Radio Astronomy Fundamentals II

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Spring 2010

The "signal" from an astronomical radio source is hard to distinguish from the random thermal noise present in a receiver. Power measurements of both, as a function of time, would look like randomly varying functions, with some mean and some standard deviation. The power when pointing at a radio source might just be a little stronger, and therefore noticeable in beam switching, or drift scans, or it might be noticeable in some part of the spectrum (e.g., thermal noise has a "white noise" spectrum --- it has a "flat" spectrum, while a 21cm emission line source will certainly not have a flat spectrum).

The statistical nature of both the signal and the noise are the same. Stated in terms of voltage measured in the system, the voltage varies with a random amplitude and phase, with frequencies limited by the bandpass of the system (of bandwidth B). To take account of the randomly varying phase, one often mathematically describes the voltage as made up of real and imaginary components, each of which varies as an independent Gaussian variable of zero mean. The variations of the amplitude of the voltage can be described by a Rayleigh distribution (the Maxwell-Boltzmann distribution of speeds in a gas is a Rayleigh distribution). Finally, received power is proportional to the square of the voltage, so the power follows an exponential distribution, which has an rms equal to the mean value.

An observation is done by "integrating" over some integration time τ, i.e., averaging the power over time τ. The averaged power for the observation is an estimate (or measurement) of the true mean power. The uncertainty in the measurement will be reduced for longer integration times, as usual for measurements of means. For a bandwidth of B, the time scale on which the output power varies is $1/B$. Therefore averaging over time τ means averaging about $N = B\tau$ independent measurements of the power. Thus the uncertainty in the average is equal to rms of the output power, reduced by the factor $1/\sqrt{B\tau}$. Because the output power has an exponential probability distribution, the rms is equal to the mean. Finally, since powers can be expressed as equivalent noise temperatures T, and the system temperature T_{sys} is the power being measured, this result can be written as

$$\sigma_{\overline{T}} = \frac{\overline{T}}{\sqrt{B\tau}} = \frac{T_{sys}}{\sqrt{B\tau}}$$

where $\sigma_{\overline{T}}$ is the rms in the measured mean temperature, thus the uncertainty in the measured mean temperature. This equation is the so-called "radiometer equation."

The signal you want to measure is the antenna temperature T_A --- the contribution to the system temperature due to the radio source you are observing. The system temperature has contributions due to the receiver temperature, the sky temperature, and a contribution due to the source. Some or one of these contributions may dominate the system temperature. The signal-to-noise ratio is

$$\frac{T_A}{\sigma_{\overline{T}}} = \frac{T_A}{T_{sys}/\sqrt{B\tau}}$$

Note that the signal-to-noise ratio for a measurement is proportional to $\sqrt{B\tau}$, and thus the usual result has been obtained: the longer the integration time, and/or the larger the bandwidth, the larger is the signal-to-noise ratio (this is true in optical astronomy also). The following pages give the physical and mathematical details of these concepts.

Radio Astronomy

Signal Detection and Noise

As in all measurements, in optical astronomy or radio astronomy (or any quantitative science) the signal-to-noise ratio determines the "quality" of the measurement, or detection.

Radio source (in astronomy)

$\cong$ ∞ collection of oscillators $\Rightarrow$ EM waves

random amplitudes
random frequencies
random phases

Thermal noise produced by receiver

$\simeq$ same sort of thing

How to distinguish the two?

- beam switching

- drift scans

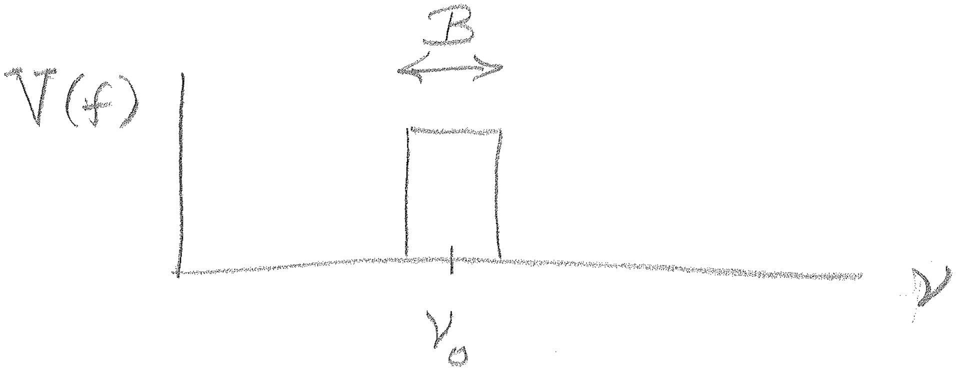
- spectral details
  (thermal noise is "white" = flat spectrum)

... all these measure "power" or "antenna temperature"

$\propto$ power.

Antennas/receivers respond to voltage which is converted to power by squaring ("square-law detector")

Receiver set up to receive:



... a finite bandpass of bandwidth B centred on frequency ν_0.

Consider one frequency in the bandpass : ν_0

$$v_{\nu_0}(t) = v(t)\left[\cos(2\pi\nu_0 t + \phi(t))\right]$$

amplitude, changes in time, slower than ν_0

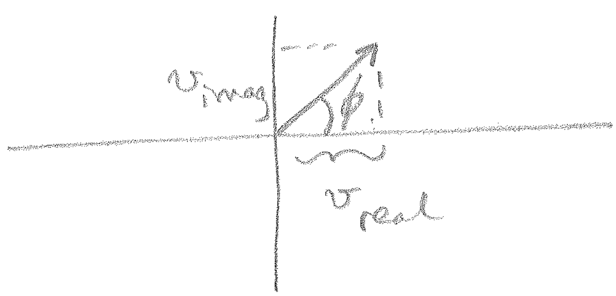
changes in time also

or,

$$v_{\nu_0}(t) = \text{Real}\left\{ \underset{\sim}{v}(t)\, e^{i 2\pi \nu_0 t} \right\}$$

where

$$\underset{\sim}{v}(t) = v_{real}(t) + i\, v_{imag}(t)$$



Both v_{real} and v_{imag} <u>vary randomly</u> with a <u>Gaussian</u> <u>distribution</u>, e.g.,

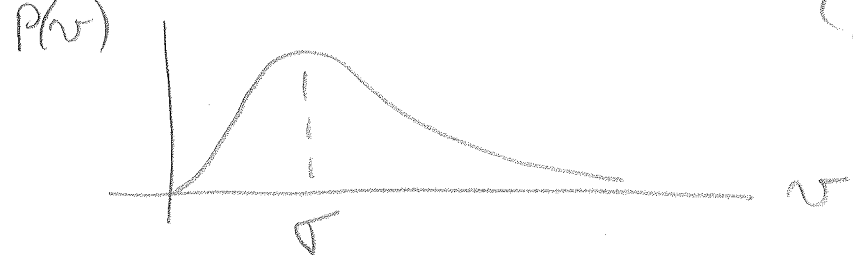
$$P(v_{real}) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{v_{real}}{2\sigma^2}\right)$$

look at $v = \sqrt{v_{real}^2 + v_{imag}^2}$

$$P(v) = \frac{v}{\sigma^2} e^{-v^2/2\sigma^2}$$

(same σ as above)

"Rayleigh distribution"



(like Maxwell-Boltzmann dist. of speeds)

Power $I = \left[v_{\nu_0}(t) \right]^2$ = output signal

$$I = v^2(t) \cos^2 \left[2\pi\nu_0 t + \phi(t) \right]$$

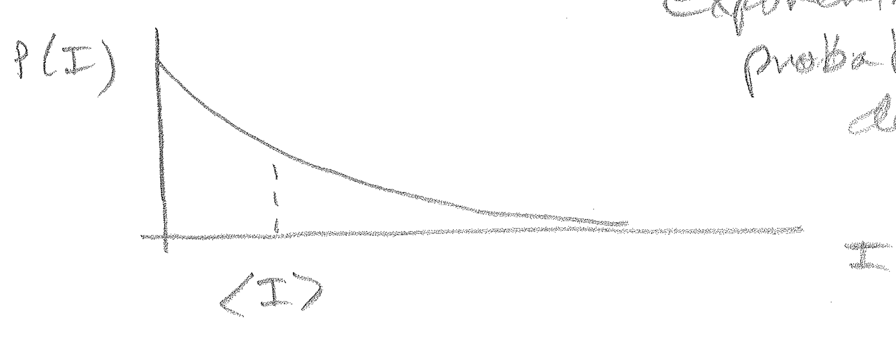
$$= \frac{1}{2} v^2(t) \left[1 + \underbrace{\cos 2(2\pi\nu_0 t + \phi(t))}_{\text{fast variation, integrated away during observation}} \right]$$

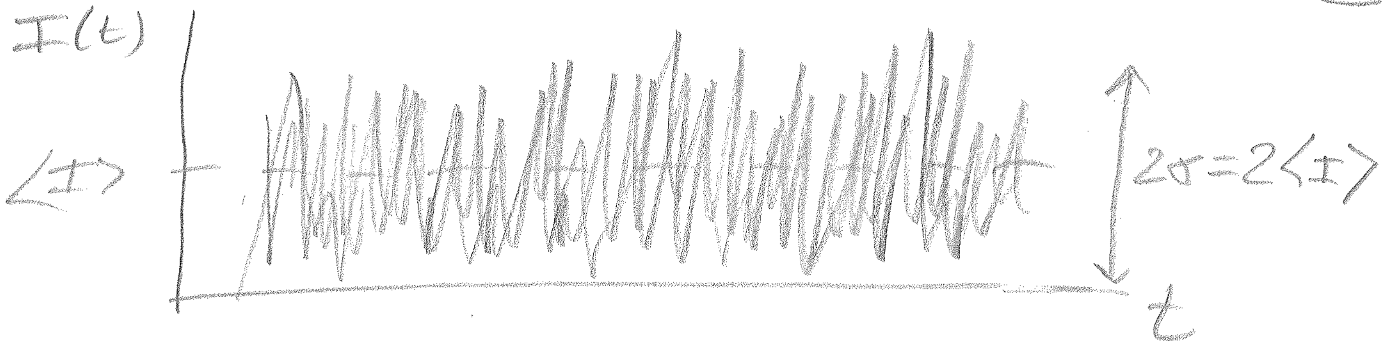
$$\therefore \quad I = \frac{1}{2} v^2(t)$$

$$P(I) = \frac{1}{\langle I \rangle} e^{-I/\langle I \rangle}$$

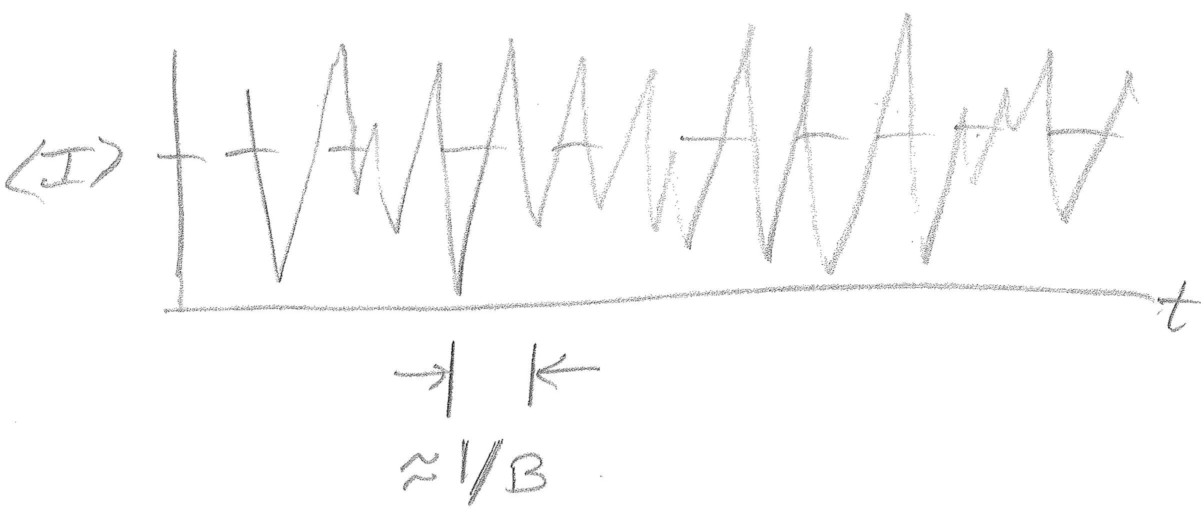
and $\sigma_I = \langle I \rangle = 2\sigma^2$

"exponential probability density"





In reality, a range of frequencies contribute (i.e., different $\mathcal{E}_\nu(t)$ add, smoothing the result):



An observation = an "integration" over time τ
= an average over time τ, producing

$$\overline{I} = \frac{1}{N} \sum_{i=1}^{N} I_i$$

= an estimate of $\langle I \rangle$, the mean of I.

During time τ, there are

$$N \simeq \frac{\tau}{1/B} = B\tau$$

independent samples of I that go into calculating $\overline{I}$.

Therefore, uncertainty in estimate of $\langle I \rangle$ is

$$\sigma_{\overline{I}} \simeq \frac{\sigma_I}{\sqrt{B\tau}} \simeq \frac{\overline{I}}{\sqrt{B\tau}}$$

where that last equality is due to the nature of exponential probability dist, i.e., $\sigma_I = \langle I \rangle$.

In terms of temperatures

$$T \propto I$$

So,

$$\sigma_T \simeq \frac{\sigma_I}{\sqrt{B\tau}} \simeq \frac{I}{\sqrt{B\tau}}$$

often written as

$$\sigma_T \simeq \frac{T_{sys}}{\sqrt{B\tau}}$$ "radiometer equation"

$T_{sys} = T_{receiver} + T_{sky} + T_{\text{due to source}} + \cdots$

often one of these terms dominates the sum.

The signal one wants to measure is

$T_{\text{due to source}} = T_A$ ("antenna Temp.", due to source)

Signal-to-noise ratio is

$$\frac{T_A}{\sigma_T} = \frac{T_A}{T_{sys}/\sqrt{B\tau}} \propto \sqrt{B\tau}$$