Physics 3304
Final Review solutions

Problem 1
The time elapsed in the laboratory is given by the time dilation expression:
\[ \Delta t = \gamma \Delta t_0 \]
\[ = \frac{1}{\sqrt{1 - (0.99)^2}} (10^{-7} \text{ sec}) \]
\[ = 7.09 \times 10^{-7} \text{ sec} \]
So the particle goes a distance
\[ x = v \Delta t = 0.99c (7.09 \times 10^{-7} \text{ s}) \]
\[ = 211 \text{ meters} \]
in the lab before decaying.

Problem 2
Let the time elapsed in the airplane clock be \( \Delta t_0 \) and the time elapsed as read on the ground clock be \( \Delta t \). We want to determine how big \( \Delta t \) is such that \( \Delta t - \Delta t_0 = 1 \text{ sec} \). The time dilation expression relates the two:
\[ \Delta t = \gamma \Delta t_0 \]
\[ \Delta t_0 = \frac{1}{\gamma} \Delta t = \sqrt{1 - \beta^2} \Delta t \]
\[ = (1 - \frac{1}{2} \beta^2) \Delta t \]
where we have used the binomial expansion since \( \beta^2 \) is too small for most calculators to deal with. So we have
\[ \Delta t - \Delta t_0 = \frac{1}{2} \beta^2 \Delta t = 1 \text{ sec} \]
\[ \Delta t = \frac{2 \text{ sec}}{\beta^2} = \frac{2 \text{ sec} (3.0 \times 10^8 \text{ m/s})^2}{(300 \text{ m/s})^2} = 2.0 \times 10^{12} \text{ sec} \]
\[ = 6.3 \times 10^4 \text{ years} \]

Problem 3
The total energy of the electron can be written as:
\[ E = \gamma m_e c^2 \]
Equating this to the rest energy of the proton gives:
\[ m_p c^2 = \gamma m_e c^2 \]
\[ \gamma = \frac{m_p c^2}{m_e c^2} \]
\[ 1 - \beta^2 = \left( \frac{m_e c^2}{m_p c^2} \right)^2 = \left( \frac{511 \times 10^3 \text{ eV}}{938 \times 10^6 \text{ eV}} \right)^2 \]
\[ \beta = 0.9999998 c \]
Problem 4

The work will increase the kinetic energy of the electron from \( K_i \) to \( K_f \). In general, the kinetic energy of the electron can be written as:

\[
K = E - m_e c^2
\]

where \( E \) is the total energy of the electron. The work done can be written as:

\[
K_f - K_i = E_f - E_i = m_e c^2 (\gamma_f - \gamma_i)
\]

\[
= (511 \times 10^3 \text{ eV}) \left( \frac{1}{\sqrt{1 - (2.4 \times 10^8 \text{ m/s} / 3.0 \times 10^8 \text{ m/s})^2}} - \frac{1}{\sqrt{1 - (1.2 \times 10^8 \text{ m/s} / 3.0 \times 10^8 \text{ m/s})^2}} \right)
\]

\[
= 0.294 \text{ MeV}
\]

Problem 5

The de Broglie wavelength of the electron is written as:

\[
\lambda = \frac{h}{p}
\]

The kinetic energy of the electron is given in terms of the potential difference as:

\[
K = eV
\]

Now we need the relativistic relation between the kinetic energy and the momentum:

\[
K = E - m_e c^2
\]

\[
p_c = \sqrt{(K + m_e c^2)^2 - (m_e c^2)^2}
\]

\[
= \sqrt{(eV + m_e c^2)^2 - (m_e c^2)^2}
\]

\[
= \sqrt{(eV)^2 + (eV)(2m_e c^2)}
\]

\[
= \sqrt{2m_e c^2 eV} \sqrt{V \left( \frac{eV}{2m_e c^2} + 1 \right)}
\]

so the de Broglie wavelength is:

\[
\lambda = \frac{hc}{p_c}
\]

\[
= \frac{hc}{\sqrt{2m_e c^2 eV}} \left[ V \left( \frac{eV}{2m_e c^2} + 1 \right) \right]^{-1/2}
\]

\[
= \frac{1240 \text{ eV-nm}}{\sqrt{2(511 \times 10^3 \text{ eV})(1 \text{ eV})}} \left[ V \left( \frac{eV}{2m_e c^2} + 1 \right) \right]^{-1/2}
\]

\[
= (1.227 \text{ nm}) \left[ V \left( \frac{eV}{2m_e c^2} + 1 \right) \right]^{-1/2}
\]
where $V$ is assumed to be in volts.

**Problem 6**

The expression for the energy levels in a hydrogen-like atom is given by:

$$E_n = -\frac{m(Ze^2)^2}{32\pi^2\epsilon_0^2\hbar^2 n^2} = -(13.6 \text{ eV}) \frac{Z^2}{n^2}$$

For the muonic atom we replace the electron mass by the muon mass so we have:

$$E_n = -\left(\frac{m_u}{m_e}\right)(13.6 \text{ eV}) \frac{Z^2}{n^2} = -2815 \text{ eV} \frac{Z^2}{n^2}$$

The energy radiated in the $n = 2$ to $n = 1$ transition will be:

$$\Delta E = E_2 - E_1 = -2815 \text{ eV}(22)^2\left(\frac{1}{2^2} - \frac{1}{1^2}\right)$$

$$= 1.02 \text{ MeV}$$

**Problem 7**

The relative number of hydrogen atoms in a state of energy $E_n$ relative to the number in the ground state can be written as:

$$\frac{p(E_n)}{p(E_1)} = \frac{g(E_n)e^{-E_n/kT}}{g(E_1)e^{-E_1/kT}} = \frac{2n^2e^{-E_n/kT}}{2(1)^2e^{-E_1/kT}}$$

$$= n^2e^{-(E_n - E_1)/kT}$$

where we have used the classical Maxwell-Boltzmann distribution for this ideal hydrogen gas. Using the fact that $kT = (8.617 \times 10^{-5} \text{ eV/K})(5000 \text{ K}) = 0.431 \text{ eV}$ and that the hydrogen atom energies are given by $E_n = -13.6 \text{ eV}/n^2$ we have:

$$\frac{p(E_2)}{p(E_1)} = (2)^2e^{-(10.2 \text{ eV}/0.431 \text{ eV})} = 2.10 \times 10^{-10}$$

$$\frac{p(E_3)}{p(E_1)} = (3)^2e^{-(12.1 \text{ eV}/0.431 \text{ eV})} = 5.8 \times 10^{-12}$$

$$\frac{p(E_4)}{p(E_1)} = (4)^2e^{-(12.75 \text{ eV}/0.431 \text{ eV})} = 2.3 \times 10^{-12}$$

**Problem 8**

a) In a Fermi gas the occupancy probability of any level is given by the Fermi-Dirac distribution:

$$f = 0.90 = \frac{1}{e^{(E-E_F)/kT} + 1}$$

$$\ln \left(\frac{1}{0.90} - 1\right) = \left(\frac{E - E_F}{kT}\right)$$

$$E = -2.1972kT + E_F = -(2.1972)(8.617 \times 10^{-5} \text{ eV/K})(1000 \text{ K}) + 7.04 \text{ eV}$$

$$= 6.83 \text{ eV}$$
b) The number of conduction electrons in an energy interval of width \( dE \) about an energy \( E \) is given by Equation 10.36 in the text:

\[
p(E)dE = \frac{8\sqrt{2\pi} V m^{3/2}}{h^3} \frac{E^{1/2}dE}{e^{E-E_F}/kT + 1}
\]

The number per unit volume is just the above expression divided by the volume \( V \):

\[
p(E)dE = \frac{8\sqrt{2\pi}(mc^2)^{3/2}}{(hc)^3} \frac{E^{1/2}dE}{e^{E-E_F}/kT + 1} = \frac{8\sqrt{2\pi}(511 \times 10^3 \text{ eV})^{3/2}}{(1240 \text{ ev-nm})^3} \frac{(6.85 \text{ eV})^{1/2}(0.1 \text{ eV})}{e^{(6.85-7.01) \text{ eV}/0.0862 \text{ eV}} + 1} = 1.60 \times 10^{21} \text{ cm}^{-3}
\]

**Problem 9**

The process of removing one neutron from \(^{16}\text{O}\) can be written as:

\[^{16}\text{O} \rightarrow^{15}\text{O} + n + Q\]

The \( Q \) value for this process is:

\[
Q = [m(^{16}\text{O}) - m(^{15}\text{O}) - m(n)]c^2
= [15.994915 - 15.003066 - 1.008665](u)(c^2)(931.5 \text{ MeV}/uc^2)
= -15.7 \text{ MeV}
\]

Thus, at least \(-Q = 15.7 \text{ MeV}\) of energy is necessary to remove the neutron.

**Problem 10**

We use the radioactive decay law:

\[
N(t) = N_0 e^{-\lambda t} = N_0 e^{-\ln 2 t/\lambda}
\]

\[
\frac{N(t)}{N_0} = e^{-\ln 2 (35 \text{ years})/(12.5 \text{ years})} = 0.144
\]

**Problem 11**

The reaction is:

\[^{3}\text{He} \rightarrow^{12}\text{C} + Q\]

\[
Q = [3m(^{4}\text{He}) - m(^{12}\text{C})]c^2
= [(3)(4.002603) - 12.000000](u)(c^2)(931.5 \text{ MeV}/uc^2)
= 7.27 \text{ MeV}
\]

**Problem 12**

a) This \( \beta^- \) decay can be written as:

\[^{12}\text{B} \rightarrow^{12}\text{C} + e^- + p\]
The Q value for a $\beta^-$-decay (in terms of atomic masses) can be written as:

$$Q = [m(\text{^{12}B}) - m(\text{^{12}C})]c^2$$
$$= [12.014352 - 12.000000](u)(c^2)(931.5 \text{ MeV}/uc^2)$$
$$= 13.4 \text{ MeV}$$

b) This $\beta^+$ decay can be written as:

$$\text{^{11}_6C} \rightarrow \text{^{11}_5B} + e^+ + \nu$$

The Q value for a $\beta^+$-decay (in terms of atomic masses) can be written as:

$$Q = [m(\text{^{11}C}) - m(\text{^{11}B}) - 2m_e]c^2$$
$$= [11.011433 - 11.009306](u)(c^2)(931.5 \text{ MeV}/uc^2) - (2)(0.511 \text{ MeV})$$
$$= 0.959 \text{ MeV}$$

Problem 13

a) Given the occupation probability at a given temperature ($kT = 0.0258 \text{ eV}$), we can determine where the Fermi level is. Let $E_d$ be the energy of the donor level.

$$f_{FD}(E_d) = \frac{1}{e^{(E_d-E_F)/kT} + 1}$$
$$5.00 \times 10^{-5} = \frac{1}{e^{(E_d-E_F)/(0.0258 \text{ eV})} + 1}$$
$$9.903 = \frac{(E_d-E_F)}{(0.0258 \text{ eV})}$$
$$E_d - E_F = 0.255 \text{ eV}$$

Let $E_v$ be the energy of a state at the top of the valence band. We can conclude from the information given in the problem that the energy difference between the donor level and the top of the valence band is $E_d - E_v = 1.11 - 0.110 = 1.00 \text{ eV}$. Since we have determined that $E_d - E_F = 0.255 \text{ eV}$ then $E_F - E_v = 0.745 \text{ eV}$ is the Fermi energy relative to the top of the valence band.

b) In this case, $E_c - E_F = 1.11 - 0.745 = .365 \text{ eV}$, so the occupation probability is:

$$f_{FD}(E_d) = \frac{1}{e^{(E_d-E_F)/kT} + 1}$$
$$= \frac{1}{e^{(0.365 \text{ eV})/(0.0258 \text{ eV})} + 1}$$
$$= 7.18 \times 10^{-7}$$

Problem 14

This is a Compton scattering problem:

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos \theta)$$
Here we are given that the scattered wavelength is $\lambda' = 0.0022$ nm and we want to determine the incident wavelength:

$$
\lambda = \lambda' - \frac{\hbar}{m_e c} (1 - \cos \theta)
$$

$$
= 0.0022 \text{ nm} - \frac{1240 \text{ eV-nm}}{511 \times 10^3 \text{ eV}} (1 - \cos 45) 
$$

$$
= 0.0015 \text{ nm}
$$

**Problem 15**

The Heisenberg uncertainty principle says:

$$
\Delta x \Delta p = \frac{\hbar}{c}
$$

$$
\Delta p = \frac{\hbar c}{\Delta x c} = \frac{1240 \text{ eV-nm}}{2\pi(0.1 \text{ nm})} 
$$

$$
= 1974 \text{ eV/c}
$$

Since the kinetic energy of the electron (1 keV) is much less than its rest energy (511 keV) we can use the non-relativistic relation between momentum and kinetic energy:

$$
K = \frac{p^2}{2m}
$$

$$
p = \frac{\sqrt{2mc^2K}}{c} = \frac{\sqrt{2(511 \times 10^3 \text{ eV})(1 \text{ keV})}}{c}
$$

$$
= 3.2 \times 10^4 \text{ eV/c}
$$

So the fractional uncertainty in the measured momentum is:

$$
\frac{\Delta p}{p} = \frac{1974 \text{ eV/c}}{3.2 \times 10^4 \text{ eV/c}}
$$

$$
= 0.062
$$

**Problem 16**

The process here is:

$$
\gamma + \alpha \rightarrow t + p \text{ or equivalently}
$$

$$
\gamma + ^2 \text{He} \rightarrow ^3 \text{H} + ^1 \text{H}
$$

The $Q$ value for this process is:

$$
Q = [m(^2 \text{He}) - m(^3 \text{H}) - m(^1 \text{H})]c^2
$$

$$
= [4.002603 - 3.016049 - 1.007825](u)(c^2)(931.5 \text{ MeV}/u c^2)
$$

$$
= -19.8 \text{ MeV}
$$

Thus the gamma ray must have at least an energy of $-Q = 19.8$ MeV in order to initiate this reaction.
Problem 17

a) The energy levels for a particle trapped in an infinite potential well are given by:

\[ E_n = n^2 E_0 \]

where \( E_0 \) is a constant that depends on the electron mass and the width of the well. Here we want to consider an adjacent pair of energy levels \((n \text{ and } n+1)\) such that the energy difference is three times the energy difference between the \(n = 4\) and \(n = 3\) levels:

\[
E_{n+1} - E_n = (3)(E_4 - E_3) \\
E_0((n + 1)^2 - n^2) = (3)E_0((4)^2 - (3)^2) \\
2n + 1 = (3)(7) \\
n = 10
\]

So the pair of adjacent levels is \(n = 10\) and \(n = 11\).

b) From the second to last equation above, the condition for twice the energy difference can be written as:

\[
2n + 1 = (2)(7) \\
n = \frac{13}{2}
\]

so there is no pair here, since \(n\) only takes on integral values.

Problem 18

In terms of the band theory picture, an insulator consists of a completely full band (the valence band) which is typically separated by several eV from the empty conduction band. Since this energy gap is large compared to \(kT = .025\) eV at room temperature, there is a negligible number of electrons in the conduction band so an insulator is a poor conductor. A metal, on the other hand, has its highest band only half full so there are a large number of nearby states that the electrons can be thermally excited to, so there are many electrons available for conduction.

Problem 19

a) Let the energy when the electron and proton are separated be defined to be \(E = 0\), as usual. So the initial state has an energy \(E_i = -0.85\) eV. The final state has an energy that is \(10.2\) eV greater than the ground state energy (-13.6 eV) so its energy is \(E_f = -3.4\) eV. The energy of the emitted photon is \(E_i - E_f = -0.85\) eV - (-3.4 eV) = 2.55 eV.

b) The energy levels of the hydrogen atom are given as:

\[
E_n = -\frac{13.6 \text{ eV}}{n^2}
\]

so \(n = 4\) for the \(E = -0.85\) eV state and \(n = 2\) for the \(E = -3.4\) eV state.
Problem 20

First we use the given value of \((x^2)_{av}\) to determine which state the electron is in. The wavefunctions for the 1D infinite square well are given by:

\[
\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \text{ where } n = 1, 2, 3, \ldots
\]

for \(x = 0\) to \(L\), with \(\psi = 0\) otherwise.

Knowing the wavefunctions, the average value of \(x^2\) can be written as:

\[
(x^2)_{av} = \int_{-\infty}^{+\infty} x^2 |\psi(x)|^2 dx = \frac{2}{L} \int_0^L x^2 \sin^2 \frac{n\pi x}{L} dx
\]

Let \(u = \frac{n\pi x}{L}\) then

\[
(x^2)_{av} = \frac{2}{L} \left( \frac{L}{n\pi} \right)^3 \int_0^{n\pi} u^2 \sin^2 u \, du = \frac{2}{L} \left( \frac{L}{n\pi} \right)^3 \left[ \frac{(n\pi)^3}{6} - \frac{n\pi}{4} \right] = L^2 \left( \frac{1}{3} - \frac{1}{2n^2\pi^2} \right)
\]

Solving for \(n\), we have:

\[
n = \sqrt{\frac{1}{(2\pi)^2 \left( \frac{1}{3} - \frac{(x^2)_{av}}{L^2} \right)}}
\]

Thus the photon energy emitted in the transition from the \(n = 3\) state to the ground state \((n = 1)\) is:

\[
E_\gamma = E_3 - E_1 = E_0 (3^2 - 1^2) = \frac{\hbar^2 \pi^2}{2mL^2} = 8483 \text{ eV}
\]

The ionization energy of a hydrogen-like ion in its ground state is:

\[
E_{ion} = E_\infty - E_1 = Z^2 (13.6 \text{ eV})
\]

Problem 21

It is easiest to first determine what the average kinetic energy of a nucleon in a \(^{20}\text{Ne}\) nucleus is. The nucleus can be described as two separate \(T = 0\) Fermi gases of 10
neutrons and 10 protons. The average kinetic energy is given by:

\[ E_{av} = \frac{3}{5} E_F \]

\[ E_F = \frac{\hbar^2}{2m} \left( \frac{3}{8\pi} \frac{N}{V} \right)^{2/3} \]

\[ V = \frac{4}{3} \pi R^2 \]

where \( R = (1.2 \text{ fm})A^{1/3} = 3.26 \text{ fm} \)

\[ V = \frac{4}{3} \pi (3.26 \text{ fm})^3 = 145 \text{ fm}^3 \]

\[ E_F = \frac{\hbar^2 c^2}{2mc^2} \left( \frac{3}{8\pi} \frac{N}{V} \right)^{2/3} \]

\[ = \frac{(1240 \text{ MeV fm})^2}{2(940 \text{ MeV})} \left( \frac{3}{8\pi} \right)^{2/3} \left( \frac{10}{145 \text{ fm}^3} \right)^{2/3} = 33.3 \text{ MeV} \]

\[ E_{av} = \frac{3}{5} E_F = 20.0 \text{ MeV} \]

The relation between the scattered photon energy, \( E' \), and the initial photon energy, \( E \), in Compton scattering is given by:

\[ \frac{1}{E'} = \frac{1}{E} + \frac{1}{mc^2} (1 - \cos \theta) \]

\[ \frac{1}{E'} - \frac{1}{E} = 5.73 \times 10^{-7} \text{ eV}^{-1} \]

The kinetic energy of the scattered electron (which we want to be 20.0 MeV) is given by:

\[ K = 20.0 \text{ MeV} = E - E' \]

\[ E' = E - 20.0 \text{ MeV} \]

Combining the above two equations gives:

\[ \frac{1}{E - 20.0 \times 10^6 \text{ eV}} - \frac{1}{E} = 5.73 \times 10^{-7} \text{ eV}^{-1} \]

\[ 20.0 \times 10^6 \text{ eV} = 5.73 \times 10^{-7} \text{ eV}^{-1} (E - 20.0 \times 10^6 \text{ eV})(E) \]

\[ E^2 - (20.0 \times 10^6 \text{ eV})E - 3.5 \times 10^{13} \text{ eV}^2 = 0 \]

The two roots of this equation are 21.6 MeV and -1.6 MeV. Only the positive one is physically meaningful, so the energy of the incident photon has to be

\[ E = 21.6 \text{ MeV} \]