Elliptic Calabi-Yau classification update: Weierstrass tunings and exotic representations

Physics and Mathematics of F-theory
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Based on work with L. Anderson, M. Cvetic, A. Grassi, J. Gray, J. Halverson, Y. Huang, S. Johnson, D. Klevers, G. Martini, D. Morrison, H. Piragua, N. Raghuram, J. Shaneson, A. Turner, Y. Wang

Outline

1. Classification of elliptic Calabi-Yau manifolds

— Review: threefolds

— Results on tunings

— Fourfolds

2. Exotic matter: singular fibers over singular curves

— Examples

— General structure

Review: 6D F-theory and elliptic Calabi-Yau threefolds

Using tools from algebraic geometry and F-theory, we have a systematic approach to constructing and classifying elliptic Calabi-Yau threefolds

Classifying elliptic CY threefolds

Elliptic CY3 $\pi : X_3 \rightarrow B_2$
Weierstrass model $y^2 = x^3 + fx + g$,
$f \in \Gamma(\mathcal{O}(-4K_B)), g \in \Gamma(\mathcal{O}(-6K_B))$

Gross: finite number of topological types (up to birational isomorphism)
Desire: explicit construction

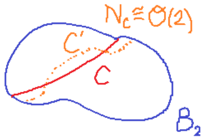
- Basic idea: classify bases B, then tune Weierstrass for each base
  Focus on Weierstrass models on smooth bases (*e.g.* not SCFT)
- Minimal models + work of Grassi:
  $B = \mathbb{P}^2, \mathbb{F}_m$ or blowup thereof (or Enriques)
  $\Rightarrow$ constructive approach to enumeration:
  finite Weierstrass strata over each base [KMT].

Non-Higgsable groups in F-theory $\Leftrightarrow$ forced Kodaira singular fibers

- "Non-Higgsable clusters" give lower bound on normal bundle of divisors

The base B_2 is a complex surface.

Contains homology classes of complex curves C_i



For $C \cong \mathbb{P}^1 \cong S^2$, local geometry encoded by *normal bundle* $\mathcal{O}(m)$

$C \cdot C = m$; *e.g.*, $N_C \cong \mathcal{O}(2) \cong TC$: deformation has 2 zeros, $C \cdot C = +2$

If $N_C \cong \mathcal{O}(-n), n > 0, C$ is *rigid* (no deformations)

For $\mathcal{O}(-n), n > 2$, base space is so curved that singularities must pile up to preserve Calabi-Yau structure on total space
$\Rightarrow$ non-Higgsable gauge group = forced singular fiber

Example: $N = \mathcal{O}(-3) \Rightarrow$ type IV fiber (SU(3) group)

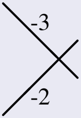
Classification of 6D "Non-Higgsable Clusters" (NHC's) [Morrison/WT]

Clusters of curves imposing generic nontrivial codimension one singularities:

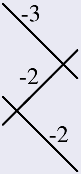
$-m$

$(m = 3, 4, 5, 6, 7, 8, 12)$

$\mathfrak{su}(3), \mathfrak{so}(8), \mathfrak{f}_4$
$\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$



$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$



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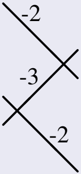


$\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$

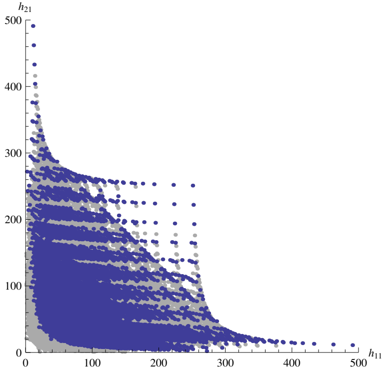
- Any other combination including -3 or below $\Rightarrow$ (4, 6) at point/curve

NHC's a useful tool in classifying bases B_2 for EF CY3's
– Also useful for 6D SCFT's [Heckman/Morrison/(Rudelius)/Vafa, ...]

Classifying bases: toric B_2

Start with $\mathbb{P}^2, \mathbb{F}_m$, blow up torically, constrain by NHC's

Generic EF Hodge #'s [Morrison/WT, WT]



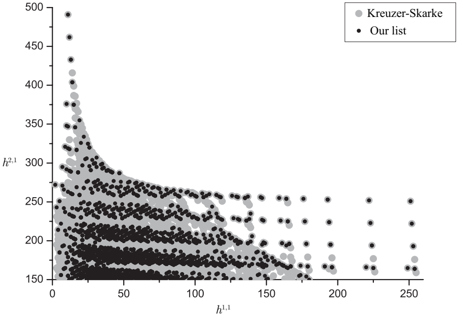
- 61,539 toric bases (some not strictly toric: -9, -10, -11 curves)
- Reproduces large subset of Kreuzer-Skarke database of CY3 Hodge #'s
  Boundary of "shield" from generic elliptic fibrations over blowups of $\mathbb{F}_{12}$.

Beyond toric: approach allows construction of general (non-toric) bases

– Computed all 162, 404 "semi-toric" bases w/ 1 $\mathbb{C}^*$-structure [Martini/WT]

Generally: Keep track of cone of effective divisors as combinatorial data

- All bases for EF CY threefolds w/ $h^{2,1}(X) \geq 150$ [WT/Wang]



Technical issues at large $h^{1,1}(X)$, small $h^{2,1}(X)$:
Infinite generators for cone, Multiply intersecting -1 curves

Upshot: modest expansion of possibilities beyond toric, semi-toric

Classifying elliptic CY3's: Tuning [Johnson/WT '14, '16; for SCFT: HMRV]

Given generic Weierstrass on B_2: $y^2 = x^3 + fx + g$,

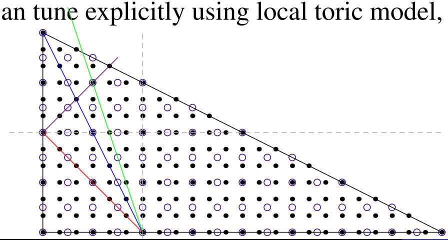
tune $f, g \Rightarrow$ enhanced gauge group G, matter

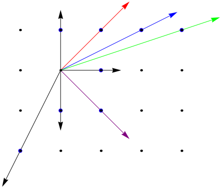
$e.g.$ on $\mathbb{P}^2$, $f = f_3 z^3 + \cdots, g = g_3 z^4 + \cdots \Rightarrow E_6$ on $\{z = 0\}$.

- Can do systematically, finite number of tunings on each B_2
- Many complications w/interesting physics, some subtle outstanding issues
- Hodge numbers from base physics/geometry [Morrison/Vafa]

$$\begin{aligned} h^{1,1}(X) &= \mathrm{rk}(G) + T + 2 = \mathrm{rk}(G) + h^{1,1}(B) + 1 \quad \text{(Shioda-Tate-Wazir)} \\ h^{2,1}(X) &= H_{\mathrm{neutral}} - 1 = 272 + \dim(G) - 29T - H_{\mathrm{charged}} \,. \end{aligned}$$

- Can tune explicitly using local toric model, globally in toric bases





Systematic classification of EFS CY3s: tuning

- Blowups, tuning from minimal bases $\mathbb{P}^2, \mathbb{F}_m$ decrease $h^{2,1}$
- Codimension 1, 2 sing.'s $\rightarrow G$, matter $\rightarrow r, V, H_{\text{ch}} \rightarrow h^{1,1}, h^{1,2}$
- Upshot: given base, what you can tune $\cong$ allowed by anomaly cancellation (with some interesting exceptions!)

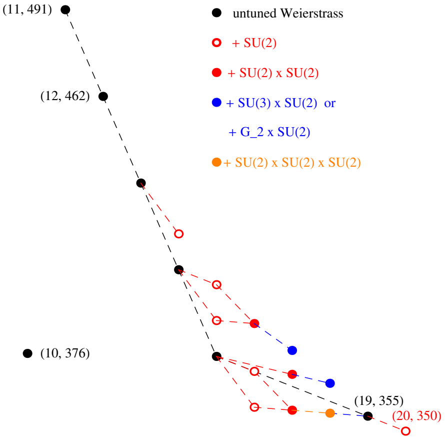
Local swampland candidates

- SU(21), SU(23) on +1 curve, SU(15) on + 2 curve, …(?)
- SU($2j+1$), SU($2j+8$) on -1, -2 [*e.g.* SU(3) $\times$ SU(10)] (? No Tate form)

…

SO(8) $\times$ SU(2) on *, -2: low-energy inconsistency [Ohmori/Shimizu/Tachikawa/Yonekura]

EFCY3's w/ $h^{2,1} \geq 350$, $\mathbb{F}_m$+ tuning $\rightarrow$ WM classification [Johnson/WT '14]



- Matches KS; non-toric + toric at (19, 355); new non-toric below 350

Systematic tunings on toric models [Huang/WT, TA]

Tate tunings $\leftrightarrow$ reflexive polytopes (modulo some technical caveats)

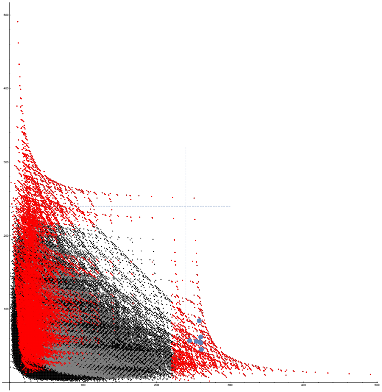
Can do a branched search, find all allowed tunings on B_{toric}

– Use local rules [JT], + global group constraints [Bertolini/Merkx/Morrison]

Tunings $\Rightarrow$

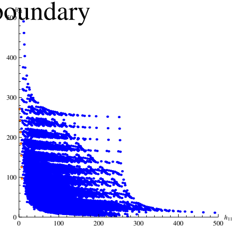
- All KS Hodge #'s $h^{2,1} \geq 240$
- All but 8 $h^{1,1} \geq 240$

$\Rightarrow$ *most* known CY3's w/large h's are EF
(cf. [Gray/Haupt/Lukas,
Candelas/Constantin/Skarke,
Anderson/Apruzzi/Gao/Gray/Lee])



Results on CY3's: global picture

- Systematic approach to construction
- Complete control at large $h^{2,1}(X)$ (*e.g.*, proof $h^{2,1} \leq 491$)
- Toric bases give good representative global picture, capture boundary
- Finite number of bases, minimal $\mathbb{P}^2, \mathbb{F}_m$ on left boundary



- "Most" bases B_2 have non-Higgsable $G_{\rm NA}$

  (all but weak Fano = gdP)

Some outstanding/ongoing issues:

- Difficult regime: large $h^{1,1}(X)$, small $h^{2,1}(X)$
- Mordell-Weil (much recent work)

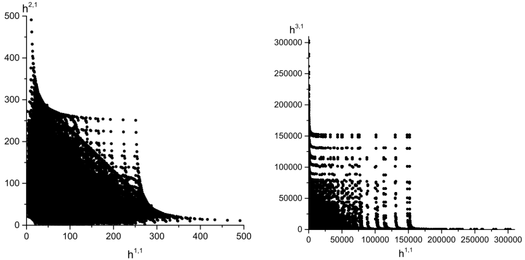
Possible further issues: singular bases, Enriques

4D F-theory compactifications

Story parallel in many ways:

– Compactify on Calabi-Yau fourfold, base B_3 = complex threefold

– Empirical data suggest similar structure (though less complete for CY4's)



No proof of finiteness
Mori theory threefold analog of minimal model bases more subtle

All evidence so far: moduli space of CY 4's quite parallel to CY3 story

4D non-Higgsable clusters [Morrison/WT]
(see also: Anderson/WT, Grassi/Halverson/Shaneson/WT, cf. Halverson talk)

At level of geometry/complex structure, similar to 6D but more complicated

Expanding in coordinate z, around divisor (surface) $S = \{z = 0\}$,

$$f = f_0 + f_1 z + f_2 z^2 + \cdots$$

Compute using geometry of *surfaces*: up to leading non-vanishing term,
$f_k \in \mathcal{O}(-4K_S + (4-k)N_S), \quad g_k \in \mathcal{O}(-6K_S + (6-k)N_S)$

Single group clusters: $\boxed{SU(2), SU(3), G_2, SO(7), SO(8), F_4, E_6, E_7, E_8}$

(cannot have: non-Higgsable $SU(5), SO(10)$)

the only 2-factor products that can appear are:

$$\boxed{\begin{array}{lll} G_2 \times SU(2), & SO(7) \times SU(2), & SU(2) \times SU(2), \\ SU(3) \times SU(2), & SU(3) \times SU(3) & \end{array}}$$

4D clusters can have chains, loops, branching ...

Classification of elliptic Calabi-Yau fourfolds

Mathematical minimal models $\rightarrow$ Mori theory.
No proofs, but finite classification seems manageable.
Rough "physicist's" picture – ignore various subtleties
Focus on classifying bases B_3, finite number [Di Cerbo/Svaldi]

"minimal models" $\sim \mathbb{F}_m$ but more complex – populate LHS of Hodge plot
Roughly, min $B_3 = \{\mathbb{P}^1$ (conic) bundle over B_2, B_2 bundle over $\mathbb{P}^1$, Fano$\}$

Blow up curves, points: $h^{3,1} \downarrow, h^{1,1} \uparrow$; finite # of options on each minimal B_3

w/Halverson: $\mathbb{P}^1$ bundles over toric bases B_2
(w/Anderson: B_2 = gdP, smooth heterotic dual)
Finite # $\mathbb{P}^1$ bundles over fixed B_2

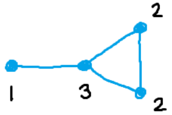
w/Wang: B_2 bundles over $\mathbb{P}^1$, B_2 supports EF CY3, finite # B_2, bundles
Max $h^{3,1}$ = 303,148

Fano: 105 Fano bases $< \infty$

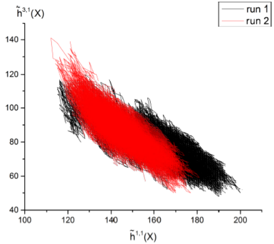
Possible issue: irrational bases (Morrison/WT TA: similar (?))

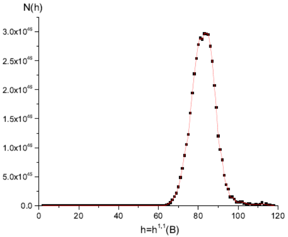
Monte Carlo on threefold bases for EF CY4's (w/ Yinan Wang)

Random walk on a graph: $p_i \propto \nu_i = \#$ of neighbors, *e.g.*



Explore connected toric threefold bases from $\mathbb{P}^3$ by blow-up, -down transitions





Estimate number of connected toric threefold bases $\sim 10^{48\pm2}$

Elliptic Calabi-Yau fourfolds:

Seems closely parallel to EF CY3, but mostly empirical evidence

- Semi-systematic classification for toric B_3
- $h^{3,1} \leq 303, 148$ (?)
- At large Hodge numbers, most known CY4's elliptic (?)
- Toric $\rightarrow$ good global picture ??
  (more exceptions *e.g. irrational*, but appear similar [Morrison/WT])
- Most bases not weak Fano $\Rightarrow$ have non-Higgsable G
- Need clearer analog of minimal model/Grassi result
- Include singularities (??)

Codimension 2 singularities: tricky, not completely classified

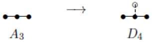
Simple case: rank one enhancement [Katz-Vafa]

e.g.

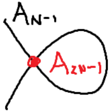
$SU(N)$ enhanced $\Rightarrow SU(N+1)$
$SU(N+1)$ adjoint $\rightarrow SU(N)$ $\square$

Similar for *e.g.* $A_{N-1} \rightarrow D_N$: $SU(N)$ $\begin{array}{c}\square\\[-4pt]\square\end{array}$

Realized by embedding of Dynkin diagrams in singularity



Other cases more complex [Sadov, Morrison/WT, Esole/Yau]



$$A_{N-1} \Rightarrow A_{2N-1}$$

gives $SU(N)$ $\square\!\square + \begin{array}{c}\square\\[-4pt]\square\end{array}$ or adj + 1

Analyze through explicit resolution in specific cases [MT, Esole/Yau, HLMS, ...]
[Grassi/Halverson/Shaneson: alternative: local deformation?]

"Topology" of matter representations

Some reps are not understood in F-theory but seem okay from supergravity
Genus classification from anomaly structure [Kumar/Park/WT]

$$
\begin{aligned}
2g - 2 &= (K + C) \cdot C = \sum_R x_R g_R - 2 \\
g_R &= \frac{1}{12}(2C_R + B_R - A_R)
\end{aligned}
$$

(group theory coefficients: $\mathrm{tr}_R F^2 = A_R \mathrm{tr} F^2$, $\mathrm{tr}_R F^4 = B_R \mathrm{tr} F^4 + C_R (\mathrm{tr} F^2)^2$)
e.g. for SU(N) $\square\!\square$, $g = 1$

Structure suggests $\sim$ Kodaira: Rep. theory $\leftrightarrow$ cod. 2 sing.'s
Conjecture: non-adjoint $g > 0 \Rightarrow$ arithmetic genus of discriminant singularity

Matter on singular divisors

Examples:

SU(3) $\square\!\square$ on double points

– From unHiggsing U(1) $\times$ U(1) models [Cvetic/Klevers/Piragua/WT]

– From SU(6) $\rightarrow$ SU(3) $\times$ SU(3) $\rightarrow$ SU(3) and transitions via SCFT's [Anderson/Gray/Raghuram/WT]

SU(2) $\square\!\square\!\square$ on triple points

– From unHiggsing U(1) $q = 3$ models [Klevers/WT]

All examples have nontrivial Weierstrass form, rely on non-UFD ring on C.

General construction for SU(N) $\square\square$, ... on singular divisor
[Klevers/Morrison/Raghuram/WT]

Follow term by term in discriminant *a la* [Morrison/WT '11]

Allow component functions not in $R(C)$ but in extension

Example:
SU(2) on C : $\sigma = \xi^3 - A\eta^3 = 0$ (A, ξ, η in $R(C)$); triple points at $\xi = \eta = 0$

Include α : $\alpha^3 = A$, $\quad \Rightarrow \; \xi = \alpha\eta$

Expanding $f = f_0 + f_1\sigma + f_2\sigma^2 + \cdots, g = g_0 + g_1\sigma + \cdots,$

Need $4f_0^3 + 27g_0^2 = \sigma(\cdots)$. In UFD $f_0 = -3\phi^2, g_0 = 2\phi^3 \Rightarrow \Delta_0 = 0$.

Choose $\phi = \alpha^2\eta$, not in $R(C)$!

$f_0 = -3\phi^2 = -3\alpha^4\eta^2 = -3A\xi\eta, g_0 = 2\phi^3 = 2\alpha^6\eta^3 = A^2\eta^3$

$\Delta_0 = 4f_0^3 + 27g_0^2 = 108(-A^3\xi^3\eta^3 + A^4\eta^6) = -108A^3\eta^3\sigma$!

Can continue in this fashion, similar for SU(N) on double points.

Exotic matter representations on singular divisors

This approach gives SU(N) $\square\!\square$, SU(2) $\square\!\square\!\square$

Claim: that's it for SU(N) $g > 0$ representations.

Others: affine Dynkin diagram with $G \to$ extra node

e.g. SU(2) $\square\!\square\!\square\!\square \to \hat{D}_4$

SU(3) $\square\!\square\!\square \to \hat{E}_6$

Some swamp questions: for which combinations of matter find Weierstrass models?

Conclusions

– Systematic control of construction and classification of elliptic CY3's, including systematic tunings in toric cases

– All KS Hodge numbers with $h^{2,1} \geq 240$ reproduced by tunings

– Appear to be 8 KS examples with $h^{1,1} > 240$ not (torically) elliptically fibered

– Big picture for fourfolds appears similar to CY3, but additional complexity. Would like clearer understanding of threefold bases via Mori theory

– Systematic construction of exotic matter structure over singular divisors