Grading: The problems that will be graded in detail for this assignment are Ch. 9, problems 9a and 21 (each worth 5 points total). You will receive 1 point for each of the other problems if you have made a reasonable attempt at a solution. (Total points for this assignment=18)

Ch. 9, Problem 7

a) Let \( E_2 = E, E_1 = 0 \). The states are nondegenerate, so \( g(E_2) = g(E_1) = 1 \). Let \( N_1 \) be the number of states with energy \( E_1 \) and \( N_2 \) be the number of states with energy \( E_2 \). Then we have:

\[
\frac{N_2}{N_1} = \frac{p(E_2)}{p(E_1)} = \frac{g(E_2) f_{MB}(E_2)}{g(E_1) f_{MB}(E_1)} = e^{-E/kT}
\]

b) \[
E_m = \frac{N_1 E_1 + N_2 E_2}{N_1 + N_2} = \frac{E_1 + (N_2/N_1) E_2}{1 + (N_2/N_1)} = \frac{0 + e^{-E/kT} E}{1 + e^{-E/kT}} = \frac{E}{e^{E/kT} + 1}
\]

c) \[
E_{\text{total}} = N_1 E_1 + N_2 E_2 = N E_m = \frac{N E}{e^{E/kT} + 1}
\]

d) Assume \( N = N_A \) (corresponding to one mole of the substance):

\[
\frac{dE_{\text{total}}}{dT} = N_A E \frac{1}{(e^{E/kT} + 1)^2} \left( -\frac{E}{kT^2} e^{E/kT} \right) = k N_A (E/kT)^2 \frac{e^{E/kT}}{(e^{E/kT} + 1)^2}
\]

\[
C = R(E/kT)^2 \frac{e^{E/kT}}{(e^{E/kT} + 1)^2}
\]

Ch. 8, Problem 8

a) At height \( h \), \( E = K + mgh \), where \( K \) is the kinetic energy.

\[
\frac{p(E_2)}{p(E_1)} = \frac{g(E_2) e^{-E_2/kT}}{g(E_1) e^{-E_1/kT}} = \frac{e^{-(K+mgh)/kT}}{e^{-K/kT}} = e^{-mgh/kT}
\]

b) The density at height \( h \) is proportional to the probabllity to find a molecule at height \( h \). Thus

\[
\frac{\rho(h)}{\rho(0)} = e^{-mgh/kT}
\]

c) Probably not, because the atmosphere does not have a constant temperature as a function of height.

Ch. 8, Problem 9
a) The energy eigenvalues in the magnetic field are given by:

\[ E = \mu_B B m_l \] so

\[ \Delta E = E_1 - E_0 = E_0 - E_{-1} = (9.27 \times 10^{-24} \text{ J/T}) (5.0 \text{ T}) = 4.64 \times 10^{-23} \text{ J} \]

The ratio of the occupation probabilities of the levels is given by:

\[ \frac{p(+1)}{p(0)} = \frac{p(0)}{p(-1)} = e^{-\Delta E/kT} = e^{-\frac{4.64 \times 10^{-23}}{1.38 \times 10^{-23}}} \frac{\text{J}}{\text{K}}(293 \text{ K}) = 0.9886 \]

The fraction of atoms in the \( m_l = +1 \) state is:

\[
\begin{align*}
f(+1) &= \frac{p(+1)}{p(+1) + p(0) + p(-1)} = \frac{p(+1)/p(0)}{0.9886} = 0.3295 \\
f(0) &= \frac{p(0)}{p(+1) + p(0) + p(-1)} = \frac{1}{0.9886 + 1 + 1/0.9886} = 0.3333 \\
f(-1) &= 1 - f(0) - f(+1) = 1 - 0.3333 - 0.3295 = 0.3372
\end{align*}
\]

Ch. 8, Problem 10

If we analyze the problem in the rotating frame of the centrifuge, the molecules will be at rest (zero kinetic energy), but they will be subject to a centrifugal force due to the fact that they are in a non-inertial frame. The centrifugal force is \( F = mx\omega^2 \). Let us assume that the forces in the liquid that give rise to this force can be associated with a potential energy \( U(x) \), where \( F = -dU/dx \):

\[
U = -\int F dx = -\int_0^x m x \omega^2 dx = -\frac{m \omega^2 x^2}{2}
\]

where we take \( U = 0 \) at \( x = 0 \) (on the axis of rotation). The Boltzmann factors then give:

\[
\begin{align*}
\frac{\rho(x)}{\rho(0)} &= \frac{f_{MB}(x)}{f_{MB}(0)} = e^{-U/kT} = e^{m \omega^2 x^2 / 2kT} \\
\rho(x) &= \rho(0) e^{m \omega^2 x^2 / 2kT} \text{ where } \rho_0 = \rho(0).
\end{align*}
\]

Ch. 8, Problem 17

We assume each copper atom contributes one free electron to the metal.

\[
\frac{N}{V} = \frac{\rho N_A}{M} = \frac{(8.95 \text{ g/cm}^3)(6.02 \times 10^{23} \text{ atoms/mole})}{63.5 \text{ g/mole}} = 8.48 \times 10^{28} \text{ atoms/m}^3
\]
$$E_F = \frac{\hbar^2}{2m} \left( \frac{3}{8\pi} N \right)^{2/3} = \frac{\hbar^2 c^2}{2mc^2} \left( \frac{3}{8\pi} N \right)^{2/3}$$
$$= \frac{(1240 \text{ eV-nm})^2}{2(0.511 \times 10^6 \text{ eV})} \left( \frac{3}{8\pi} \right)^{2/3} \left( 8.48 \times 10^{28} \text{ m}^{-3} \right)^{2/3} = 7.04 \text{ eV}$$

$$E_m = \frac{3}{5} E_F = 4.22 \text{ eV}$$

**Ch. 8, Problem 19**

At room temperature (293 K), $kT = 0.02525 \text{ eV}$. For sodium, $E_F = 3.15 \text{ eV}$.

$$f_{FD}(E) = \frac{1}{e^{(E-E_F)/kT} + 1} = 0.1$$

$$e^{(E-E_F)/kT} = 9 \quad \text{or} \quad \frac{E - E_F}{kT} = \ln 9 = 2.20$$

$$E = E_F + 2.20kT = 3.15 \text{ eV} + 2.20(0.02525 \text{ eV}) = 3.21 \text{ eV}$$

$$f_{FD}(E) = \frac{1}{e^{(E-E_F)/kT} + 1} = 0.9$$

$$e^{(E-E_F)/kT} = \frac{1}{9} \quad \text{or} \quad \frac{E - E_F}{kT} = -\ln 9 = -2.20$$

$$E = E_F - 2.20kT = 3.15 \text{ eV} - 2.20(0.02525 \text{ eV}) = 3.09 \text{ eV}$$

The energy difference is 3.21 eV - 3.09 eV = 0.12 eV, so the distribution is reasonably sharp. The occupation probability drops from near maximum (0.9) to near minimum (0.1) within ±2% of $E_F$.

**Ch. 8, Problem 21**

a) From Equation 10.36, the number of electrons with energy between $E$ and $E + dE$ is:

$$p(E)dE = \frac{8\sqrt{2\pi V} m^{3/2}}{h^3} \frac{\sqrt{E}dE}{e^{(E-E_F)/kT} + 1}$$

The probability to find an electron in this energy interval is given by $p(E)dE/N$ where $N$ is the total number of electrons. We aren’t given $N$ and $V$ in this problem, only the Fermi energy $E_F$. $p(E)dE/N$ can be written in terms of the Fermi energy using Equation 10.39. At $T = 295 \text{ K}$, $kT = 0.0254 \text{ eV}$:

$$\frac{p(E)dE}{N} = \frac{3 E^{1/2}dE}{2} \frac{1}{E_F^{3/2}} \frac{1}{e^{(E-E_F)/kT} + 1}$$

$$= \frac{3 (5.00 \text{ eV})^{1/2}(0.10 \text{ eV})}{2} \frac{1}{(3.00 \text{ eV})^{3/2}} \frac{1}{e^{2.00 \text{ eV}/(0.0254 \text{ eV})} + 1} = 4.37 \times 10^{-36}$$

b) At $T = 2500 \text{ K}$, $kT = 0.215 \text{ eV}$:

$$\frac{p(E)dE}{N} = \frac{3 E^{1/2}dE}{2} \frac{1}{E_F^{3/2}} \frac{1}{e^{(E-E_F)/kT} + 1}$$

$$= \frac{3 (5.00 \text{ eV})^{1/2}(0.10 \text{ eV})}{2} \frac{1}{(3.00 \text{ eV})^{3/2}} \frac{1}{e^{2.00 \text{ eV}/(0.215 \text{ eV})} + 1} = 6.00 \times 10^{-6}$$
Ch. 8, Problem 22

There are 20 protons and 20 neutrons in the nuclear volume, which is

\[ V = \frac{4}{3} \pi R^3 = \frac{4}{3} \pi (4.1 \text{ fm})^3 = 289 \text{ fm}^3 \]

\[ E_F = \frac{\hbar^2}{2m} \left( \frac{3}{8\pi} \frac{N}{V} \right)^{2/3} = \frac{\hbar^2 c^2}{2mc^2} \left( \frac{3}{8\pi} \frac{N}{V} \right)^{2/3} \]

\[ = \frac{(1240 \text{ eV-nm})^2}{2(940 \text{ MeV})(3)} \left( \frac{20}{289 \text{ fm}^3} \right)^{2/3} = 33.4 \text{ MeV} \]

\[ E_m = \frac{3}{5} E_F = 20.1 \text{ MeV} \]

These are quite reasonable numbers for the motion of nucleons in the nucleus where the potential well depth is typically about 50 MeV.

Ch. 8, Problem 23

For a mass of \( 2.00 \times 10^{30} \text{ kg} \), the number of nucleons is about

\[ N = \frac{2.00 \times 10^{30} \text{ kg}}{1.67 \times 10^{-27} \text{ kg/nucleon}} = 1.20 \times 10^{57} \text{ nucleons} \]

The number of electrons is thus about \( 6 \times 10^{56} \), and the electron density is

\[ \frac{N}{V} = \frac{6.00 \times 10^{56}}{\frac{4}{3} \pi (6.40 \times 10^6 \text{ m})^3} = 5.46 \times 10^{35} \text{ m}^{-3} \]

\[ E_F = \frac{\hbar^2 c^2}{2mc^2} \left( \frac{3}{8\pi} \frac{N}{V} \right)^{2/3} \]

\[ = \frac{(1240 \text{ eV-nm})^2}{2(0.511 \times 10^6 \text{ eV})(3)} \left( \frac{5.46 \times 10^{35} \text{ m}^{-3}}{8\pi} \right)^{2/3} = 244 \text{ keV} \]

(This is rather close to the electron’s rest energy of 511 keV, and thus the nonrelativistic expression \( E = p^2/2m \) used to derive Equation 10.35 may not be valid.)

Ch. 8, Problem 24

\[ N = \frac{2m_{\text{Sun}}}{m_{\text{nucleon}}} = \frac{4.00 \times 10^{30} \text{ kg}}{1.67 \times 10^{-27} \text{ kg/nucleon}} = 2.40 \times 10^{57} \text{ nucleons} \]

\[ \frac{N}{V} = \frac{2.40 \times 10^{57}}{\frac{4}{3} \pi (10^4 \text{ m})^3} = 5.72 \times 10^{44} \text{ m}^{-3} \]

\[ E_F = \frac{\hbar^2 c^2}{2mc^2} \left( \frac{3}{8\pi} \frac{N}{V} \right)^{2/3} \]

\[ = \frac{(1240 \text{ eV-nm})^2}{2(940 \text{ MeV})(3)} \left( \frac{5.72 \times 10^{44} \text{ m}^{-3}}{8\pi} \right)^{2/3} = 137 \text{ MeV} \]