An introduction to heterotic mirror symmetry

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I’ll begin today by reminding us all of ordinary mirror symmetry.

Most basic incarnation:

String theory on a Calabi-Yau $X$

$$= \text{String theory on a Calabi-Yau } Y$$

Ex: $X = \text{quintic threefold}, \mathbb{P}^4[5] \quad Y = \mathbb{P}^4[5]/\mathbb{Z}_5^3$

$$\quad \text{dim}(X) = \text{dim}(Y)$$

Relates Hodge numbers: $h^{p,q}(X) = h^{p,n-q}(Y)$

Also swaps perturbative & nonpert’ corrections: made computing GW invariants easy.
Plan for today:

Outline a generalization of mirror symmetry, (involving heterotic strings,) that is perhaps not so well-known.

• Brief review of ordinary mirrors, then heterotic analogues

• Some other more exotic dualities

• Heterotic version of quantum cohomology: quantum sheaf cohomology
Let’s quickly review some of the reasons physicists believe in and think about mirror symmetry, en route to talking about the `heterotic’ generalization.

Some of the original checks....
Numerical checks of mirror symmetry

Plotted below are data for a large number of Calabi-Yau 3-folds.

Vertical axis: \( h^{1,1} + h^{2,1} \)

Horizontal axis: \( 2(h^{1,1} - h^{2,1}) = 2 (\# \text{Kahler} - \# \text{cpx def's}) \)

Mirror symmetry exchanges \( h^{1,1} \leftrightarrow h^{2,1} \)

\[ \Rightarrow \text{symm' across vert' axis} \]

(Klemm, Schimmrigk, NPB 411 (’94) 559-583)
Constructions of mirror pairs

One of the original methods:
in special cases, can quotient by a symmetry group.
“Greene-Plesser orbifold construction”

Example: quintic

\[ Q_5 \subset \mathbb{P}^4 \quad \text{mirror} \quad Q_5 / \mathbb{Z}_5^3 \]

More general methods exist....
Constructions of mirror pairs

Batyrev’s construction:

For a hypersurface in a toric variety, mirror symmetry exchanges

polytope of ambient toric variety ↔ dual polytope for ambient t.v. of mirror
Constructions of mirror pairs

Example of Batyrev’s construction:

$T^2$ as degree 3 hypersurface in $\mathbb{P}^2$

\[ \mathbb{P}^2 = \{ y \mid \langle x, y \rangle \geq -1 \ \forall x \in P \} = \mathbb{P}^2 / \mathbb{Z}_3 \]

Result:

degree 3 hypersurface in $\mathbb{P}^2$, mirror to $\mathbb{Z}_3$ quotient of degree 3 hypersurface

(matching Greene-Plesser '90)
Ordinary mirror symmetry is pretty well understood nowadays.

- lots of constructions
- both physics and math proofs
  
  Givental / Yau et al in math
  
  Morrison-Plesser / Hori-Vafa in physics

However, there are some extensions of mirror symmetry that are still being actively studied....
Ordinary mirror symmetry is a property of type II strings, or worldsheets with “(2,2) supersymmetry.”

It is also believed to apply to heterotic strings, whose worldsheets have “(0,2) supersymmetry.”

(2,2): specified, in simple cases, by a Kahler mfld \( X \)

(0,2): specified, in the same simple cases, by a Kahler manifold \( X \) together with a holomorphic bundle \( \mathcal{E} \rightarrow X \) such that

\[
\text{ch}_2(\mathcal{E}) = \text{ch}_2(TX)
\]

(Recover (2,2) in special case that \( \mathcal{E} = TX \).)

Heterotic aka (0,2) mirror symmetry involves bundles + spaces.
Analogues of topological field theories:

True TFT’s based on (0,2) theories do not exist, but, there do exist pseudo-topological field theories with closely related properties, at least in special cases.

\[ \text{A/2 model: } \exists \text{ when } \det E^* \cong K_X \]

States counted by \( H^\bullet(X, \wedge^\bullet E^*) \)

Reduces to A model on (2,2) locus (\( E = TX \))

\[ \text{B/2 model: } \exists \text{ when } \det E \cong K_X \]

States counted by \( H^\bullet(X, \wedge^\bullet E) \)

Reduces to B model on (2,2) locus (\( E = TX \))

\[ A/2(X, E) \cong B/2(X, E^*) \]
(0,2) mirror symmetry ( (0,2) susy )

How should this work?

Nonlinear sigma models with (0,2) susy defined by space $X$, with hol’ vector bundle $E \to X$

(0,2) mirror defined by space $Y$, w/ bundle $F$.

\[
\text{dim } X = \text{ dim } Y \\
\text{rk } E = \text{ rk } F \\
A/2( X, E ) = B/2( Y, F ) \\
H^p(X, \wedge^q E^*) = H^p(Y, \wedge^q F) \\
\text{(moduli)} = \text{(moduli)}
\]

When $E=TX$, should reduce to ordinary mirror symmetry.
**(0,2) mirror symmetry**

Not as much known about heterotic/(0,2) mirror symm’, though a few basics have been worked out.

Example: numerical evidence

Horizontal: $h^1(\mathcal{E}) - h^1(\mathcal{E}^*)$

Vertical: $h^1(\mathcal{E}) + h^1(\mathcal{E}^*)$

where $\mathcal{E}$ is rk 4

(Blumenhagen, Schimmrigk, Wisskirchen, NPB 486 ('97) 598-628)
Constructions include:

- Blumenhagen-Sethi ’96 extended Greene-Plesser orbifold construction to (0,2) models — handy but only gives special cases
- Adams-Basu-Sethi ’03 repeated Hori-Vafa-Morrison-Plesser-style GLSM duality in (0,2) — but results must be supplemented by manual computations; (0,2) version does not straightforwardly generate examples

More recent progress includes a version of Batyrev’s construction…. 
(0,2) mirror symmetry

- Melnikov-Plesser ’10 extended Batyrev’s construction & monomial-divisor mirror map to include def’s of tangent bundle, for special (‘reflexively plain’) polytopes

Progress, but still don’t have a general construction.
Now let’s turn to a few other dualities, which may or may not be related....
Gauge bundle dualization duality ( (0,2) susy )

(Nope, not a typo....)

Nonlinear sigma models with (0,2) susy defined by space $X$, with hol’ vector bundle $E \rightarrow X$

Duality: $\text{CFT}(X, E) = \text{CFT}(X, E^*)$

ie, replacing the bundle with its dual is an invariance of the theory.
Gauge bundle dualization duality \((0,2)\) susy

How is this related to \((0,2)\) mirrors?

Maybe orthogonal:

\[
\begin{align*}
(X, E) & \leftrightarrow_{(0,2)\text{ mirror}} (Y, F) \\
(X, E^*) & \leftrightarrow_{(0,2)\text{ mirror}} (Y, F^*)
\end{align*}
\]

On the other hand, both exchange A/2, B/2 models, both flip sign of left U(1)… …maybe it’s also a sort of \((0,2)\) mirror.

More exotic variations….
Triality (0,2 susy)
(Gadde-Gukov-Putrov ’13-'14)

It has been proposed that triples of certain (0,2) theories might be equivalent.

\[
\begin{align*}
S^A \oplus (Q^*)^{2k+A-n} \oplus (\det S^*)^2 &\rightarrow G(k,n) \\
S^{2k+A-n} \oplus (Q^*)^n \oplus (\det S^*)^2 &\rightarrow G(n-k,A) \\
S^n \oplus (Q^*)^A \oplus (\det S^*)^2 &\rightarrow G(A-n+k,2k+A-n)
\end{align*}
\]

are conjectured to all be equivalent, for n, k, A such that the geometries above are all sensible.

Moving on....
Triality \((0,2) \text{ susy}\)

How is this related to \((0,2)\) mirrors?

Maybe notion of \((0,2)\) mirrors is richer, & more variations exist to be found:

\[(X_1, E_1) \leftrightarrow (X_2, E_2) \leftrightarrow (X_3, E_3) \leftrightarrow (X_4, E_4) \leftrightarrow \]

Triality seems to be in this spirit.
So far I’ve outlined (0,2) mirrors and some possibly related dualities.

Next: analogue of curve counting, Gromov-Witten….
Review of quantum sheaf cohomology

Quantum sheaf cohomology is the heterotic version of quantum cohomology — defined by space + bundle. (Katz-ES ’04, ES ’06, Guffin-Katz ’07, ….)

On the (2,2) locus, where bundle = tangent bundle, encodes Gromov-Witten invariants.

Off the (2,2) locus, Gromov-Witten inv’ts no longer relevant. Mathematical GW computational tricks no longer apply. No known analogue of periods, Picard-Fuchs equations.

New methods needed…. … and a few have been developed. (A Adams, J Distler, R Donagi, J Guffin, S Katz, J McOrist, I Melnikov, R Plesser, ES, …. )
Minimal area surfaces:
standard case ("type II strings")

Schematically: For $X$ a space, 
$\mathcal{M}$ a space of holomorphic $S^2 \rightarrow X$
we compute a "correlation function" in A model TFT

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle = \int_{\mathcal{M}} \omega_1 \wedge \cdots \wedge \omega_k$$

where $\mathcal{O}_i \sim \omega_i \in H^{p_i,q_i}(\mathcal{M})$

$$= \int_{\mathcal{M}} \text{(top form on } \mathcal{M})$$

which encodes minimal area surface information.

Such computations are at the heart of Gromov-Witten theory.
Minimal area surfaces: heterotic case

Schematically: For $X$ a space, $\mathcal{E}$ a bundle on $X$, $\mathcal{M}$ a space of holomorphic $S^2 \rightarrow X$

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle = \int_{\mathcal{M}} \tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_k$$

where $\mathcal{O}_i \sim \tilde{\omega}_i \in H^{q_i} (\mathcal{M}, \wedge^{p_i} \mathcal{F}^*)$

$\mathcal{F} = \text{sheaf of 2d fermi zero modes over } \mathcal{M}$

anomaly cancellation $\xRightarrow{\text{GRR}} \wedge^{\text{top}} \mathcal{F}^* \cong K_{\mathcal{M}}$

hence, again,

$$= \int_{\mathcal{M}} (\text{top form on } \mathcal{M})$$

(S Katz, ES, 2004)

This computation takes place in “A/2 model,” a pseudo-topological field theory.
Correlation functions are often usefully encoded in `operator products’ (OPE’s).

**Physics:** Say $\mathcal{O}_A \mathcal{O}_B = \sum_i \mathcal{O}_i$ ("operator product") if all correlation functions preserved:

$$\langle \mathcal{O}_A \mathcal{O}_B \mathcal{O}_C \cdots \rangle = \sum_i \langle \mathcal{O}_i \mathcal{O}_C \cdots \rangle$$

**Math:** if interpret correlation functions as maps

$\text{Sym} \cdot W \rightarrow \mathbb{C}$

(where $W$ is the space of $\mathcal{O}$’s)

then OPE’s are the kernel, of form $\mathcal{O}_A \mathcal{O}_B = \sum_i \mathcal{O}_i$
Examples:

Ordinary ("type II") case:

\[ X = \mathbb{P}^1 \times \mathbb{P}^1 \quad W = H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{C}^2 = \mathbb{C}\{\psi, \tilde{\psi}\} \]

OPE's: \[ \psi^2 = q, \quad \tilde{\psi}^2 = \tilde{q} \]

where \( q, \tilde{q} \sim \exp(-\text{area}) \)

\[ \longrightarrow 0 \quad \text{in classical limit} \]

Looks like a deformation of cohomology ring, hence called "quantum cohomology"
Examples:

Ordinary ("type II") case: \( X = \mathbb{P}^1 \times \mathbb{P}^1 \)

\[ \text{OPE's: } \psi^2 = q, \quad \tilde{\psi}^2 = \tilde{q} \]

Heterotic case:

\[ X = \mathbb{P}^1 \times \mathbb{P}^1 \quad \mathcal{E} \text{ a deformation of } T(\mathbb{P}^1 \times \mathbb{P}^1) \]

Def'n of \( \mathcal{E} \):

\[ 0 \to W^* \otimes \mathcal{O} \to \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \to \mathcal{E} \to 0 \]

where \( * = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix} \) \( A, B, C, D \) const' 2x2 matrices

\( x, \tilde{x} \) vectors of homog' coord's

Here, \( W = H^1(X, \mathcal{E}^*) = \mathbb{C}^2 = \mathbb{C}\{\psi, \tilde{\psi}\} \)

OPE's:

\[ \det \left( A\psi + B\tilde{\psi} \right) = q, \quad \det \left( C\psi + D\tilde{\psi} \right) = \tilde{q} \]

Check: \( \mathcal{E} = TX \) when \( A = D = I_{2 \times 2}, \quad B = C = 0 \)

& in this limit, OPE's reduce to those of ordinary case quantum sheaf cohomology
Review of quantum sheaf cohomology

To make this more clear, let’s consider an example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle $E$ a deformation of the tangent bundle:

$$0 \to W^* \otimes O \to O(1,0)^2 \oplus O(0,1)^2 \to E \to 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$

and $W = \mathbb{C}^2$

Operators counted by $H^1(E^*) = H^0(W \otimes O) = W$

n-pt correlation function is a map

$\text{Sym}^n H^1(E^*) = \text{Sym}^n W \to H^n(\wedge^n E^*)$

OPE’s = kernel

Plan: study map corresponding to classical corr’ f’n
Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle $E$ a deformation of the tangent bundle:

$$0 \to W^* \otimes O \to O(1,0)^2 \oplus O(0,1)^2 \rightarrow E \rightarrow 0$$

where $*$ = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix} $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Since this is a rk 2 bundle, classical sheaf cohomology defined by products of 2 elements of $H^1(E^*)=H^0(W \otimes O)=W$

So, we want to study map $H^0(\text{Sym}^2 W \otimes O) \to H^2(\wedge^2 E^*) = \text{corr' f'n}$

This map is encoded in the resolution

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \text{Sym}^2 W \otimes O \to 0$$
Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \text{Sym}^2 W \otimes O \to 0$$

Break into short exact sequences:

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0$$

$$0 \to S_1 \to Z \otimes W \to \text{Sym}^2 W \otimes O \to 0$$

Examine second sequence:

induces $H^0(Z \otimes W) \to H^0(\text{Sym}^2 W \otimes O) \to H^1(S_1) \to H^1(Z \otimes W)$$

Since $Z$ is a sum of $O(-1,0)$'s, $O(0,-1)$'s,

hence $\delta: H^0(\text{Sym}^2 W \otimes O) \to H^1(S_1)$ is an iso.

Next, consider the other short exact sequence at top....
Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

\[
0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \text{Sym}^2 W \otimes O \to 0
\]

Break into short exact sequences:

\[
0 \to S_1 \to Z \otimes W \to \text{Sym}^2 W \otimes O \to 0
\]

\[
\delta : H^0(\text{Sym}^2 W \otimes O) \to H^1(S_1)
\]

Examine other sequence:

\[
0 \to \wedge^2 E^* \to \wedge^2 Z \to S_1 \to 0
\]

induces \[
H^1(\wedge^2 Z) \to H^1(S_1) \xrightarrow{\delta} H^2(\wedge^2 E^*) \to H^2(\wedge^2 Z)
\]

Since $Z$ is a sum of $O(-1,0)$’s, $O(0,-1)$’s,

\[
H^2(\wedge^2 Z) = 0 \quad \text{but} \quad H^1(\wedge^2 Z) = \mathbb{C} \oplus \mathbb{C}
\]

and so \[
\delta : H^1(S_1) \to H^2(\wedge^2 E^*) \quad \text{has a 2d kernel.}
\]

Now, assemble the coboundary maps....
Review of quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \to \wedge^2 E^* \to \wedge^2 Z \to Z \otimes W \to \text{Sym}^2 W \otimes O \to 0$$

Now, assemble the coboundary maps....

A classical (2-pt) correlation function is computed as

$$H^0(\text{Sym}^2 W \otimes O) \xrightarrow{\sim} H^1(S_1) \xrightarrow{\delta} H^2(\wedge^2 E^*)$$

where the right map has a 2d kernel, which one can show is generated by

$$\det(A\psi + B\tilde{\psi}), \det(C\psi + D\tilde{\psi})$$

where $A$, $B$, $C$, $D$ are four matrices defining the def’ $E$, and $\psi, \tilde{\psi}$ correspond to elements of a basis for $W$.

Classical sheaf cohomology ring:

$$\mathbb{C}[[\psi, \tilde{\psi}]] / (\det(A\psi + B\tilde{\psi}), \det(C\psi + D\tilde{\psi}))$$
Review of quantum sheaf cohomology

Quantum sheaf cohomology

\[ = \text{OPE ring of the A/2 model} \]

Instanton sectors have the same form, except X replaced by moduli space M of instantons, E replaced by induced sheaf F over moduli space M.

Must compactify M, and extend F over compactification divisor.

\[ \wedge^{\text{top}} E^* \cong K_X \quad \text{GRR} \quad \wedge^{\text{top}} F^* \cong K_M \]

Within any one sector, can follow the same method just outlined....
Review of quantum sheaf cohomology

In the case of our example, one can show that in a sector of instanton degree \((a,b)\), the `classical’ ring in that sector is of the form

\[
\text{Sym}^\cdot \mathcal{W} / (Q^{a+1}, \tilde{Q}^{b+1})
\]

where

\[
Q = \det(A\psi + B\bar{\psi}), \quad \tilde{Q} = \det(C\psi + D\bar{\psi})
\]

Now, OPE’s can relate correlation functions in different instanton degrees, and so, should map ideals to ideals.

To be compatible with those ideals,

\[
\langle O \rangle_{a,b} = q^{a' - a} \tilde{q}^{b' - b} \langle OQ^{a' - a} \tilde{Q}^{b' - b} \rangle_{a',b'}
\]

for some constants \(q, \tilde{q}\)  

\(\Rightarrow\) OPE’s \(Q = q, \quad \tilde{Q} = \tilde{q}\)

— quantum sheaf cohomology rel’ns
Review of quantum sheaf cohomology

General result: (Donagi, Guffin, Katz, ES, ’11)

For any toric variety, and any def’ E of its tangent bundle,

\[ 0 \rightarrow W^* \otimes O \rightarrow \bigoplus O(\vec{q}_i) \rightarrow E \rightarrow 0 \]

the chiral ring is

\[ \prod_{\alpha} (\det M_{(\alpha)})^{Q^a_\alpha} = q_a \]

where the M’s are matrices of chiral operators built from \(*\).
Review of quantum sheaf cohomology

So far, I’ve outlined mathematical computations of quantum sheaf cohomology, but GLSM-based methods also exist:

• Quantum cohomology ( (2,2) ): Morrison-Plessser ‘94
• Quantum sheaf cohomology ( (0,2) ): McOrist-Melnikov ’07, ’08

Briefly, for (0,2) case:

One computes quantum corrections to effective action of form

\[ L_{\text{eff}} = \int d\theta^+ \sum_a Y_a \log \left[ \prod_\alpha (\det M_{(\alpha)})^{Q_\alpha} / q_a \right] \]

from which one derives

\[ \prod_\alpha (\det M_{(\alpha)})^{Q_\alpha} = q_a \]

— these are q.s.c. rel’ns — match math’ computations
The Future

Next Exit
More general constructions of (0,2) mirrors, & related duals, as current methods are limited

Generalize quantum sheaf cohomology computations to arbitrary compact Calabi-Yau manifolds
Generalize quantum sheaf cohomology…

State of the art: computations on toric varieties

To do: compact CY’s

Intermediate step: Grassmannians (work in progress)

Briefly, what we need are better computational methods.

Conventional GW tricks seem to revolve around idea that A model is independent of complex structure, not necessarily true for A/2.

- McOrist-Melnikov ’08 have argued an analogue for A/2
- Despite attempts to check (Garavuso-ES ‘13), still not well-understood
Mathematics

Geometry:
Gromov-Witten
Donaldson-Thomas
quantum cohomology
etc

Homotopy, categories:
derived categories
stacks
derived spaces
categorical equivalence

Physics

Supersymmetric, topological quantum field theories

D-branes
gauge theories
sigma models w/ potential
renormalization group flow