Weyl Groups and Minimal Models

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Motivation 1

Classical result: Weyl groups associated to Del Pezzo surfaces
(surfaces S with $-K_S$ ample)

Example: E_6 associated to a cubic surface

Question 1. Can we generalize this result to have
Weyl groups associated to Fano 3-folds ?
(3-folds X with $-K_X$ ample)

Problem: Classical Weyl groups are constructed
via the intersection form on surfaces.
No apparent intersectiion form on 3-folds.

Motivation 2

Well-known Weyl groups showing up in algebraic geometry

- Weyl groups associated to Del Pezzo surfaces
- Weyl groups associated to Rational Double points

Question 2. Can we give a unified view point to interpret these together ?

Problem: No apparent relation.

My Answers to Q1 and Q2

Q1: YES. Weyl groups associated to Fano 3-folds !
Q2: YES. There is a unified view point !

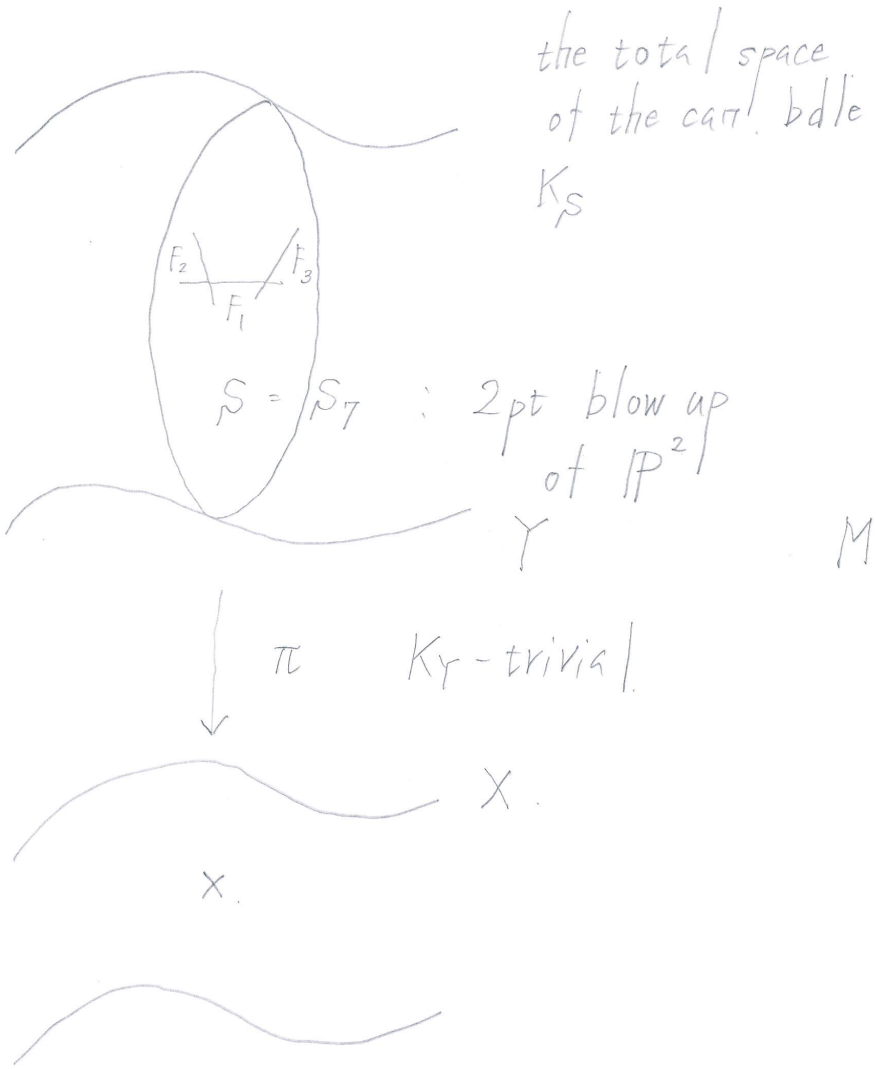
Both in terms of

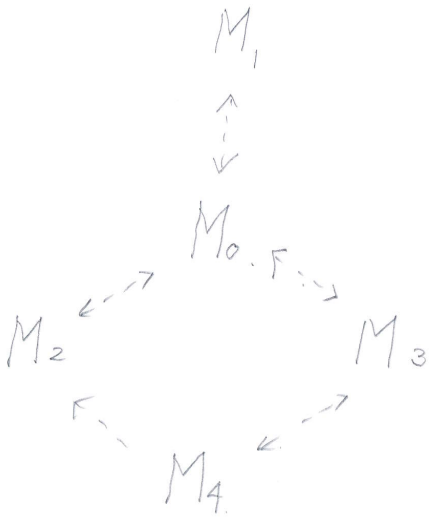
Symmetries among the Minimal Models.

Note: My answer was labled 30 years ago
by M. Reid as

a combinatorial Mumbo-Jumbo.

Explanation of my answer (Del Pezzo Case): 1



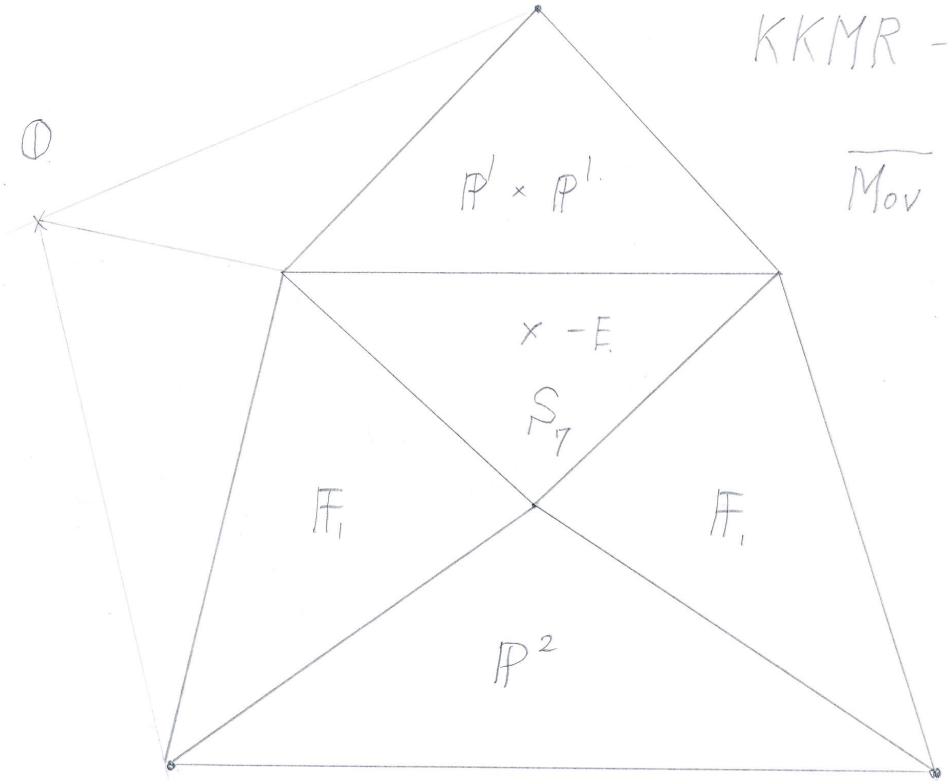


Explanation of my answer (Del Pezzo Case): 2

Consider

$$
\begin{array}{ccc}
\mathrm{Pic}(S) & \subset & N^1(S) \\
\| & & \| \\
\mathrm{Pic}(Y/X) & \subset & N^1(Y/X) \\
\| & & \| \\
L : \text{lattice} & \subset & N^1 : \text{f.d. vector space} \\
 & & \cup \\
 & & \overline{\mathrm{Mov}} = \amalg\, \overline{\mathrm{Amp}}(M_i/X) \\
 & & \uparrow \\
 & & \text{KKMR decomposition}
\end{array}
$$

Explanation of my answer (Del Pezzo Case): 3



KKMR - decomp.

$$\overline{\mathrm{Mov}} = \coprod \overline{\mathrm{Amp}}(M_i/X)$$

$$\cap$$

$$N'$$

$$\dim N' = 3$$

Explanation of my answer (Del Pezzo Case): 4

Symmetry
$=$ Weyl Groups
$$= \left\{ g \in \mathrm{Aut}(L);\ \begin{array}{l} g \text{ preserves } E \\ g \text{ preserves KKMR-decomposition.} \end{array} \right\}$$

This interpretation makes sense also for Fano 3-folds.

Remark: Related to (???)
"Quantum Periods for 3-dimensional Fano Manifolds"
by Coates-Corti-Galkin-Kasprzyk

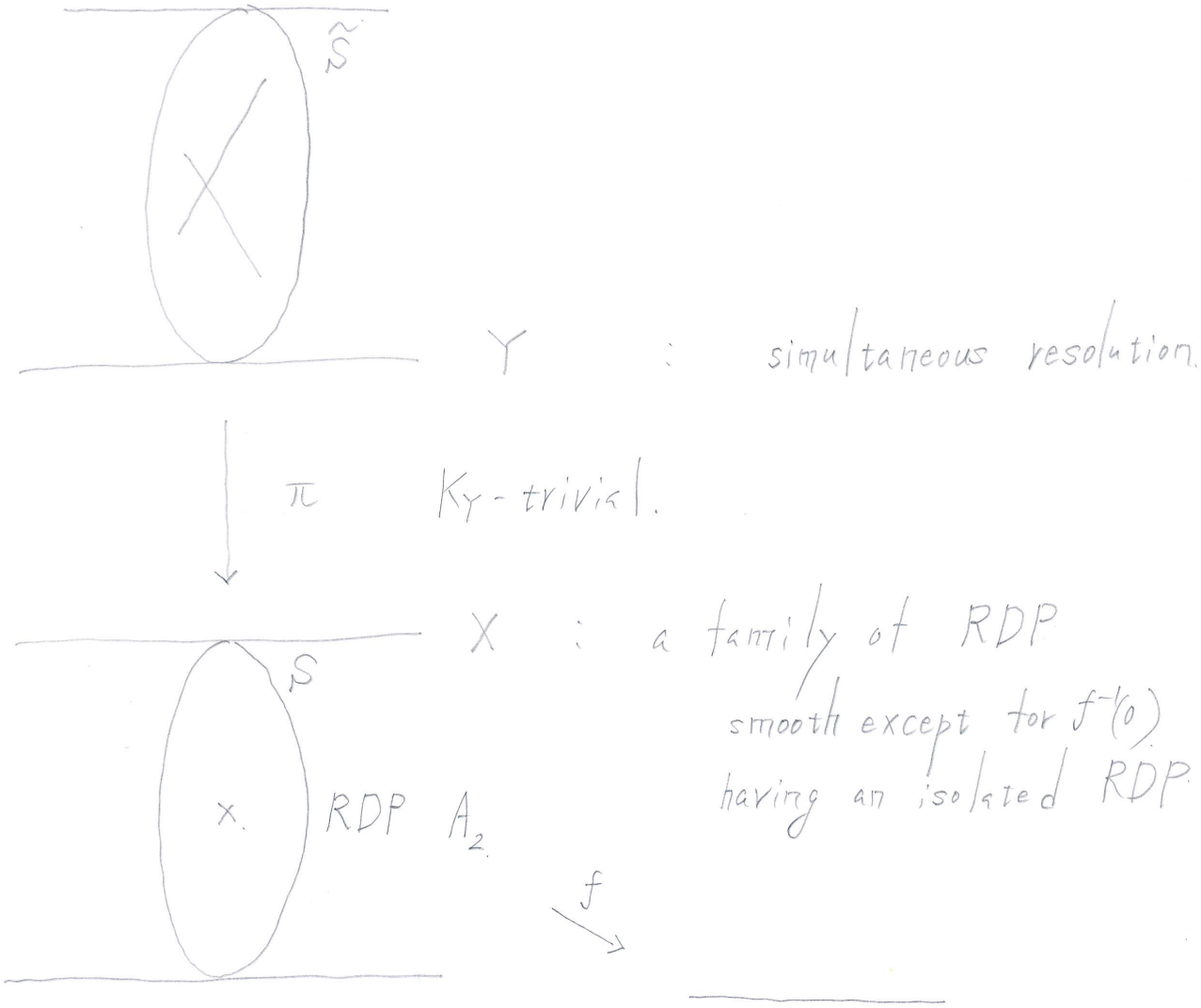
Table of Weyl Groups for Fano 3-folds

In the table below, we use the same notation and numbering as in MM[39,40].

TABLE III-3-6.

B_2	$\{e\}$	A_1	$A_1 \times A_1$	A_2	$A_1 \times A_2$	A_3	A_4	D_5	E_6	E_7	E_8
1	index 1: Q, $\mathbb{P}^3$ V_1, V_2, V_3, V_4, V_5										
2	1 3 4 5 7 8 9 10 11 13 14 15 16 17 18 19 20 22 23 24 25 26 27 28 29 30 31 33 34 35 36	2 6 12 21 32									
3	2 4 5 6 8 11 12 14 15 16 18 21 22 23 24 26 28 29 30	3 7 9 10 13 17 19 20 25 31		1 27							
4	5 9 11	3 4 7 8 10 12	2	6		1					
5		2		1	3						
6							$\mathbb{P}^1 \times S_5$				
7								$\mathbb{P}^1 \times S_4$			
8									$\mathbb{P}^1 \times S_3$		
9										$\mathbb{P}^1 \times S_2$	
10											$\mathbb{P}^1 \times S_1$

Explanation of my answer (RDP Case): 1

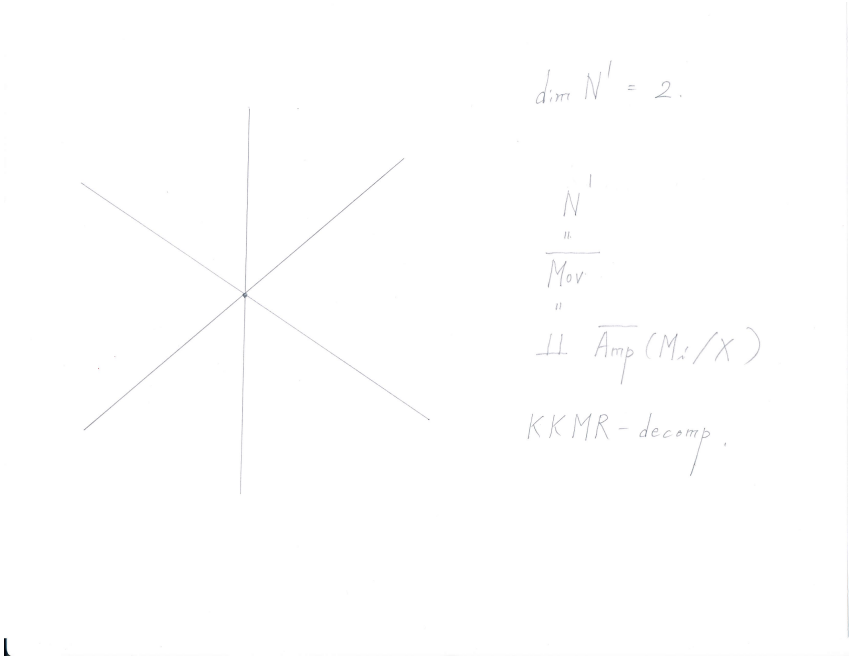


Explanation of my answer (RDP Case): 2

Consider

$$
\begin{array}{ccc}
\mathrm{Pic}(\widetilde{S}/S) & \subset & N^1(\widetilde{S}/S) \\
\| & & \| \\
\mathrm{Pic}(Y/X) & \subset & N^1(Y/X) \\
\| & & \| \\
L : \text{lattice} & \subset & N^1 : \text{f.d. vector space} \\
 & & \cup \\
 & & \text{(actually } \| \text{ in this case)} \\
 & & \overline{\mathrm{Mov}} = \amalg\, \overline{\mathrm{Amp}}(M_i/X) \\
 & & \uparrow \\
 & & \text{KKMR decomposition}
\end{array}
$$

Explanation of my answer (RDP Case): 3



Symmetry
$=$ Weyl Groups
$= \{g \in \mathrm{Aut}(L); g$ preserves KKMR-decomposition$\}$

Table according to κ : the Kodaira dimension

Table of symmetries among the Minimal Models
according to $\kappa =$ dimension of the canonical model

κ	Geometric Objects	Type of W.G.	Quad. Form
3	RDP Del Pezzo	finite	pos. def.
2	Kodaira's deg. of Elliptic Curves	affine	pos. semi. def.
1	K3 surfaces	hyperbolic	indef. with sign $(1, \rho - 1)$
0	Calabi-Yau 3-folds	ref. group inf. generators ? with no relation	N/A

Speculation: Geometric Objects $\leftrightarrow$ Weyl (algebraic) Groups

Geometric Objects	$\longrightarrow$ $\longleftarrow$	Weyl (algebraic) Groups
RDP	$\longleftarrow$ Brieskorn-Grothendieck	
Del Pezzo	$\longleftarrow$ S.B.-Grojnowski	
Fano 3-folds	$\longleftarrow$ NO	
Deg.of ellip. curve	$\longleftarrow$ Helmke-Slowdowy	
K3 surface	$\longleftarrow$???	
Calabi-Yau 3-fold	$\longleftarrow$???	

Questions?