex Kahler parameters.

In the mirror, these become complex structure parameters.

Another example: 4d Yang-Mills with a restriction on instantons

(Tanizaki-Unsal '19)

Start with an ordinary 4d Yang-Mills theory

and add a scalar field B of periodicity 2π and a 3-form potential $C^{(3)}$:

$$S = \frac{1}{2g^2}\int \mathrm{Tr} F \wedge {}^*F \; + i\int B\left(\frac{1}{8\pi^2}\mathrm{Tr} F \wedge F - \frac{k}{2\pi}H^{(4)}\right)$$

where locally $H^{(4)} = dC^{(3)}$, and k is an integer.

EOM for B: $\quad \frac{1}{8\pi^2}\mathrm{Tr} F \wedge F = \frac{k}{2\pi}H^{(4)}$ $\quad$ so the instanton number is divisible by k

As discussed, restrictions on instantons violate cluster decomposition, but note....

there is a global 3-form symmetry: $C^{(3)} \mapsto C^{(3)} + \Lambda^{(3)}$ for $d\Lambda^{(3)} = 0$

so we expect a decomposition....

Another example: 4d Yang-Mills with a restriction on instantons

(Tanizaki-Unsal '19)

$$S = \frac{1}{2g^2}\int \mathrm{Tr} F\wedge *F \quad + i\int B\left(\frac{1}{8\pi^2}\mathrm{Tr} F\wedge F - \frac{k}{2\pi}H^{(4)}\right)$$

EOM for B: $\quad \frac{1}{8\pi^2}\mathrm{Tr} F\wedge F = \frac{k}{2\pi}H^{(4)}$ $\quad$ so the instanton number is divisible by k

Global 3-form symmetry: $C^{(3)} \mapsto C^{(3)} + \Lambda^{(3)}$ for $d\Lambda^{(3)} = 0$

EOM for $C^{(3)}$: $\quad dB = 0$ $\quad$ so B is constant. (In fact, $B = 2\pi m/k$ for m an integer.)

Next, integrate out B and $C^{(3)}$....

Since B can take finitely-many values, the path integral must sum over those values.

Result: on a connected spacetime,

$$Z = \sum_{m=0}^{k-1}\int [DA]\exp\left[-\left(\frac{1}{2g^2}\int \mathrm{Tr} F\wedge *F + \frac{i}{8\pi^2}\frac{2\pi m}{k}\int \mathrm{Tr} F\wedge F\right)\right]$$

Another example: 4d Yang-Mills with a restriction on instantons

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This is the partition function of a decomposition,
a sum over k universes, each with a shifted theta angle.

Recap: Started with a 4d theory with restriction on instantons,
whose construction had global 3-form symmetry,
and we've discovered a decomposition.

Another example: Noninvertible symmetries

(A Perez-Lona, D Robbins, ES,
T Vandermeulen, X Yu, 2311.16230 & to appear)

There also exist noninvertible symmetries, which sometimes can be gauged.
Consider a 2d theory with a $\text{Rep}(S_3) = \{1, X, Y\}$ symmetry,
and a trivially-acting $\text{Rep}(\mathbb{Z}_2)$ subsymmetry.

$$1 \longrightarrow \mathbb{Z}_3 \longrightarrow S_3 \longrightarrow \mathbb{Z}_2 \longrightarrow 1 \qquad \text{Rep}(\mathbb{Z}_2) \longrightarrow \text{Rep}(S_3) \longrightarrow \text{Rep}(\mathbb{Z}_3)$$

Results:

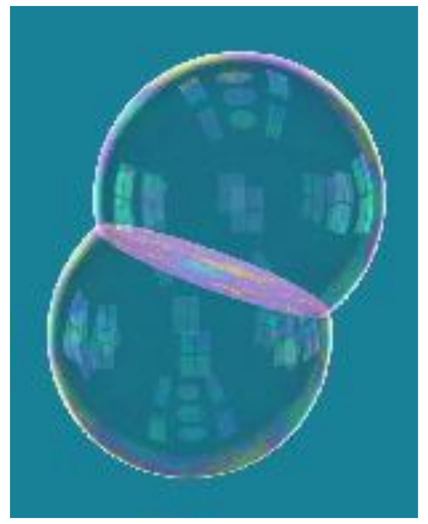
$$\left[\mathcal{T}/(1+X)\right] = \coprod_2 \mathcal{T}$$

$$\left[\mathcal{T}/(1+Y)\right] \cong \left[\mathcal{T}/\text{Rep}(\mathbb{Z}_3)\right]$$

$$\left[\mathcal{T}/\text{Rep}(S_3)\right] \cong \coprod_2 \left[\mathcal{T}/\text{Rep}(\mathbb{Z}_3)\right]$$

Application: GW invariants

The Gromov-Witten (GW) invariants count minimal-area surfaces in a given space, and form the instantons of 2d sigma models.



There exists a def'n of GW invariants of generalizations of spaces called gerbes.

(Chen, Ruan; Abramovitch, Graber, Vistoli ~2000)

Gerbes have 1-form symmetries geometrically;
a 2d sigma model with target a gerbe has a 1-form symmetry.

Decomposition predicts,
GW invariants of a gerbe = sum of GW invariants of universes

Checked by (H-H Tseng, Y Jiang, et al '08 on)

Application: Quantum K theory

We can make a similar prediction for quantum K theory,
which arises as OPE's of Wilson lines in 3d gauge theories.

Consider 3d gauge theories with 1-form symmetries (re: QK on gerbes).
The whole theory doesn't decompose,
but parallel Wilson lines are described by an effectively 2d theory,
and that 2d theory decomposes.

In addition, the same 1-form symmetry generates a multiplicity of Wilson lines.

Result: If the 3d gauge theory has a $B\mathbb{Z}_k$ one-form symmetry,
then the QK ring is isomorphic to that of a disjoint union of
k^2 copies of an underlying space, not just k.

So: Decomposition^2 !

(Gu, du Pei, Zhang 2105.02247)

Application: GLSMs

(Caldararu et al '07)

Consider the GLSM for e.g. $\mathbb{P}^3[2,2] = T^2$.

This is a $U(1)$ gauge theory, with ϕ_i charge $+1$, p_a charge -2.

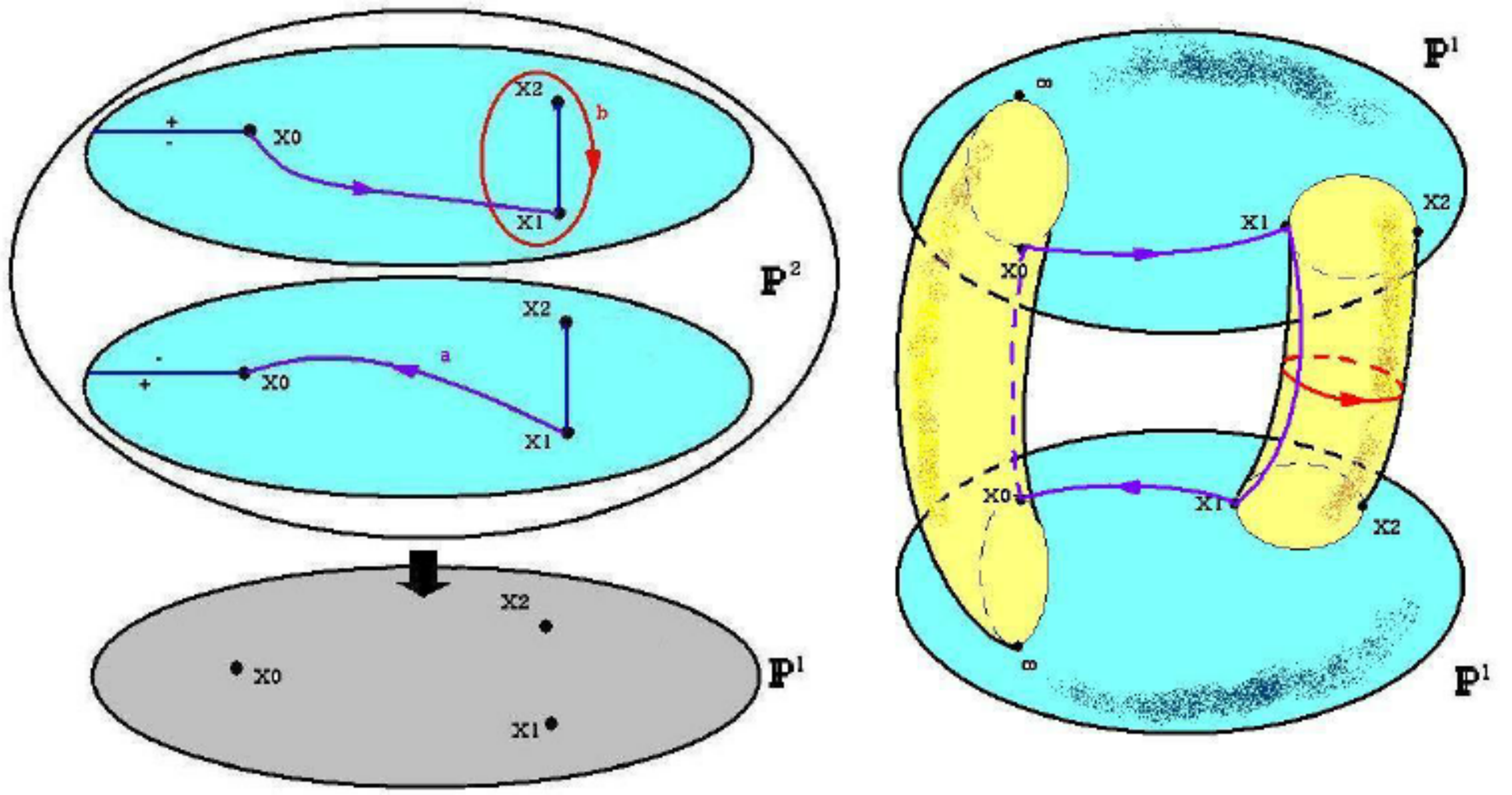
The LG point has superpotential

$$W = \sum_{ij} A^{ij}(p)\,\phi_i\phi_j \qquad \text{— mass matrix for } \phi \text{ fields.}$$

Away from zeroes of eigenvalues of A^{ij},

looks like sigma model on $\mathbb{P}^1 = \mathrm{Proj}\,\mathbb{C}[p_1, p_2]$, with $B\mathbb{Z}_2$ symmetry.

Decomposition $\Rightarrow$ Double cover of $\mathbb{P}^1$, branched over $\{\det A = 0\} = \{4 \text{ points}\}$



Another T^2 !
geometry
realized
nonperturbatively
via decomposition

Application: elliptic genera of pure susy gauge theories

(R Eager, ES '20)

We can use decomposition to predict elliptic genera of pure (2,2) susy gauge theories, using knowledge of IR susy breaking for various discrete theta angles.

Example: for $SU(k)/\mathbb{Z}_k$, susy unbroken only for discrete theta $\theta = -(1/2)k(k-1) \mod k$

(as derived from 2d nonabelian mirrors)

$$\mathrm{EG}(G/K, \theta) = 0 \quad \text{if susy broken in IR}$$

$$\textbf{Decomposition} \Rightarrow \mathrm{EG}(G) = \sum_{\theta} \mathrm{EG}(G/K, \theta)$$

Can then algebraically recover elliptic genera.

Example: $\mathrm{EG}(SU(k)/\mathbb{Z}_k, \theta) = (1/k)\mathrm{EG}(SU(k)) \sum_{m=0}^{k-1} (-)^{m(k+1)} \exp(im\theta)$

For $k = 2$, matches (Kim, Kim, Park '17).

Numerous other low-rank exs checked with susy localization.

Application: anomalies

Consider a finite G-gauge theory, $[X/G]$, with a gauge anomaly
(so that the theory does not actually exist).

Two methods to resolve the anomaly:

1) Make G bigger. (Wang-Wen-Witten '17, Tachikawa '17)

Replace G by Γ, $\quad 1 \longrightarrow K \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1$

where $\pi^*\alpha$ trivial for $\alpha \in H^3(G, U(1))$ the anomaly,

and replace original orbifold with $[X/\Gamma]_B$ for suitable phases $B \in H^1(G, H^1(K, U(1)))$.

2) Make G smaller.

Replace original orbifold with $[X/\ker f]$ for some hom' $f: G \to H$ s.t. $\alpha|_{\ker f} = 0$

Decomposition: $[X/\Gamma]_B \; = \;$ (copies of) $[X/\ker B]$ (Robbins, ES, Vandermeulen '21)

So the two possibilities are equivalent.

Application: moduli spaces

Gerbe structures are common on moduli spaces of SCFTs.

Moduli stack of susy sigma models = $\mathbb{Z}_2$ gerbe over moduli stack of CYs

Bagger-Witten line bundle = `fractional' bundle over that gerbe

(a bundle on the gerbe that is not a pullback from the underlying moduli space)

(Donagi et al '17, '19)

Example: moduli space of elliptic curves

$$\mathscr{M} = [\mathfrak{h}/SL(2,\mathbb{Z})]$$ for $\mathfrak{h}$ the upper half plane

However, the Bagger-Witten line bundle lives on $\mathscr{N} = [\mathfrak{h}/Mp(2,\mathbb{Z})]$

where $$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Mp(2,\mathbb{Z}) \longrightarrow SL(2,\mathbb{Z}) \longrightarrow 1$$ (Gu, ES '16)

which reflects a subtle $\mathbb{Z}_2$ extending T-duality in susy theories.

(Pantev, ES '16)

(Debray, Dierigl, Heckman, Montero, Torres '22-'23)

Summary

Decomposition: sometimes one QFT secretly $= \sum$ QFTs $= \coprod$ universes

Restrictions on instantons arise from such sums as
interference effect between universes

Examples include gauge theories w/ trivially-acting subgroups

Applications include Gromov-Witten theory, GLSMs, elliptic genera, anomalies.

Thank you for your time!

Details of another 2d example, involving orbifolds

Let's first construct a family of examples in $d = 2$ spacetime dimensions.

We'll gauge a noneffectively-acting $(d-2) = 0$-form symmetry,
to get a global 1-form symmetry (& hence a decomposition).

Specifically, consider the orbifold $[X/\Gamma]$, where

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad\qquad \sim\ \omega \in H^2(G, K)$$

is a central extension, and K, Γ, G are finite, K abelian, and K acts trivially.
(Decomposition exists more generally, but today I'll stick w/ easy cases.)

The orbifold $[X/\Gamma]$ has a global $BK \; = \; K^{(1)}$ symmetry, & should decompose.

I'm going to outline one way to see that

$$\mathrm{QFT}\left([X/\Gamma]\right) \; = \; \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)} \right)$$

where $\begin{array}{ccc} H^2(G, K) & \longrightarrow & H^2(G, U(1)) \\ \omega & \mapsto & \rho(\omega) \end{array}$ gives the discrete torsion on universe ρ

Claim: $\quad \mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)} \right)$

Let's establish this in partition functions on T^2.

Universally, for any Γ orbifold on T^2,

$$Z_{T^2}\left([X/\Gamma]\right) = \frac{1}{|\Gamma|} \sum_{\gamma_1 \gamma_2 = \gamma_2 \gamma_1} Z_{\gamma_1, \gamma_2}(X) \quad \text{where } Z_{g,h} = \left(g \quad \longrightarrow X \right)$$

h

("twisted sectors")

(Think of $Z_{g,h}$ as sigma model to X with branch cuts g, h.)

We need to count commuting pairs of elements in Γ ....

Claim: $\mathrm{QFT}([X/\Gamma]) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$

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Universally, for any Γ orbifold on T^2, $Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|} \sum_{\gamma_1\gamma_2 = \gamma_2\gamma_1} Z_{\gamma_1,\gamma_2}(X)$

We need to count commuting pairs of elements in Γ ....

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad \sim \ \omega \in H^2(G, K)$$

Write $\gamma \in \Gamma$ as $\gamma = (g \in G, k \in K)$ where $\gamma_1\gamma_2 = (g_1 g_2, k_1 k_2 \omega(g_1, g_2))$

Then, $\gamma_1\gamma_2 = \gamma_2\gamma_1 \Leftrightarrow g_1 g_2 = g_2 g_2$ and $\omega(g_1, g_2) = \omega(g_2, g_1)$

commuting pairs in G such that $\omega(g_1, g_2) = \omega(g_2, g_1)$

Restriction on nonperturbative sectors

(In an orbifold, nonperturbative sectors = twisted sectors)

Claim: $\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$

Let's establish this in partition functions on T^2.

Universally, for any Γ orbifold on T^2, $\quad Z_{T^2}\left([X/\Gamma]\right) = \frac{1}{|\Gamma|} \sum_{\gamma_1\gamma_2=\gamma_2\gamma_1} Z_{\gamma_1,\gamma_2}(X)$

We need to count commuting pairs of elements in Γ .... $\quad 1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$

These are commuting pairs in G such that $\omega(g_1, g_2) = \omega(g_2, g_1)$

So:

$$Z_{T^2}\left([X/\Gamma]\right) = \frac{1}{|\Gamma|} \sum_{\gamma_1\gamma_2=\gamma_2\gamma_1} Z_{\gamma_1,\gamma_2}(X) = \frac{|K|^2}{|\Gamma|} \sum_{g_1 g_2 = g_2 g_1} \delta\left(\frac{\omega(g_1,g_2)}{\omega(g_2,g_1)} - 1 \right) Z_{g_1,g_2}$$

where we have used $Z_{\gamma_1,\gamma_2} = Z_{g_1,g_2}$ since K acts trivially.

Claim: $\quad \mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho\in\hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$

Let's establish this in partition functions on T^2.

So far:
$$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|}\sum_{\gamma_1\gamma_2=\gamma_2\gamma_1} Z_{\gamma_1,\gamma_2}(X) = \frac{|K|^2}{|\Gamma|}\sum_{g_1g_2=g_2g_1}\delta\left(\frac{\omega(g_1,g_2)}{\omega(g_2,g_1)}-1\right) Z_{g_1,g_2}$$

Next, write
$$\delta\left(\frac{\omega(g_1,g_2)}{\omega(g_2,g_1)}-1\right) = \frac{1}{|\hat{K}|}\sum_{\rho\in\hat{K}}\frac{\rho\circ\omega(g_1,g_2)}{\rho\circ\omega(g_2,g_1)}$$
where $\rho\circ\omega \in H^2(G,U(1))$

(discrete torsion!)

so that, after rearrangement,
$$Z_{T^2}([X/\Gamma]) = \frac{|G||K|^2}{|\Gamma||\hat{K}|}\sum_{\rho\in\hat{K}} Z_{T^2}\left([X/G]_{\rho\circ\omega}\right) = \sum_{\rho\in\hat{K}} Z_{T^2}\left([X/G]_{\rho\circ\omega}\right)$$
consistent with decomposition !

Adding the universes projects out some sectors — interference effect.

So far we have demonstrated that for T^2 partition functions,

$$\mathrm{QFT}\left([X/\Gamma]\right) \;=\; \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)} \right)$$

which is the statement of decomposition in this case ($K \subset \Gamma$ central).

Similar computations can be done at any genus,
and for local operators, etc.

Next, we'll walk through details in a simple example....

To make this more concrete, let's walk through an example, where everything can be made completely explicit.

Example: Orbifold $[X/D_4]$ in which the $\mathbb{Z}_2$ center acts trivially.

— has $B\mathbb{Z}_2$ (1-form) symmetry

(T Pantev, ES '05)

$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ so this is closely related to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Decomposition predicts

$$\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$$

which here means

$$\mathrm{QFT}\left([X/D_4]\right) = \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Let's check this explicitly....

Example, cont'd

$$\mathrm{QFT}\left([X/D_4]\right) \;=\; \mathrm{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

At the level of operators, one reason for this is that the theory admits projection operators:

Let $\hat{z}$ denote the (dim 0) twist field associated to the trivially-acting $\mathbb{Z}_2$:

$\hat{z}$ obeys $\hat{z}^2 \;=\; 1.$

Using that relation, we form projection operators:

$$\Pi_\pm \;=\; \frac{1}{2}(1 \pm \hat{z})$$ (= specialization of general formula)

$$\Pi_\pm^2 = \Pi_\pm \qquad \Pi_\pm\Pi_\mp = 0 \qquad \Pi_+ + \Pi_- = 1$$

Note: untwisted sector lies in both universes; universes = lin' comb's of twisted & untwisted.

Next: compare partition functions....

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

Take the (1+1)-dim'l spacetime to be T^2.

The partition function of any orbifold $[X/\Gamma]$ on T^2 is

$$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|} \sum_{gh=hg} Z_{g,h} \qquad \text{where } Z_{g,h} = \left(g \, \underset{h}{\blacksquare} \longrightarrow X \right)$$

("twisted sectors")

(Think of $Z_{g,h}$ as sigma model to X with branch cuts g, h.)

We're going to see that

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) + Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

Compute the partition function of $[X/D_4]$ (T Pantev, ES '05)

$$D_4 \;=\; \{1, z, a, b, az, bz, ab, ba = abz\}$$

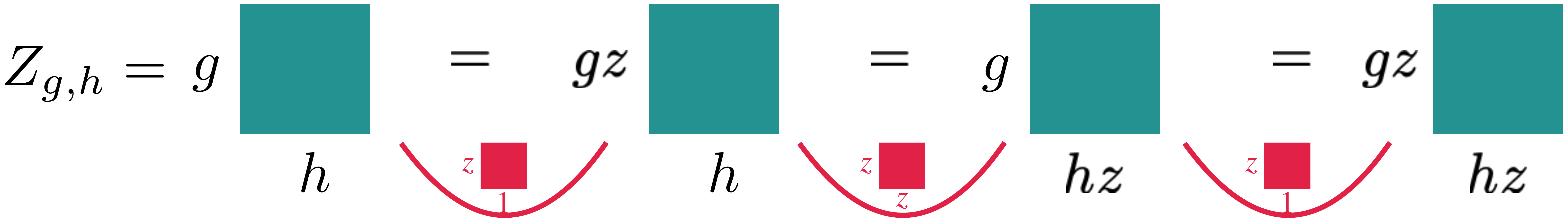
where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \;=\; \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

$$Z_{T^2}\left([X/D_4]\right) = \frac{1}{|D_4|} \sum_{g,h \in D_4,\ gh=hg} Z_{g,h}$$ where $Z_{g,h} = \left(g \; \boxed{}_{\,h} \longrightarrow X \right)$

Since z acts trivially,

$Z_{g,h}$ is symmetric under multiplication by z



This is the $B\mathbb{Z}_2$ 1-form symmetry.

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 \;=\; \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

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$$Z_{T^2}\left([X/D_4]\right) = \frac{1}{|D_4|} \sum_{g,h \in D_4,\; gh=hg} Z_{g,h} \qquad \text{where } Z_{g,h} = \left(g \underset{h}{\blacksquare} \longrightarrow X \right)$$

Each D_4 twisted sector ($Z_{g,h}$) that appears is the same as a $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sector,

appearing with multiplicity $|\mathbb{Z}_2|^2 = 4$,

except for the sectors $\overline{a}\underset{\overline{b}}{\blacksquare} \qquad \overline{a}\underset{\overline{ab}}{\blacksquare} \qquad \overline{b}\underset{\overline{ab}}{\blacksquare}$ which do **not** appear.

Restriction on nonperturbative sectors

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right) \right)$$

$$= 2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right) \right)$$

Different theory than $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Physics knows when we gauge even a trivially-acting group!

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - (\text{some twisted sectors}) \right)$$

$$= 2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - (\text{some twisted sectors}) \right)$$

Fact: given any one partition function $Z_{T^2}([X/G]) = \frac{1}{|G|} \sum_{gh=hg} Z_{g,h}$

we can multiply in $SL(2,\mathbb{Z})$-invariant phases $\epsilon(g,h)$

to get another consistent partition function (for a different theory)

$$Z' = \frac{1}{|G|} \sum_{gh=hg} \epsilon(g,h) \, Z_{g,h}$$

There is a universal choice of such phases, determined by elements of $H^2(G, U(1))$

This is called "discrete torsion."

Example, cont'd

Compute the partition function of $[X/D_4]$

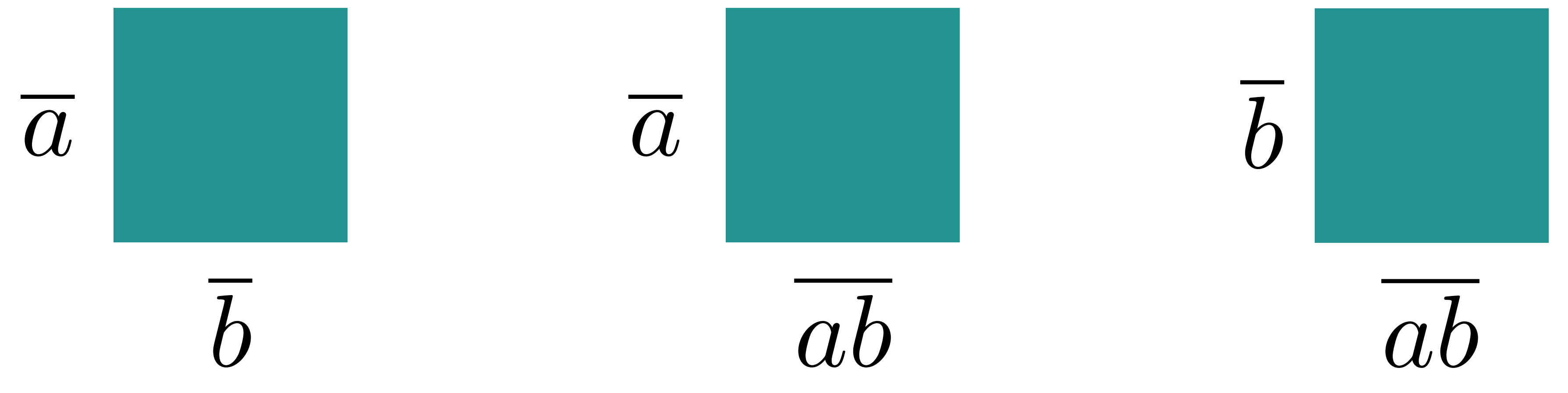
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|}\,|\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors}) \right)$$

$$= \; 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors}) \right)$$

In a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, discrete torsion $\in \; H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \; = \; \mathbb{Z}_2$,

and the nontrivial element acts as a sign on the twisted sectors



the same sectors which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Adding the universes projects out some sectors — interference effect.

Example, cont'd

Compute the partition function of $[X/D_4]$

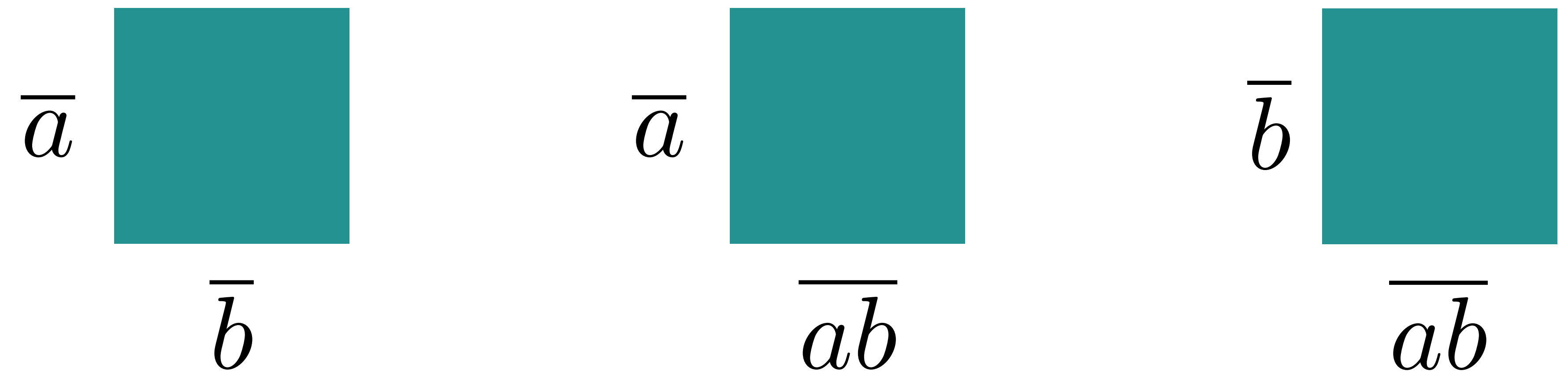
(T Pantev, ES '05)

$$Z_{T^2}([X/D_4]) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \Big(Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors}) \Big)$$

$$= 2 \Big(Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors}) \Big)$$

Discrete torsion is $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and acts as a sign on the twisted sectors



which were omitted above.

$$Z_{T^2}([X/D_4]) = Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}) + Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}})$$

Matches prediction of decomposition

$$\text{QFT}([X/D_4]) = \text{QFT}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}) \coprod \text{QFT}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}})$$

Example, cont'd

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) \;=\; \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

The computation above demonstrated that the partition function on T^2
has the form predicted by decomposition.
The same is also true of partition functions at higher genus
— just more combinatorics.
(see hep-th/0606034, section 5.2 for details)

Only slightly novel aspect: in gen'l, one finds dilaton shifts,
which mostly I'll suppress in this talk.

Example, cont'd

Massless states of $[X/D_4]$ for $X = T^6$ (T Pantev, ES '05)

Massless states of $[T^6/D_4]$

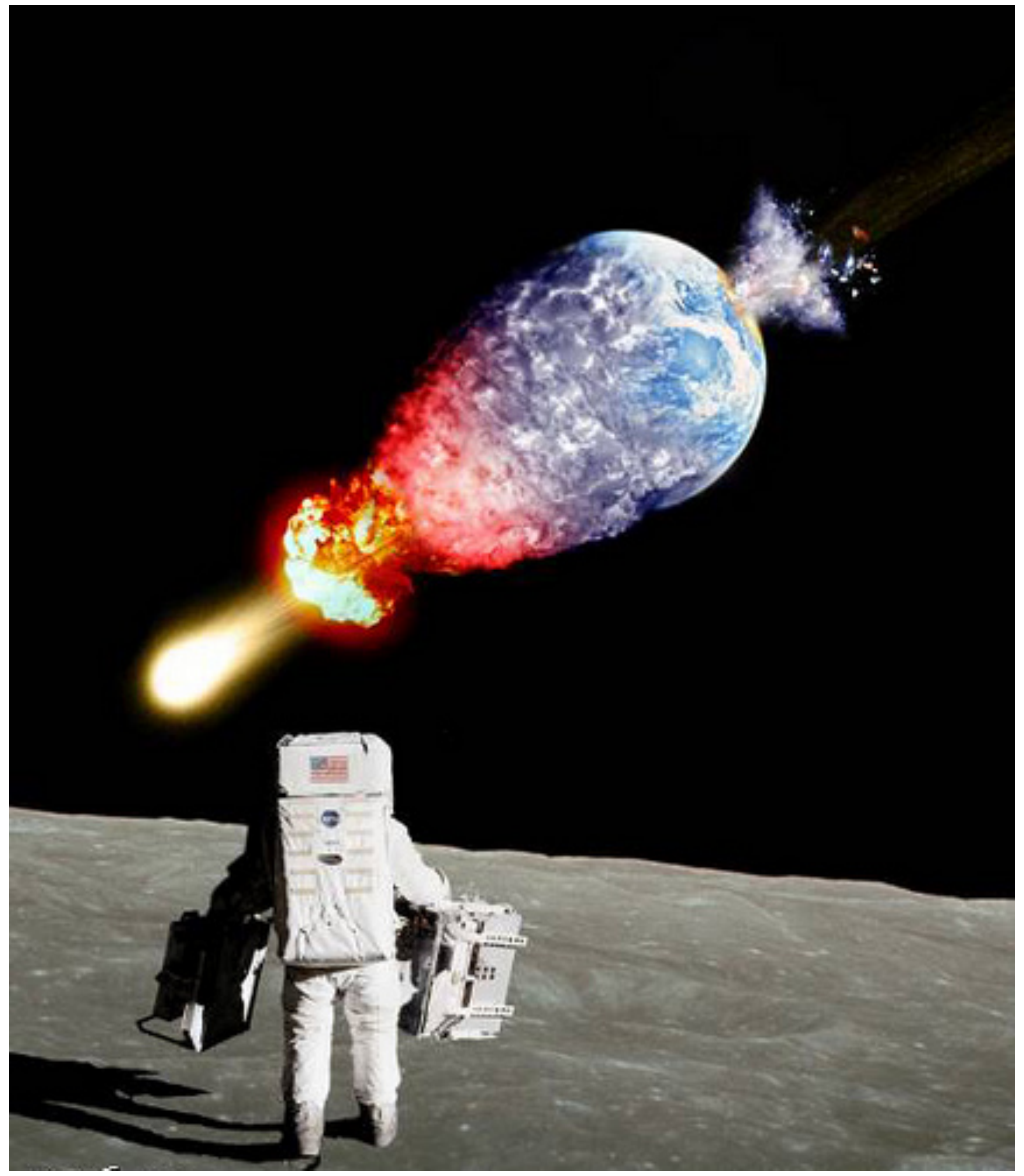
$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}
$$

Signals mult' components / cluster decomp' violation

If we didn't know about decomposition, the 2's in the corners would be a problem...

A big problem!

They signal a violation of cluster decomposition, the same axiom that's violated by restricting instantons.



Ordinarily, I'd assume that the computation was wrong.

However, decomposition saves the day....

Example, cont'd

Massless states of $[X/D_4]$ for $X = T^6$ (T Pantev, ES '05)

Massless states of $[T^6/D_4]$

$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 51 & & 0 & \\
1 & & 3 & & 3 & & 1 \\
 & 0 & & 51 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\quad + \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 51 & & 51 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Signals mult' components / cluster decomp' violation

spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb' w/o d.t.

spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb' w/ d.t.

matching the prediction of decomposition

$$\mathrm{CFT}\left([X/D_4]\right) = \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{w/o\,d.t.}}\right) \coprod \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{d.t.}}\right)$$