**Problem 1**

a) The impact parameter is given by:

\[ b = \frac{z Z}{2 K} \frac{e^2}{4 \pi \epsilon_0} \cot \frac{\theta}{2} \]
\[ = \frac{2(47)}{2(10.00 \text{ MeV})} \left( 1.440 \text{ MeV-fm} \right) \cot 55^\circ = 4.74 \text{ fm} \]

The minimum distance for a given impact parameter is given by:

\[ \frac{1}{2} m v^2 = \frac{1}{2} m v^2 \frac{b^2}{r_{min}^2} + \frac{e^2}{4 \pi \epsilon_0 r_{min}} z Z \]
\[ (10.00 \text{ MeV}) r_{min}^2 = (10.00 \text{ MeV}) (4.73 \text{ fm})^2 + (1.440 \text{ MeV-fm})(2)(47) r_{min} \]
\[ r_{min}^2 - 13.5 r_{min} - 22.4 = 0 \]
\[ r_{min} = 15.0 \text{ fm} \text{ or } -1.5 \text{ fm} \]

Only the positive root is physically meaningful, so \( r_{min} = 15.0 \text{ fm} \).

\( c) \)

\[ U = \frac{e^2}{4 \pi \epsilon_0} \frac{z Z}{r_{min}} \left( \frac{1.440 \text{ MeV-fm})(2)(47)}{15.0 \text{ fm}} \right) = 9.02 \text{ MeV} \]
\[ K = E - U = 10.00 \text{ MeV} - 9.02 \text{ MeV} = 0.98 \text{ MeV} \]

**Problem 2**

a) From the \( n = 8 \) level, downward transitions are possible to any level of smaller \( n \). The transitions with the longest wavelengths are those with the smallest energy difference.

\( 8 \rightarrow 7 : \Delta E = E_8 - E_7 = (-13.6 \text{ eV}) Z^2 \left( \frac{1}{64} - \frac{1}{49} \right) = 0.260 \text{ eV} \]

\[ \lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV-nm}}{0.260 \text{ eV}} = 4.77 \mu \text{m} \]

\( 8 \rightarrow 6 : \Delta E = E_8 - E_6 = (-13.6 \text{ eV}) Z^2 \left( \frac{1}{64} - \frac{1}{36} \right) = 0.661 \text{ eV} \]

\[ \lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV-nm}}{0.661 \text{ eV}} = 1.88 \mu \text{m} \]

\( 8 \rightarrow 5 : \Delta E = E_8 - E_5 = (-13.6 \text{ eV}) Z^2 \left( \frac{1}{64} - \frac{1}{25} \right) = 1.33 \text{ eV} \]

\[ \lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV-nm}}{1.33 \text{ eV}} = 0.935 \mu \text{m} \]

b) The transition with the shortest wavelength is the one with the largest energy difference.

\( 8 \rightarrow 1 : \Delta E = E_8 - E_1 = (-13.6 \text{ eV}) Z^2 \left( \frac{1}{64} - 1 \right) = 53.6 \text{ eV} \)
\[ \lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV-nm}}{53.6 \text{ eV}} = 23.2 \text{ nm} \]

c) From the \( n = 8 \) level, the atom can absorb a photon and the electron will jump to a state of larger \( n \). The longest absorption wavelengths correspond to the smallest energy differences.

\[ 8 \to 9 : \Delta E = E_9 - E_8 = (-13.6 \text{ eV}) Z^2 \left( \frac{1}{81} - \frac{1}{64} \right) = 0.178 \text{ eV} \]

\[ \lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV-nm}}{0.178 \text{ eV}} = 6.95 \mu\text{m} \]

\[ 8 \to 10 : \Delta E = E_{10} - E_8 = (-13.6 \text{ eV}) Z^2 \left( \frac{1}{100} - \frac{1}{64} \right) = 0.306 \text{ eV} \]

\[ \lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV-nm}}{0.306 \text{ eV}} = 4.05 \mu\text{m} \]

\[ 8 \to 11 : \Delta E = E_{11} - E_8 = (-13.6 \text{ eV}) Z^2 \left( \frac{1}{121} - \frac{1}{64} \right) = 0.400 \text{ eV} \]

\[ \lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV-nm}}{0.400 \text{ eV}} = 3.10 \mu\text{m} \]

d) The shortest wavelength corresponds to the largest energy difference.

\[ 8 \to \infty : \Delta E = E_{\infty} - E_8 = (-13.6 \text{ eV}) Z^2 \left( 0 - \frac{1}{64} \right) = 0.850 \text{ eV} \]

\[ \lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV-nm}}{0.850 \text{ eV}} = 1.46 \mu\text{m} \]

**Problem 3**

The usual Bohr radius is:

\[ a_0 = \frac{4\pi \epsilon_0 \hbar^2}{me^2} = 0.0529 \text{ nm} \]

This problem consists of replacing the electric charge \( e \) by \( e/10 \), so in the other world the Bohr radius is increased by a factor of \((10)^2 = 100\), or 5.29 nm. The usual expression for the energy levels for the hydrogen atom is:

\[ E_n = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2 \cdot n^2} = -\frac{13.6 \text{ eV}}{n^2} \]

In the new world this energy is decreased by \((1/10)^4 = 10^{-4}\), so

\[ E_n = -\frac{0.00136 \text{ eV}}{n^2} \]

\[ \Delta E = E_2 - E_1 = -(0.00136 \text{ eV}) \left( \frac{1}{4} - 1 \right) \]

So the energy difference is 0.00102 \text{ eV}.

**Problem 4**
For \( l = 4, m_l = +4, +3, +2, +1, 0, -1, -2, -3, -4 \)

\[
\cos \theta = \frac{L_z}{|L|} = \frac{m_l \hbar}{\hbar \sqrt{l(l + 1)}} = \frac{m_l}{\sqrt{20}}
\]

\[
m_l = +4 \quad \cos \theta = \frac{2}{\sqrt{5}} \quad \theta = 27^\circ
\]

\[
m_l = +3 \quad \cos \theta = \frac{3}{\sqrt{20}} \quad \theta = 48^\circ
\]

\[
m_l = +2 \quad \cos \theta = \frac{1}{\sqrt{5}} \quad \theta = 63^\circ
\]

\[
m_l = +1 \quad \cos \theta = \frac{1}{\sqrt{20}} \quad \theta = 77^\circ
\]

\[
m_l = 0 \quad \cos \theta = 0 \quad \theta = 90^\circ
\]

\[
m_l = -1 \quad \cos \theta = -\frac{1}{\sqrt{20}} \quad \theta = 103^\circ
\]

\[
m_l = -2 \quad \cos \theta = -\frac{1}{\sqrt{5}} \quad \theta = 117^\circ
\]

\[
m_l = -3 \quad \cos \theta = -\frac{3}{\sqrt{20}} \quad \theta = 132^\circ
\]

\[
m_l = -4 \quad \cos \theta = -\frac{2}{\sqrt{5}} \quad \theta = 153^\circ
\]

**Problem 5**

For the \( n = 2, l = 0 \) state the radial probability density is:

\[
P(r) = r^2 \left| R_{n,l}(r) \right|^2 = r^2 \frac{1}{8a_0^3} \left( 2 - \frac{r}{a_0} \right)^2 e^{-r/a_0}
\]

The total probability to find the electron beyond \( 5a_0 \) is

\[
P = \frac{1}{8a_0^3} \int_{5a_0}^{\infty} \left( 4r^2 - \frac{4r^3}{a_0} + \frac{r^4}{a_0^4} \right) e^{-r/a_0} dr
\]

\[
= \frac{1}{8} \int_{5}^{\infty} (4x^2 - 4x^3 + x^4) e^{-x} dx
\]

where the substitution \( x = r/a_0 \) has been used. At this point you can use standard integrals from the integral tables (I would include the relevant integrals on the front of the exam):

\[
\int x^4 e^{-x} dx = (-x^4 - 4x^3 - 12x^2 - 24x - 24) e^{-x}
\]

\[
\int x^3 e^{-x} dx = (-x^3 - 3x^2 - 6x - 6) e^{-x}
\]

\[
\int x^2 e^{-x} dx = (-x^2 - 2x - 2) e^{-x}
\]
Using these integrals yields:

\[ P = \frac{1}{8} [10.5718 - 4(1.5902) + 4(0.2493)] = 0.651 \]

For the \( n = 2, l = 1 \) state the radial probability density is:

\[ P(r) = r^2 |R_{n,l}(r)|^2 = r^2 \frac{1}{24a_0^3} \frac{r^2}{a_0} e^{-r/a_0} \]

\[ P = \frac{1}{24a_0^3} \int_0^\infty r^4 e^{-r/a_0} dr = \frac{1}{24} \int_0^\infty x^4 e^{-x} dx = 0.440 \]

Thus the \( n = 2, l = 0 \) electron is more likely to be found beyond \( 5a_0 \) than the \( n = 2, l = 1 \) electron.

**Problem 6**

The energy of the photon emitted by the He\(^+\) ion is:

\[ \Delta E = Z^2 (-13.6 \text{ eV}) \left( \frac{1}{4} - 1 \right) = 40.8 \text{ eV} \]

The ionization energy for the hydrogen atom in its ground state is \( E_\infty - E_1 = 13.6 \text{ eV} \). So after ionization the electron has a kinetic energy of \( 40.8 - 13.6 \text{ eV} \). This is much less than the rest energy of the electron so we can use the nonrelativistic expression:

\[ K = \frac{1}{2} mc^2 \frac{v^2}{c^2} \]

\[ 27.2 \text{ eV} = \frac{1}{2} (0.511 \times 10^6 \text{ eV}) \frac{v^2}{c^2} \]

\[ v = 0.01c = 3 \times 10^6 \text{ m/s} \]

**Problem 7**

All transitions with \( \Delta l = \pm 1 \) are allowed.
Problem 8

Solving Equation 8.1 with $\lambda = c/\nu$ gives:

$$Z - 1 = \sqrt{\frac{4}{3R_\infty \lambda}} = \sqrt{\frac{1.33}{(1.0974 \times 10^7 \text{ m}^{-1})(0.1935 \times 10^{-9} \text{ m})}} = 25.02$$

so $Z = 26$ for $\lambda = 0.1935$ nm

Similarly

- $Z = 27$ for $\lambda = 0.1787$ nm
- $Z = 28$ for $\lambda = 0.1656$ nm
- $Z = 30$ for $\lambda = 0.1434$ nm

The missing atomic number is $Z = 29$ which will have a $K_\alpha$ wavelength of 0.1543 nm.

Problem 9

a) Pa: ([Rn]5f^26d7s^2)

The 7s subshell is full. The two 5f electrons can have $m_s = +1/2$ as can the one 6d electron. So $S = 3(1/2) = 3/2$. Maximizing $m_l$ requires $m_l = +3,+2$ for the two 5f electrons and $m_l = +2$ for the single 6d electron. So $L = 7$. Thus $L = 7$ and $S = 3/2$ for the ground state.

b) U: ([Rn]5f^36d7s^2)

The 7s subshell is full. The three 5f electrons can have $m_s = +1/2$ as can the one 6d electron. So $S = 4(1/2) = 2$. Maximizing $m_l$ requires $m_l = +3,+2,+1$ for the three 5f electrons and $m_l = +2$ for the single 6d electron. So $L = 8$. Thus $L = 8$ and $S = 2$ for the ground state.

c) Cm: ([Rn]5f^76d7s^2)

The 7s subshell is full. The seven 5f electrons can have $m_s = +1/2$ as can the one 6d electron. So $S = 8(1/2) = 4$. Maximizing $m_l$ requires $m_l = +3,+2,+1,0,-1,-2,-3$ for the seven 5f electrons and $m_l = +2$ for the single 6d electron. So $L = 2$. Thus $L = 2$ and $S = 4$ for the ground state.

Problem 10

$f$ corresponds to $l = 3$ so adding the orbital angular momenta for two $l = 3$ states yields $L$ ranging from $|3 - 3| = 0$ to $|3 + 3| = 6$ or $L = 0, 1, 2, 3, 4, 5, 6$. The two states can each have spin $s = 1/2$ so adding the two spin angular momenta yields $S = 0, 1$.

Problem 11

The neutrons and protons each form a separate Fermi gas, so they will each have their own Fermi energy. The nuclear volume is:

$$V = \frac{4}{3} \pi R^3 = \frac{4}{3} \pi (7.7 \text{ fm})^3 = 1912 \text{ fm}^3$$

For the neutrons ($N = 126$) we have:

$$E_F = \frac{\hbar^2}{2m} \left( \frac{3}{8\pi V} \right)^{2/3} = \frac{\hbar^2 c^2}{2mc^2} \left( \frac{3}{8\pi V} \right)^{2/3}$$
\[
\frac{(1240 \text{ MeV-fm})^2}{2(940 \text{ MeV})} \left( \frac{3}{8\pi} \right)^{2/3} \left( \frac{126}{1912 \text{ fm}^3} \right)^{2/3} = 32.3 \text{ MeV}
\]

\[
E_m = \frac{3}{5} E_F = 19.4 \text{ MeV}
\]

For the protons \((N = 82)\) we have:

\[
E_F = \frac{\hbar^2}{2m} \left( \frac{3}{8\pi} \right)^{2/3} = \frac{h^2 c^2}{2mc^2} \left( \frac{3}{8\pi} \right)^{2/3}
\]

\[
= \frac{(1240 \text{ MeV-fm})^2}{2(938 \text{ MeV})} \left( \frac{3}{8\pi} \right)^{2/3} \left( \frac{82}{1912 \text{ fm}^3} \right)^{2/3} = 24.3 \text{ MeV}
\]

\[
E_m = \frac{3}{5} E_F = 14.6 \text{ MeV}
\]

**Problem 12**

The number of conduction electrons with energy between \(E\) and \(E + dE\) is given by Equation 10.36:

\[
p(E) dE = \frac{8\sqrt{2\pi} V m^{3/2}}{\hbar^3} \frac{E^{1/2} dE}{e^{(E-E_F)/kT} + 1}
\]

The Fermi-Dirac occupation probability is used since the conduction electrons have spin 1/2. To get the number per unit volume in a given energy range, we simply divide the above expression by the volume \(V\). In this case we are given that the energy interval has width \(dE = 0.01E_F = 0.0315 \text{ eV}\) centered on the average energy per electron:

\[
E_m = \frac{3}{5} E_F = 1.89 \text{ eV}
\]

We are given that the temperature is room temperature so \(kT = (8.617 \times 10^{-5} \text{ eV/K})(300 \text{ K}) = 0.025 \text{ eV}\). So the number of conduction electrons per unit volume is given by:

\[
\frac{p(E) dE}{V} = \frac{8\sqrt{2\pi} V m^{3/2}}{\hbar^3} \frac{E_m^{1/2}(0.01E_F)}{e^{(E-E_F)/kT} + 1}
\]

\[
= \frac{8\sqrt{2\pi}(0.511 \times 10^6 \text{ eV})^{3/2}}{(1240 \text{ eV-nm})^3} \frac{(1.89 \text{ eV})^{1/2}(0.0315 \text{ eV})}{e^{(1.89 \text{ eV}-3.15 \text{ eV})/(0.025 \text{ eV})} + 1}
\]

\[
= 0.295 \text{ nm}^{-3} = 0.295 \times 10^{27} \text{ m}^{-3}
\]

**Problem 13**

I neglected to tell you what the pressure of the gas is, so you can't really get the density. Let us say you are given the fact that the density is \(n = 1.1 \times 10^{18} \text{ cm}^{-3}\). So the mean distance between the atoms in the gas will be \(n^{-1/3} \sim 10^{-6} \text{ cm} = 10 \text{ nm}\). To justify using the Maxwell-Boltzmann distribution function we have to verify that the de Broglie wavelength of the particles is much less than the interatomic spacing. Only when these two quantities are comparable (or when the de Broglie wavelength
is greater than the interparticle spacing) do we need to use the quantum distribution functions. The average kinetic energy of the atoms in the hydrogen gas is given by:

\[
K = \frac{3}{2} kT
\]

\[
\lambda = \frac{\hbar}{p} = \frac{\hbar}{\sqrt{2mK}}
\]

\[
= \frac{\hbar c}{\sqrt{3mc^2kT}}
\]

\[
= \frac{(1240 \text{ eV-nm})}{\sqrt{3(939 \text{ MeV})(8.617 \times 10^{-5} \text{ eV/K})(8000 \text{ K})}}
\]

\[
= 0.028 \text{ nm} << (d = 10 \text{ nm})
\]

so the use of the Maxwell-Boltzmann distribution is well-justified. The ratio of the number of particles in the two states of the hydrogen atom can be calculated using Equation 10.18:

\[
\frac{p(E_3)}{p(E_1)} = \frac{g(E_3)e^{-E_3/kT}}{g(E_1)e^{-E_1/kT}}
\]

\[
= \left(\frac{2(3)^2}{2(1)^2}\right) e^{-(-1.5 \text{ eV}+13.6 \text{ eV})/0.689 \text{ eV}}
\]

\[
= 2.1 \times 10^{-7}
\]

where we have used the following facts for atomic hydrogen:

\[
g(E_n) = 2n^2 \text{ is the degeneracy for energy level } E_n
\]

\[
E_n = -\frac{13.6 \text{ eV}}{n^2}
\]

and \(kT = 0.689 \text{ eV at } T = 8000\text{K.}\)