Weyl Groups and Minimal Models

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Motivation 1

Classical result: **Weyl groups** associated to **Del Pezzo surfaces**
(surfaces $S$ with $-K_S$ ample)

Example: $E_6$ associated to a cubic surface

**Question 1.** Can we generalize this result to have **Weyl groups** associated to **Fano 3-folds**?
(3-folds $X$ with $-K_X$ ample)

**Problem:** Classical Weyl groups are constructed via the **intersection form** on surfaces.
No apparent intersection form on 3-folds.
Well-known Weyl groups showing up in algebraic geometry
• Weyl groups associated to Del Pezzo surfaces
• Weyl groups associated to Rational Double points

Question 2. Can we give a unified view point to interpret these together?

Problem: No apparent relation.
My Answers to Q1 and Q2

Q1: YES. **Weyl groups** associated to **Fano 3-folds**!
Q2: YES. There is **a unified view point**!

Both in terms of

**Symmetries among the Minimal Models.**

Note: My answer was labeled 30 years ago by M. Reid as

**a combinatorial Mumbo-Jumbo**.
Explanation of my answer (Del Pezzo Case): 1

The total space of the conical bundle $K_S$.

$S = S_7 : 2$ pt blow up of $\mathbb{P}^2$.

$\pi : K_Y$ - trivial.

$X.$
Consider

\[ \text{Pic}(S) \subset \text{Pic}(Y/X) \subset L : \text{lattice} \subset N^1(S) \parallel N^1(Y/X) \parallel N^1 : \text{f.d. vector space} \]

\[ \overline{\text{Mov}} = \sqcup \overline{\text{Amp}}(M_i/X) \]

KKMR decomposition
Explanation of my answer (Del Pezzo Case): 3

KKMR - decomp.

$$\overline{\text{Mov}} = \bigcap \overline{\text{Amp}(\mathcal{M}/X)}$$

$$\cap$$

$$N'$$

$$\dim N' = 3$$
Symmetry
= Weyl Groups
= \{ g \in \text{Aut}(L); g \text{ preserves } E \\
g \text{ preserves KKMR-decomposition.} \}

This interpretation makes sense also for Fano 3-folds.

Remark: Related to (???)
“Quantum Periods for 3-dimensional Fano Manifolds”

by Coates-Corti-Galkin-Kasprzyk
In the table below, we use the same notation and numbering as in MM[39,40].

<table>
<thead>
<tr>
<th>$B_2$</th>
<th>$e$</th>
<th>$A_1$</th>
<th>$A_1 \times A_1$</th>
<th>$A_2$</th>
<th>$A_1 \times A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$D_5$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
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</table>
$X$ : a family of RDP smooth except for $f^{-1}(0)$ having an isolated RDP

$Y$ : simultaneous resolution.
Explanation of my answer (RDP Case): 2

Consider

\[
\begin{align*}
\text{Pic}(\tilde{S}/S) & \subset N^1(\tilde{S}/S) \\
\| & \\
\text{Pic}(Y/X) & \subset N^1(Y/X) \\
\| & \\
L: \text{lattice} & \subset N^1: \text{f.d. vector space} \\
\| & \\
(\text{actually } \| \text{ in this case}) & \\
\overline{\text{Mov}} = \amalg \overline{\text{Amp}}(M_i/X) & \\
\uparrow & \\
\text{KKMR decomposition}
\end{align*}
\]
Symmetry
= Weyl Groups
= \{ g \in \text{Aut}(L); g \text{ preserves KKMR-decomposition} \}
Table of symmetries among the Minimal Models according to $\kappa = \text{dimension of the canonical model}$

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>Geometric Objects</th>
<th>Type of W.G.</th>
<th>Quad. Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>RDP Del Pezzo</td>
<td>finite</td>
<td>pos. def.</td>
</tr>
<tr>
<td>2</td>
<td>Kodaira’s deg. of Elliptic Curves</td>
<td>affine</td>
<td>pos. semi. def.</td>
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<tr>
<td>1</td>
<td>K3 surfaces</td>
<td>hyperbolic</td>
<td>indef. with sign $(1, \rho - 1)$</td>
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<tr>
<td>0</td>
<td>Calabi-Yau 3-folds</td>
<td>ref. group</td>
<td>N/A</td>
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<tr>
<td></td>
<td></td>
<td>inf. generators</td>
<td></td>
</tr>
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<td></td>
<td></td>
<td>with no relation</td>
<td></td>
</tr>
<tr>
<td>Geometric Objects</td>
<td>Weyl (algebraic) Groups</td>
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<tr>
<td>RDP</td>
<td>Brieskorn-Grothendieck</td>
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<tr>
<td>Del Pezzo</td>
<td>S.B.-Grojnowski</td>
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<tr>
<td>Fano 3-folds</td>
<td>NO</td>
<td></td>
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<tr>
<td>Deg.of ellip. curve</td>
<td>Helmke-Slowdowy</td>
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<tr>
<td>K3 surface</td>
<td>???</td>
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<tr>
<td>Calabi-Yau 3-fold</td>
<td>???</td>
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Questions?