Some new torsional local models for heterotic strings

Teng Fei

Columbia University

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Overview

1. Background and Motivation
2. Calabi-Gray manifolds
   - Construction
   - Geometry of Calabi-Gray manifolds
   - Degenerate solutions
   - A new geometric description of Calabi-Gray manifolds
3. Non-perturbative solutions on a class of noncompact CY3
   - Construction
   - Strategy for finding solutions
   - A concrete example
4. Conclusion

History

- Candelas-Horowitz-Strominger-Witten'85: compactification of superstrings on metric product of 6-manifolds and maximally symmetric spacetime.
  Flux $H = 0 \rightsquigarrow$ Ricci-flat Kähler CYs
- Strominger'86 (Hull'86): compactification of heterotic superstrings on warp product of 6-manifolds and maximally symmetric spacetime.
  Flux $H \neq 0 \rightsquigarrow$ Strominger system
  Key ingredients in derivation:
  - $\mathcal{N} = 1$ SUSY
  - Green-Schwarz anomaly cancellation

Strominger system

- (X, ω, Ω): Hermitian 3-fold with canonical bundle globally trivialized by Ω
- $(E, h) \to X$: holomorphic Hermitian vector bundle
- R, F: curvature forms of $T^{1,0}X$ and E
  Note: ambiguity in choice of connections, usually use Chern connection for E and certain Hermitian connection for $T^{1,0}X$.
- $\alpha' > 0$ coupling constant

The Strominger system consists of three equations

$$
\begin{aligned}
F \wedge \omega^2 = 0, \quad F^{0,2} = F^{2,0} = 0,\\
i\partial\overline{\partial}\omega = \frac{\alpha'}{4}(\mathrm{Tr}(R \wedge R) - \mathrm{Tr}(F \wedge F)),\\
\mathrm{d}(\|\Omega\|_\omega \cdot \omega^2) = 0.
\end{aligned}
$$

The system makes sense for non-Kähler backgrounds!

Observations

Definition

On a complex n-fold, a Hermitian metric ω is called a *balanced metric* if $\mathrm{d}(\omega^{n-1}) = 0$.

$$F \wedge \omega^2 = 0, \quad F^{0,2} = F^{2,0} = 0, \tag{1}$$

$$i\partial\overline{\partial}\omega = \frac{\alpha'}{4}(\mathrm{Tr}(R \wedge R) - \mathrm{Tr}(F \wedge F)), \tag{2}$$

$$\mathrm{d}(\|\Omega\|_\omega \cdot \omega^2) = 0. \tag{3}$$

- Eq. (3): ω is conformal to a balanced metric $\widetilde{\omega}$, i.e., $\mathrm{d}(\widetilde{\omega}^2) = 0 \rightsquigarrow$ mild obstructions (Michelsohn'82)
- Eq. (1): existence of Hermitian-Yang-Mills connection is equivalent to the stability of E (Li-Yau'87)
- Eq. (2): anomaly cancellation, the hardest part

Known solutions

$$F \wedge \omega^2 = 0, \quad F^{0,2} = F^{2,0} = 0,$$
$$i\partial\bar{\partial}\omega = \frac{\alpha'}{4}(\mathrm{Tr}(R \wedge R) - \mathrm{Tr}(F \wedge F)),$$
$$\mathrm{d}(\|\Omega\|_\omega \cdot \omega^2) = 0.$$

- Kähler solution: $\mathrm{d}\omega = 0$, use Ricci-flat metric (Yau'77) and take $E = T^{1,0}X$
- Strominger'86: (infinitesimal) perturbative solutions from Kähler solution, orbifolded solutions
- Li-Yau'05: smooth perturbative solutions from Kähler solution Andreas-Garcia-Fernandez'12: more general perturbations

Known solutions cont'd

- Fu-Yau'08: non-Kähler solutions on certain T^2 bundles over K3
  Fu-Tseng-Yau'09 & Becker-Tseng-Yau'09: similar local models
- On nilmanifolds and solvmanifolds:
  Fernández-Ivanov-Ugarte-Villacampa'09, Grantcharov'11, Fernández-Ivanov-Ugarte-Vassilev'14, Ugarte-Villacampa'14, Ugarte-Villacampa'15, Otal-Ugarte-Villacampa'16
- Carlevaro-Israël'10: on blow-up of conifold $K_{\mathbb{P}^1\times\mathbb{P}^1}$

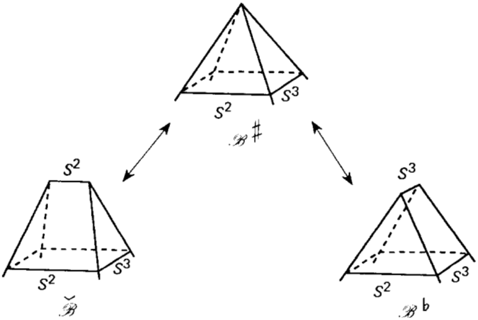
All known solutions have special structures. No general theorem has been proved.

Motivation from math

Local conifold transition (credit to Candelas-de la Ossa'90)

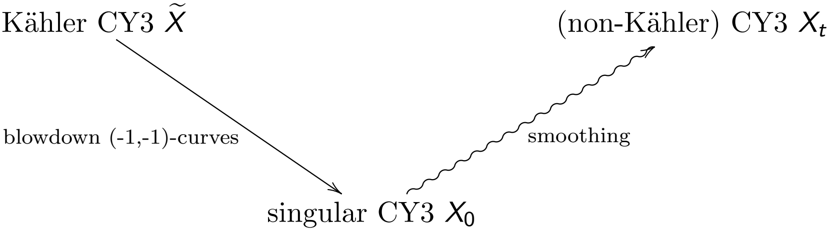
$$\sum_{i=1}^4 z_i^2 = 0$$

$$\mathcal{O}(-1,-1) \qquad\qquad \sum_{i=1}^4 z_i^2 = t$$



Motivation from math cont'd

Conifold transition (Clemens'83, Friedman'86)



Conjecture (Reid'87)

Any two reasonably nice CY3 can be connected via a sequence of conifold transitions.

Strominger system as guidance to canonical metrics on non-Kähler CY3, may be useful to understand the moduli of CY3

Motivation from math cont'd

- Candelas-de la Ossa'90: Ricci-flat Kähler metrics on conifolds

In order to work in non-Kähler category, we may want to first solve Strominger system on conifolds.

- deformed conifold $\cong SL(2, \mathbb{C})$: F.-Yau'15
  Ingredients: left-invariant ansatz, canonical 1-parameter family of Hermitian connections.
  Recently generalized by Otal-Ugarte-Villacampa'16.
- resolved conifold $\mathcal{O}(-1, -1)$: F.'15
  Included in the main theorem.

Theorem (F.'15)

N hyperkähler 4-manifold, $p : Z \to \mathbb{CP}^1$ its holomorphic twistor fibration, $X := Z \setminus p^{-1}(\infty)$. Then X is a noncompact Calabi-Yau 3-fold and we can write down explicit solutions to the Strominger system.

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Calabi's construction

First trial: find solutions to Strominger system on compact spaces
Need: compact non-Kähler CY 3-fold with balanced metric

Construction (Calabi'58)

Identify $\mathbb{R}^7$ with $\mathrm{Im}(\mathbb{O})$. For any immersed oriented hypersurface M in $\mathbb{R}^7$, we define $J : TM \to TM$ by

$$Jv = \nu \times v,$$

where ν is the unit normal, $\times$ is the cross product on $\mathrm{Im}(\mathbb{O})$. Then J is an almost complex structure and M has a natural $SU(3)$-structure.

Theorem (Calabi'58)

Let $\Sigma_g \subset T^3$ be an oriented minimal surface of genus g in flat T^3, and take $M = \Sigma_g \times T^4 \subset T^3 \times T^4$. Then the above constructed J is in fact integrable and (M, J) is non-Kähler.

Calabi's construction cont'd

$M = \Sigma_g \times T^4 \subset T^3 \times T^4$

Remark

- Meeks'90 and Traizet'08: minimal Σ_g in T^3 exists for all $g \geq 3$.
- The projection $M \to \Sigma_g$ is holomorphic.
- Calabi'58 used this construction to give an example that c_1 depends on the complex structure.
- Gray'69 generalized Calabi's construction to manifolds with vector cross product. $\mathbb{R}^7 \rightsquigarrow$ 7-manifolds with G_2-structure. $T^4 \rightsquigarrow$ any hyperkähler 4-manifold. Moreover Gray showed that the natural metric on M is balanced.
- F.'15: M has trivial canonical bundle.

Complex geometry of M

For simplicity we work with $M = \Sigma_g \times T^4$, it is a balanced non-Kähler Calabi-Yau 3-fold.

- Solve for local holomorphic frame for curvature calculation
- All the holomorphic 1-forms are pullback from Σ_g, so $h^{1,0}(M) = g$
- M does not satisfy the $\partial\overline{\partial}$-lemma, therefore not of Fujiki class $\mathcal{C}$

M is very far away from being algebraic.

The conformal balanced equation

Notations:

- Gauss map: $\nu = (\alpha, \beta, \gamma) : \Sigma_g \to S^2 \subset \mathbb{R}^3$
- hyperkähler structure on T^4: ω_I, ω_J and ω_K
- induced metric on Σ_g: ω_{Σ_g}
- induced balanced metric on M: $\omega_0 = \omega_{\Sigma_g} + \alpha\omega_I + \beta\omega_J + \gamma\omega_K$
- holomorphic (3,0)-form Ω satisfies $\|\Omega\|_{\omega_0} = \mathrm{const}$

Hence (M, ω_0, Ω) solves the conformally balanced equation

$$\mathrm{d}(\|\Omega\|_{\omega_0} \cdot \omega_0^2) = 0.$$

Observation:

$$\omega_f = e^{2f}\omega_{\Sigma_g} + e^f(\alpha\omega_I + \beta\omega_J + \gamma\omega_K)$$

solves the conformally balanced equation for any $f : \Sigma_g \to \mathbb{R}$.

The anomaly cancellation equation

Idea: varying f to solve the anomaly cancellation equation

$$i\partial\overline{\partial}\omega_f = \frac{\alpha'}{4}(\mathrm{Tr}(R_f \wedge R_f) - \mathrm{Tr}(F \wedge F)).$$

Straightforward calculation:

- $\mathrm{LHS} = i\partial\overline{\partial}(e^f(\alpha\omega_I + \beta\omega_J + \gamma\omega_K))$
- $\mathrm{Tr}(R_f \wedge R_f) = i\partial\overline{\partial}\left(\frac{\|\mathrm{d}\nu\|^2}{2e^f}(\alpha\omega_I + \beta\omega_J + \gamma\omega_K)\right)$

Therefore by setting

$$e^{2f} = \frac{\alpha'\|\mathrm{d}\nu\|^2}{8} \quad \text{and} \quad F = 0,$$

we solve the anomaly cancellation equation as well as the HYM.

Degeneracy

We need

$$e^{2f} = \frac{\alpha' \|\mathrm{d}\nu\|^2}{8}.$$

However ν is a branched cover, so $\mathrm{d}\nu$ vanishes at finitely many points.
Hence the solution metric

$$\omega_f = e^{2f}\omega_{\Sigma_g} + e^f(\alpha\omega_I + \beta\omega_J + \gamma\omega_K)$$

is degenerate (not too bad) along the the fibers over these branched points.

Problem is caused by taking branched cover!

Hyperkähler 4-manifolds

Definition

A *hyperkähler* manifold is a Riemannian manifold (M, g) with compatible complex structures I, J and K with $I^2 = J^2 = K^2 = IJK = -1$ such that (M, g) is Kähler with respect to all of I, J and K.

Examples in real dimension 4:

- compact: T^4 and K3 surface
- noncompact: Ricci-flat ALE spaces etc.

Important facts:

- for any $(\alpha, \beta, \gamma) \in S^2$, $\alpha I + \beta J + \gamma K$ is a complex structure
- hyperkähler 4-manifolds are anti-self-dual

Twistor spaces

Construction (Penrose'67, Atiyah-Hitchin-Singer'78, HKLR'87)

Let N be a hyperkähler manifold, the twistor space of N is the product $Z(N) = N \times \mathbb{P}^1$ with the almost complex structure

$$\mathfrak{I}_{(x,\zeta)} = \alpha I_x + \beta J_x + \gamma K_x \oplus j_\zeta,$$

where j is the standard complex structure on $\mathbb{P}^1$ with coordinate ζ given by

$$(\alpha, \beta, \gamma) = \left(\frac{1 - |\zeta|^2}{1 + |\zeta|^2}, \frac{\zeta + \overline{\zeta}}{1 + |\zeta|^2}, \frac{i(\overline{\zeta} - \zeta)}{1 + |\zeta|^2} \right).$$

Important facts:

- $\mathfrak{I}$ is integrable
- the projection $\pi : Z(N) \to \mathbb{P}^1$ is holomorphic
- $\wedge^2 T^* F \otimes \pi^* \mathcal{O}(2)$ has a global section which defines a holomorphic symplectic form on each fiber of π

M as pullback

Key observation:
M fits in the pullback square

$$\begin{array}{ccc} M & \longrightarrow & Z(N^4) \\ \downarrow & & \downarrow \pi \\ \Sigma_g & \xrightarrow{\ \nu\ } & \mathbb{P}^1 \end{array}$$

Notice that Σ_g is minimal in T^3 implies that ν is holomorphic.
This observation leads to many generalizations of Calabi-Gray's construction, including construction of simply-connected non-Kähler K3-fibered Calabi-Yau 4-folds with balanced metrics.

Overview

Geometric construction

Recall that the problem of solving Strominger system on M comes from taking branched cover. With our new interpretation of M as pullback, it is natural to consider Strominger system on the twistor space $Z(N)$.
Problem: A twistor space can never have trivial canonical bundle!
Remedy: Remove a divisor from $Z(N)$ to make it a noncompact Calabi-Yau

Construction

Let N be a hyperkähler 4-manifold and let $\pi : Z(N) \to \mathbb{P}^1$ be its associated twistor fibration. Let F be a fiber of π, then $X := Z(N) \setminus F$ has trivial canonical bundle.

WLOG, we may assume the fiber is over $\zeta = \infty$, then a holomorphic (3,0)-form Ω on X can be written down explicitly

$$\Omega = (-2\zeta\omega_I + (1-\zeta^2)\omega_J + i(1+\zeta^2)\omega_K) \wedge \mathrm{d}\zeta.$$

Examples and the main result

Fibration structure

$$\begin{array}{ccc} X = Z(N)\setminus F & \hookrightarrow & Z(N) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{C} & \hookrightarrow & \mathbb{P}^1 \end{array}$$

Some examples:

- $N = \mathbb{R}^4$, then X is biholomorphic to $\mathbb{C}^3$
- $N = T^*\mathbb{P}^1$ with Eguchi-Hanson geometry and carefully chosen F, then X is biholomorphic to $\mathcal{O}(-1,-1)$ (Hitchin'81)

Main result:

Theorem (F.'15)

Let N and X be described above, then a solution to the Strominger system on X can be written down explicitly.

The conformally balanced equation

Similar strategy. We begin with the equation

$$\mathrm{d}(\|\Omega\|_\omega \cdot \omega^2) = 0.$$

Check that

$$\omega_{g,h} = \frac{e^{2h+g}}{(1+|\zeta|^2)^2}(\alpha\omega_I + \beta\omega_J + \gamma\omega_K) + e^{2g}\omega_{\mathrm{FS}}$$

solves the conformally balanced equation for any $g : \mathbb{C} \to \mathbb{R}$ and $h : N \to \mathbb{R}$. Again, we are making use of the fibration $\pi : X \to \mathbb{C}$.

The anomaly cancellation

Now we turn to the anomaly cancellation

$$i\partial\bar{\partial}\omega_{g,h} = \frac{\alpha'}{4}(\mathrm{Tr}(R_{g,h} \wedge R_{g,h}) - \mathrm{Tr}(F \wedge F)).$$

Notation: $s := 1 + |\zeta|^2$, $\omega' := \alpha\omega_I + \beta\omega_J + \gamma\omega_K$ and $B := s^3/e^{2h+g}$

- Solve for a holomorphic frame
- $LHS = i\partial\bar{\partial}(s^{-2}e^{2h+g}\omega')$
- $\mathrm{Tr}(R_{g,h} \wedge R_{g,h}) = 2i\partial\bar{\partial}(s^{-2}e^{2h-g}\omega') + \mathrm{Tr}(R' \wedge R') + (\partial\bar{\partial}\log B)^2$, where R' is the curvature of relative cotangent bundle

So if

- $g = \log(\alpha'/2)/2$
- $(\partial\bar{\partial}\log B)^2 = 0$, say $h = \mathrm{const.}$
- $F = R'$

then we solve the anomaly cancellation equation!

The HYM

To solve the whole system, we need to verify that R' satisfy HYM:

$$\omega_{g,h}^2 \wedge R' = 0.$$

- type decomposition:

$$\Omega^{1,1}X = \Omega^{1,1}\mathbb{C} \oplus (\Omega^{1,0}\mathbb{C} \otimes \Omega^{0,1}N) \oplus (\Omega^{0,1}\mathbb{C} \otimes \Omega^{1,0}N) \oplus \Omega^{1,1}N.$$

- R' has no $(\Omega^{1,1}\mathbb{C})$-component
- only $(\Omega^{1,1}N)$-component contributes to $\omega_{g,h}^2 \wedge R'$
- HYM can be verified fiberwise, using the fact hyperkähler manifolds are anti-self-dual.

Done!

A concrete example

Example

Take $N = \mathbb{R}^4$, then $X \cong \mathbb{C}^3 = \{(w_1, w_2, \zeta) : w_1, w_2, \zeta \in \mathbb{C}\}$, the solution metric is

$$\begin{aligned}\omega = {} & 2is^{-2}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} + i/2 \cdot s^{-3}(\mathrm{d}w_1 \wedge \mathrm{d}\overline{w}_1 + \mathrm{d}w_2 \wedge \mathrm{d}\overline{w}_2 \\ & + (|u_1|^2 + |u_2|^2)\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} + iu_2\mathrm{d}w_1 \wedge \mathrm{d}\bar{\zeta} - iu_1\mathrm{d}w_2 \wedge \mathrm{d}\bar{\zeta} - i\overline{u}_2\mathrm{d}\zeta \wedge \mathrm{d}\overline{w}_1 \\ & + i\overline{u}_1\mathrm{d}\zeta \wedge \mathrm{d}\overline{w}_2),\end{aligned}$$

where

$$(u_1, u_2) = \left(\frac{w_1 - i\zeta\overline{w}_2}{1 + |\zeta|^2}, \frac{w_2 + i\zeta\overline{w}_1}{1 + |\zeta|^2}\right).$$

This metric is conformally flat and non-Kähler.

A final remark

In order for $(\partial\overline{\partial}\log B)^2 = 0$, h does not have to be a constant. In the case that $N = \mathbb{R}^4$, we can also take h such that

$$\exp(h) = c \cdot \|x\|^{-3}.$$

This gives a solution of Strominger system on $\mathbb{C} \times (\mathbb{C}^2 \setminus \{0\})$.
In general I do not know how to find such h because the lack of explicit knowledge about hyperkähler metrics.

Conclusion

For any hyperkähler 4-manifold N, we know that $X = Z(N) \setminus F$ is a noncompact CY3. We can construct explicit solutions to the Strominger system on X. Such manifolds include $\mathbb{C}^3$ and $\mathcal{O}(-1,-1)$ as special examples.

Hopefully these local models can be used in gluing to give more general solutions.

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