Yukawa couplings and generalizations of Gromov-Witten invariants

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Some representative papers: hep-th/0406226,
arXiv: 1110.3751, 1110.3752,

Computation

There exist generalizations of Gromov-Witten invariants and quantum cohomology that arise in charged matter interactions in heterotic string compactifications,
but for which we have only very limited computations.

Goal: compute those generalizations.

Unfortunately this is only aspirational — this is not ready to plug into Macaulay2 or Mathematica,
instead this is merely a research program goal.
Results exist for Fano spaces, but not compact Calabi-Yau's.

Computation

There exist generalizations of Gromov-Witten invariants and quantum cohomology that arise in charged matter interactions in heterotic string compactifications,
but for which we have only very limited computations.

Plan for today:

- Outline charged matter interactions
- Outline the overarching program — heterotic GW
- Describe concrete computations in toy models
  — quantum sheaf cohomology

Charged matter interactions

To "compactify" a heterotic string to 4 dimensions,
we specify a Calabi-Yau 3-fold X
and a stable holomorphic vector bundle $\mathscr{E} \to X$
such that $\wedge^{\mathrm{top}} \mathscr{E} \cong \mathcal{O}_X$, $\mathrm{ch}_2(\mathscr{E}) = \mathrm{ch}_2(TX)$.

This results in a 4d theory with N=1 susy.
(Less constrained than 4d N=2 susy,
see eg L Anderson's talk.)

Low energy gauge group is max commutant of $SU(r)$ in $E_8 \times E_8$
where $r = \mathrm{rank}\, \mathscr{E}$

Physical charged particles <-> sheaf cohomology.

(I'll elaborate momentarily.)

Physical charged particles <-> sheaf cohomology.

Example: rank $\mathcal{E}$ = 3, dim X=3, low-energy theory is $E_6 \times E_8$

Representation of E_6	Sheaf cohomology
$\overline{\mathbf{27}}$	$H^1(X, \mathcal{E}^*) \cong H^2(X, \wedge^2 \mathcal{E})^*$
$\mathbf{27}$	$H^1(X, \mathcal{E}) \cong H^2(X, \wedge^2 \mathcal{E}^*)^*$

Example: rank $\mathcal{E}$ = 4, low-energy theory is $\mathrm{Spin}(10) \times E_8$

Representation of $\mathrm{Spin}(10)$	Sheaf cohomology
$\mathbf{16}$	$H^1(X, \mathcal{E}) \cong H^2(X, \wedge^3 \mathcal{E}^*)^*$
$\mathbf{10}$	$H^1(X, \wedge^2 \mathcal{E}) \cong H^2(X, \wedge^2 \mathcal{E}^*)^*$

(Distler, Greene, Nucl Phys B 304 (1988) 1-62)

Given a string compactification,
we want to compute the interactions.

In the low-energy 4d theory,
these are invariant combinations of the charged matter fields.

Example: For $G = SU(3)$,
there is an $SU(3)$ invariant one can construct from
three copies of irrep $\mathbf{3}$.

Example: For $G = E_6$,
there is an E_6 invariant one can construct from
three copies of irrep $\mathbf{27}$,
and also from three copies of irrep $\overline{\mathbf{27}}$.

They are determined by the Calabi-Yau X and bundle $\mathscr{E}$.

These invariant couplings, denoted eg $\overline{\mathbf{27}}^3$,
arise in the low-energy theory as superpotential terms,
and can be expanded, *schematically,*

$$\overline{\mathbf{27}}^3 = f_0(X,\mathscr{E}) + \sum_{d\neq 0} f_d(X,\mathscr{E})q^d$$

where

$$f_0 = \int_X \omega_1 \cup \omega_2 \cup \omega_3$$

$$\omega_i \in H^{p_i}(X, \wedge^{q_i}\mathscr{E})$$

$$\sum p_i = \dim X \quad \sum q_i = \operatorname{rank}\mathscr{E}$$

(see J Gray's talk)

Example: rank $\mathscr{E} = 3$

$$\overline{\mathbf{27}}^3 \leftrightarrow H^1(X,\mathscr{E}^*)\times H^1(X,\mathscr{E}^*)\times H^1(X,\mathscr{E}^*) \longrightarrow H^3(X,\wedge^3\mathscr{E}^*)$$

= number, since $\wedge^3\mathscr{E}^* \cong K_X$

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$$\omega_i \in H^{p_i}(X, \wedge^{q_i} \mathscr{E})$$

$$\sum p_i = \dim X \quad \sum q_i = \operatorname{rank} \mathscr{E}$$

Example: rank $\mathscr{E} = 4$

$\mathbf{10 - 16 - 16}$

$$\leftrightarrow \quad H^1(X, \wedge^2 \mathscr{E}) \times H^1(X, \mathscr{E}) \times H^1(X, \mathscr{E}) \longrightarrow H^3(X, \wedge^4 \mathscr{E})$$

= number, since $\wedge^4 \mathscr{E} \cong K_X$

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$$\sum p_i = \dim X \quad \sum q_i = \mathrm{rank}\, \mathcal{E}$$

(see J Gray's talk)

q = quantum parameter, -> 0 classically

So the f_d ~ quantum corrections to sheaf cohomology.

Research problem: how to compute the f_d for $d \neq 0$?

In a heterotic string compactification on a space X
with holomorphic vector bundle $\mathscr{E} \to X$,
the low-energy 4d theory has superpotential couplings

$$f_0(X, \mathscr{E}) \; + \; \sum_{d \neq 0} f_d(X, \mathscr{E}) q^d$$

Research problem: how to compute the f_d for $d \neq 0$?

Happily, there are no Feynman diagrams here,
no loop integrals.

The expression above has polynomials in $q \sim \exp(-1/\hbar)$
but no Feynman loop corrections (which $\propto \hbar^n \sim (\ln q)^n$)

(Dine-Seiberg-Wen-Witten '87)

The f_d are known in the special case $\mathscr{E} = TX$….

Motivation: generalization of Gromov-Witten invariants

In a heterotic string compactification on a space X
with holomorphic vector bundle $\mathscr{E} \to X$,
the low-energy 4d theory has superpotential couplings

$$f_0(X, \mathscr{E}) \ + \ \sum_{d \neq 0} f_d(X, \mathscr{E}) q^d$$

When $\mathscr{E} = TX$, the f_d encode Gromov-Witten inv'ts.

Setup:

S'pose X is a CY 3-fold.

Sheaf cohomology: $H^1(X, \mathscr{E}^*) = H^1(X, T^*X) = H^{1,1}(X)$

$f_0 = \int_X \omega_1 \cup \omega_2 \cup \omega_3$ for $\omega_i \in H^{1,1}(X)$ ~ intersection number

So, quantum-corrected intersection numbers….

Motivation: generalization of Gromov-Witten invariants

In a heterotic string compactification on a space X
with holomorphic vector bundle $\mathscr{E} \to X$,
the low-energy 4d theory has superpotential couplings

$$f_0(X, \mathscr{E}) \; + \sum_{d \neq 0} f_d(X, \mathscr{E}) q^d$$

When $\mathscr{E} = TX$, the f_d encode Gromov-Witten inv'ts.

Example: $X = \mathbb{P}^4[5]$, $\mathscr{E} = TX$,

(Candelas, de la Ossa, Green, Parkes, '91)

$$\overline{\mathbf{27}}^{\mathbf{3}} \;=\; 5 + \sum_{k=1}^{\infty} \frac{n_k k^3 q^k}{1 - q^k} \;=\; 5 + 2875\, q + 4876875\, q^2 + \cdots$$

intersection number

(Strominger '85)

nonperturbative
contributions

n_k = Gromov-Witten
invariants

Question: what about more general $\mathscr{E}$?

Motivation: generalization of Gromov-Witten invariants

In a heteroti string compactification on a space X
with holomorphic vector bundle $\mathcal{E} \to X$,
the low-energy 4d theory has superpotential couplings

$$f_0(X, \mathcal{E}) \; + \; \sum_{d \neq 0} f_d(X, \mathcal{E}) q^d$$

When $\mathcal{E} = TX$, the f_d encode Gromov-Witten inv'ts.

Question: what about more general $\mathcal{E}$?

In principle, in more general cases,
the f_d are expected to arise similarly —
from Gromov-Witten-like computations, involving $\mathcal{E}$

But, for compact CYs, $\mathcal{E} \neq TX$, no one knows the f_d for $d \neq 0$.

Research problem: find the f_d.

In the case $X = \mathbb{P}^4[5]$:

Unknown

$\mathscr{E} \neq TX$

Unknown

Known
($\mathscr{E} = TX$),
codim 224

$\mathscr{E} \neq TX$

Unknown

Unknown

$\mathscr{E} \neq TX$

Unknown

Unknown

Unknown

$\mathscr{E} \neq TX$

Summary so far:

In general,

$$\overline{\mathbf{27}}^{\mathbf{3}} \; = \; f_0(X, \mathscr{E}) \; + \; \sum_{d \neq 0} f_d(X, \mathscr{E}) q^d$$

and in the special case $\mathscr{E} = TX$, the f_d encode GW inv'ts.

Example: $X = \mathbb{P}^4[5]$, $\mathscr{E} = TX$

$$\overline{\mathbf{27}}^{\mathbf{3}} \; = \; 5 + \sum_{k=1}^{\infty} \frac{n_k k^3 q^k}{1 - q^k} \; = \; 5 + 2875\, q + 4876875\, q^2 + \cdots$$

n_k = Gromov-Witten invariants

Goal: compute the f_d in general (for any $\mathscr{E}$, not just TX).

Math: encode a generalization of Gromov-Witten invariants.

Physics: encode the nonpert' parts of Yukawa couplings

I realize I've been a bit vague as I set up this problem, and I'll try to fix that shortly.

We don't know how to compute the f_d for $d \neq 0$ in compact Calabi-Yau's, as relevant to string theory, but we do know how to compute them in toy models on (some) Fano spaces.

In fact, what I'll outline in this case are relations between the f_d defined by a generalization of quantum cohomology known as *quantum sheaf cohomology*.





S. Katz R. Donagi I. Melnikov J. McOrist J. Distler
plus W. Gu, J. Guo, H. Zou, Z. Chen, Z. Lu, A. Adams, M. Ernebjerg, and others

Review of quantum sheaf cohomology

Briefly, on a space X (not necessarily Calabi-Yau)
with hol' vector bundle $\mathscr{E} \to X$
such that $\wedge^{\mathrm{top}} \mathscr{E}^* \cong K_X$, $\mathrm{ch}_2(\mathscr{E}) = \mathrm{ch}_2(TX)$,

there's a product

$$H^{q_1}(X, \wedge^{p_1} \mathscr{E}^*) \times H^{q_2}(X, \wedge^{p_2} \mathscr{E}^*) \longrightarrow H^{q_1+q_2}(X, \wedge^{p_1+p_2} \mathscr{E}^*)$$

Quantum sheaf cohomology is a quantum deformation,
just as quantum cohomology is a quantum-deformed version
of (ordinary) cohomology.

In the special case $\mathscr{E} = TX$, reduces to ordinary $QH^\bullet(X)$.

It arises as relations between `correlation functions'….

Review of quantum sheaf cohomology

The superpotential terms

$$f_0(X, \mathscr{E}) \; + \; \sum_{d \neq 0} f_d(X, \mathscr{E}) q^d \qquad\qquad f_0 \; = \; \int_X \omega_1 \cup \omega_2 \cup \omega_3$$

are `correlation functions' in a QFT on the string worldsheet:

$$f_0 + \sum_{d \neq 0} f_d q^d \; = \; \langle \mathscr{O}_1 \mathscr{O}_2 \mathscr{O}_3 \rangle \; = \; \sum_d q^d \langle \mathscr{O}_1 \mathscr{O}_2 \mathscr{O}_3 \rangle_d$$

Aside for experts:

For $\mathscr{E} = TX$, these are computed in the A, B model topological field theories.

For more general $\mathscr{E}$, these are computed in generalizations known as the A/2, B/2 models.

Classical computations ($\mathscr{E} = TX$):

For X a space, not necessarily Calabi-Yau,

$$f_0 = \langle \mathscr{O}_1 \cdots \mathscr{O}_k \rangle_0 = \int_X \omega_1 \wedge \cdots \wedge \omega_k$$

for 'operators' $\mathcal{O}_i \sim \omega_i \in H^{p_i, q_i}(X)$

This (classical contribution to the) correlation function is nonzero when

$$\sum p_i = \dim X = \sum q_i$$

ie, when $\omega_1 \wedge \cdots \wedge \omega_k$ is a top-form.

Classical computations ($\mathscr{E} \neq TX$):

For X a space, $\mathcal{E} \to X$ a hol' vector bundle s.t.

$$\wedge^{\mathrm{top}} \mathcal{E}^* \cong K_X, \quad \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX)$$

$$f_0 = \langle \mathscr{O}_1 \cdots \mathscr{O}_k \rangle_0 = \int_X \omega_1 \wedge \cdots \wedge \omega_k$$

for `operators' $\mathcal{O}_i \sim \omega_i \in H^{q_i}(X, \wedge^{p_i} \mathcal{E}^*)$

Now, $\omega_1 \wedge \cdots \wedge \omega_k \in H^{\sum q_i}\left(X, \wedge^{\sum p_i} \mathcal{E}^*\right)$

In order for $\langle \mathscr{O}_1 \cdots \mathscr{O}_k \rangle_0$ to be a number,

require $\sum q_i = \dim X \qquad \sum p_i = \mathrm{rank}\, \mathcal{E}$

& use $\wedge^{\mathrm{top}} \mathcal{E}^* \cong K_X$

Computations $(\mathscr{E} = TX)$:

Schematically: For X a space,

$\mathscr{M}_d$ a space of holomorphic maps $S^2 \longrightarrow X$

we compute a "correlation function" in A model TFT

$$\langle \mathscr{O}_1 \cdots \mathscr{O}_k \rangle_d \;=\; q^d \int_{\mathscr{M}} \omega_1 \wedge \cdots \wedge \omega_k \,(\wedge c_{\mathrm{top}}(\mathrm{Obs}))$$

where $\mathscr{O}_i \sim \omega_i \in H^{p_i,q_i}(\mathscr{M}_d)$

$$= \; q^d \int_{\mathscr{M}_d} (\text{ top form on } \mathscr{M}_d)$$

which encodes minimal area surface information.

Such computations are at the heart of Gromov-Witten theory.

Computations ($\mathscr{E} \neq TX$):

Schematically: For X a space, $\mathcal{E}$ a bundle on X,

$\mathscr{M}_d$ a space of holomorphic maps S² —> X

$$\langle \mathscr{O}_1 \cdots \mathscr{O}_k \rangle_d = q^d \int_{\mathscr{M}_d} \tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_k \left(\wedge \Omega^n \right)$$

where $\mathcal{O}_i \sim \tilde{\omega}_i \in H^{q_i}\left(\mathcal{M}_d, \wedge^{p_i} \mathcal{F}^*\right)$ for $\mathcal{F} \equiv R^0 \pi_* \alpha^* \mathcal{E}$

where $\begin{array}{l} \pi : \Sigma \times \mathcal{M}_d \to \mathcal{M}_d \\ \alpha : \Sigma \times \mathcal{M}_d \to X \end{array}$

$\Omega \in H^1\left(\mathcal{M}_d, \mathcal{F}^* \otimes \mathcal{F}_1 \otimes (\mathrm{Obs})^*\right)$ $\quad \mathcal{F}_1 \equiv R^1 \pi_* \alpha^* \mathcal{E}$

$$\left. \begin{array}{c} \wedge^{\mathrm{top}} \mathcal{E}^* \cong K_X \\ \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX) \end{array} \right\} \stackrel{\mathrm{GRR}}{\Longrightarrow} \wedge^{\mathrm{top}} \mathcal{F}^* \otimes \wedge^{\mathrm{top}} \mathcal{F}_1 \otimes \wedge^{\mathrm{top}} (\mathrm{Obs})^* \cong K_{\mathcal{M}_d}$$

hence, again,

$$= q^d \int_{\mathscr{M}_d} \left(\text{ top form on } \mathscr{M}_d \right)$$

(S Katz, ES, 2004)

More succinctly, whereas for $\mathscr{E} = TX$, one computes *intersection theory* on a moduli space of curves,

for $\mathscr{E} \neq TX$ one computes *sheaf cohomology* on a moduli space of curves.

In the rest of this talk, I'm going to present results for correlation functions & quantum sheaf cohomology, but, it should be emphasized that computational methods for $\mathscr{E} \neq TX$ are still very primitive by comparison to what exists for GW theory.

Ring relations are encoded in relations between correlation functions.

Physics: Say $\mathcal{O}_A \mathcal{O}_B = \sum_i \mathcal{O}_i$ ("operator product")

if all correlation functions preserved:

$$\langle \mathcal{O}_A \mathcal{O}_B \mathcal{O}_C \cdots \rangle = \sum_i \langle \mathcal{O}_i \mathcal{O}_C \cdots \rangle$$

Math: if interpret correlation functions as maps

$$\mathrm{Sym}^\bullet W \longrightarrow \mathbb{C}$$

(where W is the space of $\mathcal{O}$'s)

then rel'ns are the kernel, of form $\mathcal{O}_A \mathcal{O}_B - \sum_i \mathcal{O}_i$

Concrete examples….

Example:

$$X = \mathbb{P}^1 \times \mathbb{P}^1 \qquad \mathscr{E} = TX$$

Space of operators = $W = H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{C}^2 = \mathbb{C}\{\psi, \tilde{\psi}\}$

Correlation functions:

$$\langle \psi \tilde{\psi} \rangle = 1 \qquad \langle \psi^2 \rangle = 0 = \langle \tilde{\psi}^2 \rangle$$

$$\langle \psi^3 \tilde{\psi} \rangle = q \qquad \langle \psi \tilde{\psi}^3 \rangle = \tilde{q}$$

$$\langle \psi^5 \tilde{\psi} \rangle = q^2 \qquad \langle \psi^3 \tilde{\psi}^3 \rangle = q\tilde{q} \qquad \langle \psi \tilde{\psi}^5 \rangle = \tilde{q}^2$$

$$\cdots$$

Pattern: $\psi^2 = q, \quad \tilde{\psi}^2 = \tilde{q}$

Example:

$$X = \mathbb{P}^1 \times \mathbb{P}^1 \qquad\qquad \mathscr{E} = TX$$

$$W = H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{C}^2 = \mathbb{C}\{\psi, \tilde{\psi}\}$$

Ring relations: $\psi^2 = q, \quad \tilde{\psi}^2 = \tilde{q}$

where $q, \tilde{q} \longrightarrow 0$ in classical limit.

Looks like a deformation of cohomology ring,
hence called “quantum cohomology”

Example 2:

$X = \mathbb{P}^1 \times \mathbb{P}^1$ $\mathscr{E}$ a deformation of $T(\mathbb{P}^1 \times \mathbb{P}^1)$

Def'n of $\mathscr{E}$: $0 \longrightarrow W^* \otimes \mathscr{O} \xrightarrow{*} \mathscr{O}(1,0)^2 \oplus \mathscr{O}(0,1)^2 \longrightarrow \mathscr{E} \longrightarrow 0$

where $* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$ A, B, C, D const' 2x2 matrices

$x, \tilde{x}$ vectors of homog' coord's

and $W = \mathbb{C}^2$

Special case: $\mathscr{E} = TX$ when $A = D = I_{2\times 2}, \quad B = C = 0$

Can show space of operators $= H^1(X, \mathscr{E}^*) = W = \mathbb{C}\{\psi, \tilde{\psi}\}$

Results for correlation functions….

Example 2:

$$X = \mathbb{P}^1 \times \mathbb{P}^1 \qquad \mathscr{E} \text{ a deformation of } T(\mathbb{P}^1 \times \mathbb{P}^1)$$

The correlation functions are computable,
and have some basic patterns, for example:

$$\langle \psi^2 \det(A\psi + B\tilde{\psi}) \rangle = q \langle \psi^2 \rangle \qquad \langle \psi^2 \det(C\psi + D\tilde{\psi}) \rangle = \tilde{q} \langle \psi^2 \rangle$$

& more gen'ly in 4-pt functions,

$$\langle f_2(\psi, \tilde{\psi}) \det(A\psi + B\tilde{\psi}) \rangle = q \langle f_2(\psi, \tilde{\psi}) \rangle$$

$$\langle f_2(\psi, \tilde{\psi}) \det(C\psi + D\tilde{\psi}) \rangle = \tilde{q} \langle f_2(\psi, \tilde{\psi}) \rangle$$

which (correctly) suggests that the ring relations are

$$\det(A\psi + B\tilde{\psi}) = q, \quad \det(C\psi + D\tilde{\psi}) = \tilde{q}$$

— These are the quantum sheaf cohomology rel'ns.

Summary so far:

Example 1: $X = \mathbb{P}^1 \times \mathbb{P}^1$ $\mathscr{E} = TX$

Rel'ns: $\psi^2 = q, \quad \tilde{\psi}^2 = \tilde{q}$

Example 2:

$X = \mathbb{P}^1 \times \mathbb{P}^1$ $\mathscr{E}$ a deformation of $T(\mathbb{P}^1 \times \mathbb{P}^1)$

Def'n of $\mathcal{E}$: $0 \longrightarrow W^* \otimes \mathcal{O} \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$

where $* = \left[\begin{array}{cc} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{array} \right]$ A, B, C, D const' 2x2 matrices
$x, \tilde{x}$ vectors of homog' coord's

Here, $W = H^1(X, \mathscr{E}*) = \mathbb{C}^2 = \mathbb{C}\{\psi, \tilde{\psi}\}$

Rel'ns: $\det\left(A\psi + B\tilde{\psi}\right) = q, \quad \det\left(C\psi + D\tilde{\psi}\right) = \tilde{q}$

quantum sheaf cohomology

Check: $\mathscr{E} = TX$ when $A = D = I_{2\times 2}, \quad B = C = 0$

& in this limit, rel'ns reduce to those of ordinary case

So far:

Outlined results for correlation functions in ordinary & heterotic cases, to illustrate how in general terms quantum corrected cohomology rings arise.

However, I have not yet explained how to compute those correlation functions, or derive q.s.c. more systematically.

That's next….

Quantum sheaf cohomology

Next, I'm going to illustrate how to explicitly compute quantum sheaf cohomology (qsc) on $X = \mathbb{P}^1 \times \mathbb{P}^1$.

Recall qsc describes relations between correlation functions.

Strategy:

Write $$\langle O \rangle \; = \; \sum_{a,b} \langle O \rangle_{(a,b)} \, q^a \tilde{q}^b$$

where $(a, b) \in H_2(\mathbb{P}^1 \times \mathbb{P}^1)$ index worldsheet instanton sectors,

I'll use diagram chasing to describe each $\langle O \rangle_{(a,b)}$ as a map,

$$\langle - \rangle_{(a,b)} : \mathrm{Sym}^{\bullet}(W) \to H^{\mathrm{top}} \left(\wedge^{\mathrm{top}} \mathscr{F}^{\mathrm{top}} \right)$$

and compute its kernel.

(Yoga: QFT as homological algebra)

Quantum sheaf cohomology

Let's consider the

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$ (degree (0,0))

with gauge bundle $\mathcal{E}$ a deformation of the tangent bundle:

$$0 \longrightarrow W^* \otimes \mathcal{O} \xrightarrow{*} \underbrace{\mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2}_{Z^*} \longrightarrow \mathcal{E} \longrightarrow 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Operators counted by $H^1(\mathcal{E}^*) = H^0(W \otimes \mathcal{O}) = W$

n-pt correlation function is a map $\mathrm{Sym}^n H^1(\mathcal{E}^*) = \mathrm{Sym}^n W \longrightarrow H^n(\wedge^n \mathcal{E}^*)$

Ring relations = kernel

Plan: study map corresponding to classical corr' f'n

Quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

with gauge bundle $\mathscr{E}$ a deformation of the tangent bundle:

$$0 \longrightarrow W^* \otimes \mathscr{O} \xrightarrow{*} \underbrace{\mathscr{O}(1,0)^2 \oplus \mathscr{O}(0,1)^2}_{Z^*} \longrightarrow \mathscr{E} \longrightarrow 0$$

where $* = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix}$ $\quad x, \tilde{x}$ homog' coord's on $\mathbb{P}^1$'s

and $W = \mathbb{C}^2$

Since this is a rk 2 bundle, classical sheaf cohomology defined by products of 2 elements of $H^1(\mathscr{E}^*) = H^0(W \otimes \mathscr{O}) = W$.

So, we want to study map $H^0(\mathrm{Sym}^2 W \otimes \mathscr{O}) \longrightarrow H^2(\wedge^2 \mathscr{E}^*) =$ corr' f'n

This map is encoded in the resolution

$$0 \longrightarrow \wedge^2 \mathscr{E}^* \longrightarrow \wedge^2 Z \longrightarrow Z \otimes W \longrightarrow \mathrm{Sym}^2 W \otimes \mathscr{O} \longrightarrow 0$$

Quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \longrightarrow \wedge^2 \mathscr{E}^* \longrightarrow \wedge^2 Z \longrightarrow Z \otimes W \longrightarrow \mathrm{Sym}^2 W \otimes \mathscr{O} \longrightarrow 0$$

Break into short exact sequences:

$$0 \longrightarrow \wedge^2 \mathscr{E}^* \longrightarrow \wedge^2 Z \to S_1 \to 0$$

$$0 \to S_1 \longrightarrow Z \otimes W \longrightarrow \mathrm{Sym}^2 W \otimes \mathscr{O} \longrightarrow 0$$

Examine second sequence:

induces $\underbrace{H^0(Z \otimes W)}_{0} \to H^0(\mathrm{Sym}^2 W \otimes O) \xrightarrow{\delta} H^1(S_1) \to \underbrace{H^1(Z \otimes W)}_{0}$

Since Z is a sum of $\mathscr{O}(-1,0)$'s, $\mathscr{O}(0,-1)$'s,

hence $\delta : H^0(\mathrm{Sym}^2 W \otimes \mathscr{O}) \xrightarrow{\sim} H^1(S_1)$ is an iso.

Next, consider the other short exact sequence at top….

Quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \longrightarrow \wedge^2 \mathscr{E}^* \longrightarrow \wedge^2 Z \longrightarrow Z \otimes W \longrightarrow \mathrm{Sym}^2 W \otimes \mathcal{O} \longrightarrow 0$$

Break into short exact sequences:

$$0 \to S_1 \longrightarrow Z \otimes W \longrightarrow \mathrm{Sym}^2 W \otimes \mathcal{O} \longrightarrow 0$$

$$\delta : H^0(\mathrm{Sym}^2 W \otimes \mathcal{O}) \xrightarrow{\sim} H^1(S_1)$$

Examine other sequence:

$$0 \longrightarrow \wedge^2 \mathscr{E}^* \longrightarrow \wedge^2 Z \to S_1 \to 0$$

induces $H^1(\wedge^2 Z) \longrightarrow H^1(S_1) \xrightarrow{\delta} H^2(\wedge^2 \mathscr{E}^*) \longrightarrow H^2(\wedge^2 Z)$

0

Since Z is a sum of $\mathcal{O}(-1,0)$'s, $\mathcal{O}(0,-1)$'s,

$H^2(\wedge^2 Z) = 0$ but $H^1(\wedge^2 Z) = \mathbb{C} \oplus \mathbb{C}$

and so $\delta : H^1(S_1) \to H^2(\wedge^2 \mathscr{E}^*)$ has a 2d kernel.

Now, assemble the coboundary maps….

Quantum sheaf cohomology

Example: classical sheaf cohomology on $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \longrightarrow \wedge^2 \mathscr{E}^* \longrightarrow \wedge^2 Z \longrightarrow Z \otimes W \longrightarrow \mathrm{Sym}^2 W \otimes \mathscr{O} \longrightarrow 0$$

Now, assemble the coboundary maps….

A classical (2-pt) correlation function is computed as

$$H^0(\mathrm{Sym}^2 W \otimes \mathscr{O}) \xrightarrow[\delta]{\sim} H^1(S_1) \xrightarrow[\delta]{} H^2(\wedge^2 \mathscr{E}^*)$$

where the right map has a 2d kernel, which one can show is generated by

$$\det(A\psi + B\tilde{\psi}), \quad \det(C\psi + D\tilde{\psi})$$

where A, B, C, D are four matrices defining the def' E, and $\psi, \tilde{\psi}$ correspond to elements of a basis for W.

Classical sheaf cohomology ring:

$$\mathbb{C}[\psi, \tilde{\psi}] / \left(\det(A\psi + B\tilde{\psi}), \det(C\psi + D\tilde{\psi})\right)$$

Quantum sheaf cohomology

Now, consider maps of nonzero degree.

Instanton sectors have the same form,
except X replaced by moduli space $\mathscr{M}$ of instantons,
$\mathscr{E}$ replaced by induced sheaf $\mathscr{F} \cong R^0 \pi_* \alpha^* \mathscr{E}$
over moduli space $\mathscr{M}$.

$$\left.\begin{array}{l} \wedge^{\mathrm{top}} \mathscr{E}^* \cong K_X \\ \mathrm{ch}_2(\mathscr{E}) = \mathrm{ch}_2(TX) \end{array}\right\} \stackrel{\mathrm{GRR}}{\Longrightarrow} \wedge^{\mathrm{top}} \mathscr{F}^* \cong K_{\mathscr{M}}$$

Must compactify $\mathscr{M}$ to $\overline{\mathscr{M}}$,
and extend $\mathscr{F}$ over compactification divisor.

Within any one sector, can follow the same method just outlined….

Within any one sector, can follow the same method just outlined….

Example: Consider degree (1,0) maps.

$\overline{\mathscr{M}}_{(1,0)} = \mathbb{P}^3 \times \mathbb{P}^1$

Simple description of induced sheaf $\mathscr{F} \to \overline{\mathscr{M}}$:

$$\mathscr{E} = \mathscr{O}(a,b) \ \mapsto \ \mathscr{F} = H^0(\Sigma, \mathscr{O}(ad_1 + bd_2)) \otimes \mathscr{O}(a,b)$$

where Σ = worldsheet = $\mathbb{P}^1$ today

Similarly, a map $\mathscr{O}(a_1,b_1) \to \mathscr{O}(a_2,b_2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$

induces a map

$$H^0(\Sigma, \mathscr{O}(a_1d_1 + b_1d_2)) \otimes \mathscr{O}(a_1,b_1) \longrightarrow H^0(\Sigma, \mathscr{O}(a_2d_1 + b_2d_2)) \otimes \mathscr{O}(a_2,b_2)$$

on $\overline{\mathscr{M}}$.

Within any one sector, can follow the same method just outlined….

Example: Consider degree (1,0) maps.

$\overline{\mathcal{M}}_{(1,0)} = \mathbb{P}^3 \times \mathbb{P}^1$

Simple description of induced sheaf $\mathcal{F} \to \overline{\mathcal{M}}$:

Putting this together,

$$0 \longrightarrow W^* \otimes \mathcal{O} \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

on $\mathbb{P}^1 \times \mathbb{P}^1$ induces

$$0 \longrightarrow W^* \otimes \mathcal{O} \xrightarrow{*} \mathcal{O}(4,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{F} \longrightarrow 0$$

on $\overline{\mathcal{M}}$.

Can show the result has desired property $\wedge^{\mathrm{top}} \mathcal{F}^* \cong K_{\overline{\mathcal{M}}}$.

Example: Consider degree (1,0) maps.

$$0 \longrightarrow W^* \otimes \mathscr{O} \xrightarrow{*} \underbrace{\mathscr{O}(4,0)^2 \oplus \mathscr{O}(0,1)^2}_{Z^*} \longrightarrow \mathscr{F} \longrightarrow 0$$

implies

$$0 \longrightarrow \wedge^4 \mathscr{F}^* \longrightarrow \wedge^4 Z \longrightarrow \wedge^3 Z \otimes W \longrightarrow \wedge^2 Z \otimes \mathrm{Sym}^2 W$$
$$\longrightarrow Z \otimes \mathrm{Sym}^3 W \longrightarrow \mathrm{Sym}^4 W \otimes \mathscr{O} \longrightarrow 0$$

giving a map

$$\mathrm{Sym}^4 W = H^0(\overline{\mathscr{M}}, \mathrm{Sym}^4 W \otimes \mathscr{O}) \longrightarrow H^4(\overline{\mathscr{M}}, \wedge^4 \mathscr{F}^*)$$

with kernel

$$\left((\det(A\psi + B\tilde{\psi}))^2, \det(C\psi + D\tilde{\psi}) \right)$$

This represents the correlation function $\langle - \rangle_{(1,0)}$ in degree (1,0).

Quantum sheaf cohomology

In the case of our example,
one can show that in a sector of instanton degree (a,b),
the kernel of corr' fn's $\langle \cdots \rangle_{(a,b)}$ is of the form

$$\left(Q^{a+1}, \tilde{Q}^{b+1} \right) \subset \mathrm{Sym}^{\bullet}(W)$$

where $\quad Q = \det(A\psi + B\tilde{\psi})\,, \quad \tilde{Q} = \det(C\psi + D\tilde{\psi})$

Now, rel'ns can relate correlation functions in different instanton degrees, and so, should map ideals to ideals.

To be compatible with those ideals,

$$\langle O \rangle_{a,b} = q^{a'-a} \tilde{q}^{b'-b} \langle O Q^{a'-a} \tilde{Q}^{b'-b} \rangle_{a',b'}$$

for some constants $q, \tilde{q}$ => rel'ns $\quad Q = q, \;\; \tilde{Q} = \tilde{q}$

— quantum sheaf cohomology rel'ns

Quantum sheaf cohomology

General result:

(Math: Donagi, Guffin, Katz, ES, '11)
(Physics: McOrist, Melnikov '08)

For any toric variety, and any def' $\mathscr{E}$ of its tangent bundle,

$$0 \longrightarrow W^* \otimes \mathcal{O} \xrightarrow{*} \bigoplus \mathcal{O}(\vec{q}_i) \longrightarrow \mathscr{E} \longrightarrow 0$$

the ring rel'ns are

$$\Pi_\alpha \left(\det M_{(\alpha)} \right)^{Q^a_\alpha} = q_a$$

where the M's are matrices of chiral operators built from *.

Generalizes Batyrev's ring $\displaystyle \prod_i \left(\sum_b Q^b_i \psi_b \right)^{Q^a_i} = q_a$

Summary of quantum sheaf cohomology:

Results exist for X = toric variety, Grassmannian, flag variety, for $\mathscr{E}$ a deformation of TX.

Open problem:

- $\mathscr{E}$ not a deformation of TX

But the real goal isn't these toy models, it's compact Calabi-Yau's.
Let's go back to those….

Another approach: mirror symmetry

Historically, the f_k were computed using mirror symmetry.

Mirror symmetry (for pairs of CY's) has a generalization, known as (0,2) mirror symmetry, which involves pairs (space, bundle).

We say $(X, \mathscr{E} \to X)$ and $(Y, \mathscr{F} \to Y)$ are (0,2) mirror if they are indistinguishable to a (heterotic) string, meaning that they define the same SCFT.

Properties:

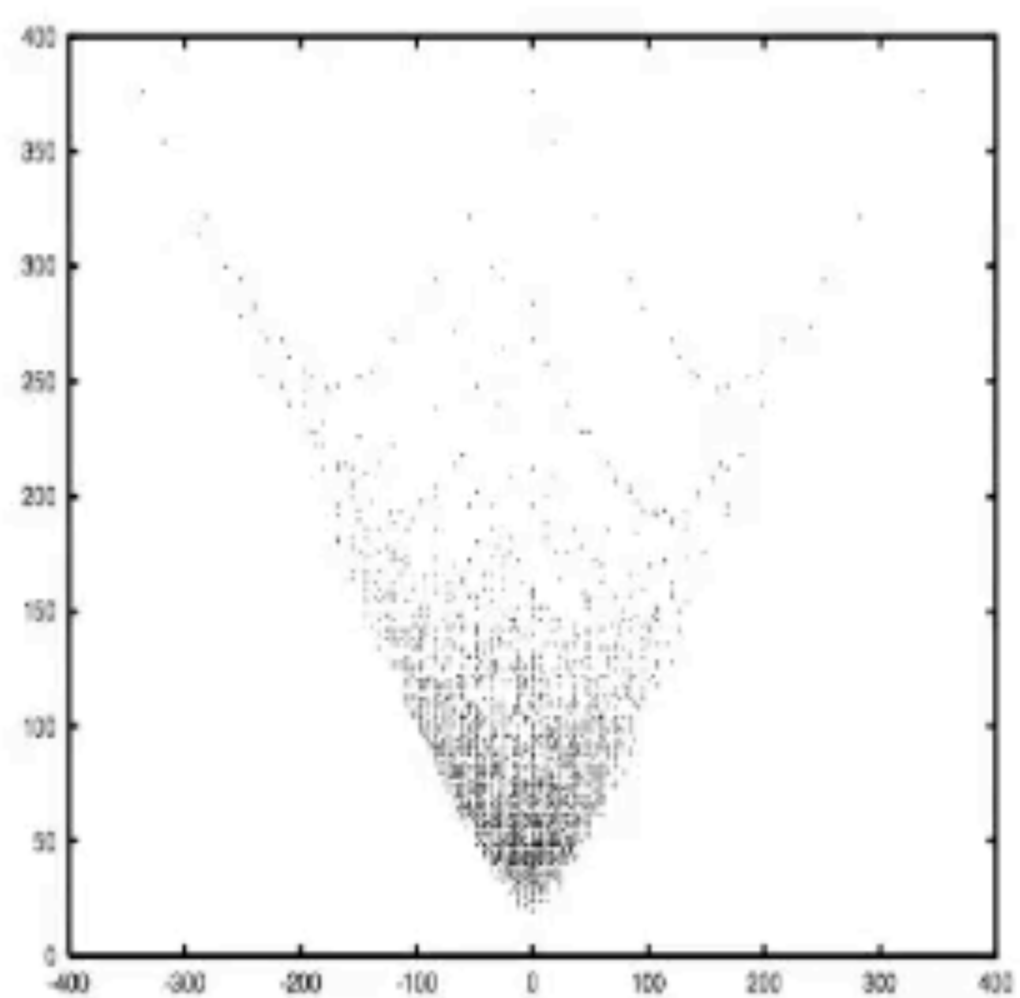
Ordinary mirror symm'	*(0,2) mirror symm'*
X, Y mirror	$(X,\mathcal{E}),(Y,\mathcal{F})$ mirror
$h^{p,q}(X) = h^{n-p,q}(Y)$	$h^{p}(X,\wedge^{q}\mathcal{E}^{*}) = h^{p}(Y,\wedge^{q}\mathcal{F})$
$\dim X = \dim Y$	$\dim X = \dim Y$ $\mathrm{rk}\,\mathcal{E} = \mathrm{rk}\,\mathcal{F}$
$\text{cpx moduli}\, X$ $= \text{Kähler moduli}\, Y$	$\{\text{cpx, Kähler, bdle moduli}\}(X,\mathcal{E})$ $= \{\text{cpx, Kähler, bdle moduli}\}(Y,\mathcal{F})$

In the special case $\mathcal{E} = TX$, (0,2) becomes ordinary.

Why in the world would we believe this exists?

Why in the world would we believe this exists?

Numerical evidence for (0,2) mirror symmetry:



(Blumenhagen, Schimmrigk, Wisskirchen, NPB 486 ('97) 598-628)

Horizontal: $h^1(\mathcal{E}) - h^1(\mathcal{E}^*)$

Vertical: $h^1(\mathcal{E}) + h^1(\mathcal{E}^*)$

where $\mathcal{E}$ is rk 4

Unlike ordinary mirror symmetry,
which is now well-understood,
(0,2) mirror symmetry is still under development.

Briefly, in special cases, some constructions exist,
but not understood to nearly the same extent
as ordinary mirror symmetry.

$\mathscr{E} \neq TX$

heterotic Gromov-Witten,
quantum sheaf cohomology,
(0,2) mirror symmetry

$\mathscr{E} = TX$

Gromov-Witten,
quantum cohomology,
mirror symmetry

Summary

- Outlined charged matter interactions
- Open problem: heterotic Gromov-Witten invariants

  (= quantum-corrected Yukawa couplings)

- Toy model: quantum sheaf cohomology

For toric varieties, Grassmannians, & flag mflds,
and $\mathscr{E}$ a deformation of the tangent bundle,
we can compute explicitly.

All other cases are open problems.

Thank you for your time!

** Extra slides **

Grassmannians

Let me quickly outline results for q.s.c. rings for Grassmannians.

(J Guo, Z Lu, ES, 1512.08586 & 1605.01410)

On G(k,n), the Grassmannian of k-planes in $\mathbb{C}^n$,

for 1 < k < n-1, the tangent bundle has moduli:

$$h^1(G(k,n), \mathrm{End}\, T) \; = \left\{ \begin{array}{cc} n^2 - 1 & 1 < k < n-1 \\ 0 & \mathrm{else} \end{array} \right.$$

We'll deform the tangent bundle,
and describe the resulting q.s.c. ring.

Deformations of tangent bundle of G(k,n)

The tangent bundle itself can be represented as the cokernel

$$0 \longrightarrow S^* \otimes S \xrightarrow{*} \mathcal{O}^n \otimes S^* \longrightarrow T \longrightarrow 0$$

We can encode a deformation $\mathcal{E}$ of the tangent bundle by modifying the map *.

$$* : \omega_\alpha^\beta \mapsto A^i_j \omega^\beta_\alpha x^j_\beta + B^i_j \omega^\beta_\beta x^j_\alpha$$

where the x^i_α are Stiefel coordinates, and S is the universal subbundle.

The tangent bundle arises in the special case that
A = I,   B = 0.

So long as A invertible, can perform GL(n) rotation to eliminate, so moduli are in (traceless part of) B.

Given a deformation $\mathcal{E}$ of T,

- we don't have a mathematical derivation/def'n of the quantum sheaf cohomology ring,

  but

- we can use physics computations to determine its form.

Since this is a mostly math audience, I'll spare you the physics details, and instead outline the results.

Structure of quantum sheaf cohomology ring for a generic deformation of T G(k,n)

$$\mathbb{C}[\sigma_{(1)}, \sigma_{(2)}, \cdots]/\Big\langle D_{k+1}, D_{k+2}, \cdots, R_{(n-k+1)}, \cdots, R_{(n-1)},$$
$$R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, R_{(n+2)} + q\sigma_{(2)}, \cdots \Big\rangle$$

where
$$D_m \ = \ \det\left(\sigma_{(1+j-i)}\right)_{1 \leq i,j \leq m}$$
$$R_{(r)} \ = \ \sum_{i=0}^{\min(r,n)} I_i \sigma_{(r-i)} \sigma_{(1)}^i$$

for I_i the char' poly's of B: $\quad \det(tI + B) \ = \ \sum_{i=0}^{n} I_{n-i} t^i$

Exs: $\quad I_0 = 1, \quad I_1 = \mathrm{Tr}\, B, \quad I_n = \det B$

Quantum sheaf cohomology ring:

$$\mathbb{C}[\sigma_{(1)}, \sigma_{(2)}, \cdots]/\Big\langle D_{k+1}, D_{k+2}, \cdots, R_{(n-k+1)}, \cdots, R_{(n-1)},$$
$$R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, R_{(n+2)} + q\sigma_{(2)}, \cdots \Big\rangle$$

If we turn off the deformation (set B=0), then

$$R_{(n)} = \sigma_{(n)}$$

and with some work it can be shown that the ring above can be presented as

$$\mathbb{C}[\sigma_{(1)}, \cdots, \sigma_{(n-k)}]/\langle D_{k+1}, \cdots, D_{n-1}, D_n + (-)^n q\rangle$$

which is a standard presentation of the (ordinary) quantum cohomology ring of G(k,n).

(Buch, Kresch, Tamvakis, Bertram, Witten, Siebert, Tian, ….)

Example: G(1,3)

This has no nontrivial deformations, so any result should be equivalent to ordinary quantum cohomology ring of $\mathbb{P}^2$.

$$\mathbb{C}[\sigma_{(1)}, \sigma_{(2)}, \cdots]/\langle D_2, \cdots, R_{(3)} + q, R_{(4)} + q\sigma_{(1)}, \cdots \rangle$$

which $\quad = \mathbb{C}[\sigma_{(1)}]/\langle R_{(3)} + q \rangle \rangle$

using $\quad D_2 = \sigma_{(1)}^2 - \sigma_{(2)}, \cdots$ to eliminate $\sigma_{(m)}$ for m>1, and

the result $\quad R_{(3+\ell)} + q\sigma_{(\ell)} = \sigma_{(\ell)}(R_{(3)} + q)$

Now, $\quad R_{(3)} = \sum_{i=0}^{3} I_i \sigma_{(3-i)} \sigma^i = \left(\sum_{i=0}^{3} I_i \right) \sigma^3 = (\det(I+B))\, \sigma^3$

so the qsc ring is $\quad \mathbb{C}[\sigma]/\langle \det(I+B)\sigma^3 + q \rangle$

which is equivalent to std quantum cohomology ring.

G(1,n), G(n-1,n) admit no deformations and so their q.s.c. rings coincide with ordinary q.c. rings

However, for 1 < k < n-1,
the q.s.c. ring of a def' of G(k,n) is not the same as the ordinary q.c. ring.

The description of the q.s.c. ring given is valid generically.

Breaks down along discriminant locus, where bundle degenerates.

This turns out to be the locus where, on G(k,n), B has k eigenvalues whose sum is -1.