Problem 1

The expression given in the textbook for the frequency of the $K\alpha$ line (Eq. 8.1) is:

$$\nu = \frac{3cR_\infty(Z-1)^2}{4}$$

So

$$\lambda = \frac{c}{\nu} = \frac{4}{3R_\infty \left(Z-1\right)^2}$$

So the ratio of the wavelength of the impurity line to the wavelength of the cobalt line is:

$$\frac{\lambda_{imp}}{\lambda_{Co}} = \frac{0.1441 \text{ nm}}{0.1793 \text{ nm}} = \frac{(27-1)^2}{(Z-1)^2}$$

$$\left(Z-1\right)^2 = 841.13$$

$$Z = 30$$

Problem 2

a) Tb: [Xe]4f$^9$6s$^2$

To find the ground state $L$ and $S$ values we use Hund’s rules. The 6s subshell is full. To maximize $M_S$, seven of the electrons in the 4f subshell go into $m_s = +1/2$ states, while the other two must go into $m_s = -1/2$ states due to the Pauli principle. Thus $S = M_{S,max}=7(+1/2)+2(-1/2)=5/2$. For the orbital angular momentum, seven of the 4f electrons get placed in the $m_l = +3, +2, +1, 0, -1, -2, -3$ states, respectively. To maximize $M_L$ the other two must go into the $m_l = +3, +2$ states. Thus $L = M_{L,max} = +3 + 2 = 5$. So the ground state quantum numbers are $S = 5/2$ and $L = 5$.

b) Np: [Rn]5f$^4$6d$^1$7s$^2$

To find the ground state $L$ and $S$ values we use Hund’s rules. The 7s subshell is full. To maximize $M_S$, all four of the 5f electrons can go into $m_s = +1/2$ states, as can the one 6d electron. Thus $S = M_{S,max}=4(+1/2)+1(+1/2)=5/2$. For the orbital angular momentum, the four 5f electrons get placed in the $m_l = +3, +2, +1, 0$ states, while the single 6d electron goes into the $m_l = +2$ state. Thus, $L = M_{L,max} = +3 + 2 + 1 + 0 + 2 = 8$. So the ground state quantum numbers are $S = 5/2$ and $L = 8$.

Problem 3

a) The radial probability density is expressed as:

$$P(r) = r^2 |R_{n,l}(r)|^2$$

and the ground state radial wavefunction is

$$R_{1,0} = \frac{2}{a_0^{3/2}} e^{-r/a_0}$$

The probability to be found at a distance of greater than $6a_0$ from the origin is:

$$P = \int_{6a_0}^{\infty} P(r) dr = \int_{6a_0}^{\infty} r^2 |R_{1,0}(r)|^2 dr$$

$$= \frac{4}{a_0^3} \int_{6a_0}^{\infty} r^2 e^{-2r/a_0} dr$$
We can convert this to a form found in integral tables with the substitution

\[
x = \frac{2r}{a_0}, \quad \text{so} \quad r = \frac{a_0}{2} x, \quad dr = \frac{a_0}{2} dx
\]

Making these substitutions, we have:

\[
P = \frac{4}{a_0^3} \frac{a_0^3}{8} \int_0^\infty x^2 e^{-x} dx
\]

\[
= \frac{1}{2} \int_0^\infty x^2 e^{-x} dx
\]

This integral can be evaluated by parts or found in the table of indefinite integrals on page 20 of the exam:

\[
P = \frac{1}{2} \left[ (-x^2 - 2x - 2) e^{-x} \right]_0^\infty = 0.0005222
\]

b) My wording of this problem was not explicit enough, so I will accept two possible answers:

Approach a): One approach is to redefine the Bohr radius for the new situation, so one could define \( a_0^{\text{new}} = \frac{a_0}{Z} \) (see Equation 6.41 in the text). Thus the wavefunction and the lower limit of integration look the same except we replace \( a_0 \) by \( a_0^{\text{new}} \) everywhere. So the final answer for the probability would remain the same since it is a dimensionless numerical value that is independent of \( a_0 \).

Approach b): In this approach one does not redefine \( a_0 \). Then the lower limit of integration stays the same (6\( a_0 \)), but the ground state wavefunction will change:

\[
R_{1,0} = \frac{2Z^{3/2}}{a_0^{3/2}} e^{-Zr/a_0}
\]

This causes the wavefunction to have smaller values at large radii than for hydrogen, so the probability will decrease compared to a). In fact, the only change ends up being that the lower limit of integration after the substitution will be \( 12Z \) instead of 12 and the exact numerical result would be \( 1.2 \times 10^{-8} \), but I didn’t expect you to evaluate that.

**Problem 4**

Recall that the starting point for deriving the allowed energy levels of the hydrogen atom was the potential energy of the electron-proton system (see Eq. 6.27 or Eq. 7.2 in the book):

\[
U = -\frac{1}{4\pi \varepsilon_0} \frac{e^2}{r}
\]

From this, the allowed energy levels of the hydrogen atom can be derived using either the Bohr model approach (section 6.5 in the text) or solving the Schrödinger
equation (section 7.2 in the text). Both methods lead to the same values for the allowed hydrogen atom energy values:

\[ E_n = -\frac{mc^4}{32\pi^2\varepsilon_0\hbar^2} \frac{1}{n^2} \]

To find the allowed energies when the gravitational force is binding the hydrogen atom, we note that the above Coulomb potential energy expression looks like the gravitational potential energy expression:

\[ U = -\frac{Gm_eM_p}{r} \]

when the quantity \( \frac{1}{4\pi\varepsilon_0} e^2 \) is replaced by \( Gm_eM_p \). Squaring these quantities implies that we want to replace:

\[ \frac{e^4}{16\pi^2\varepsilon_0^2} \text{ by } G^2 m_e^2 M_p^2 \]

So we have

\[ E_n = -\left( \frac{m_e}{2\hbar^2} \right) (G^2 m_e^2 M_p^2) \frac{1}{n^2} \]
\[ E_1 = -\left( \frac{m_e e^2}{2(\hbar c)^2} \right) (G^2 m_e^2 M_p^2) \]
\[ = -\frac{0.511 \times 10^6 \text{ eV}}{\frac{1617 \text{ eV-nm}}{2}} (6.34 \times 10^{-40} \text{ eV-nm})^2 \]
\[ = -2.65 \times 10^{-78} \text{ eV} \]

(Note: It wasn’t asked, but the Bohr radius for this gravitationally bound hydrogen atom would be huge (think how big the distance is between the Sun and Earth, and their mass is very large.) The Bohr radius becomes

\[ a_0 = \frac{4\pi \varepsilon_0 \hbar^2}{e^2 m} \rightarrow \frac{\hbar^2}{Gm_e M_p m_e} = 1.2 \times 10^{29} \text{ m} \]

**Problem 5**

In general, we have the following expression for a given impact parameter \( b \):

\[ \frac{1}{2} mv^2 = \frac{1}{2} m \left( \frac{p^2}{r_{\text{min}}^2} \right) + \frac{1}{4\pi\varepsilon_0} \frac{Z e^2}{r_{\text{min}}} \]

For \( b = 0 \), we have

\[ K = \frac{1}{2} mv^2 = \frac{e^2}{4\pi\varepsilon_0 \rho_{\text{min}}} \frac{Z e}{r_{\text{min}}} \]
\[ K = (1.44 \text{ MeV-fm}) \frac{1}{1} \frac{(1)(79)}{7 \text{ fm}} \]
\[ K = 16.3 \text{ MeV} \]
Problem 6

a) The absorbed photon energy is given by:

\[ E_\gamma = 13.06 \text{ eV} = E_n - E_1 \]
\[ E_n = E_1 + 13.06 \text{ eV} = -13.6 \text{ eV} + 13.06 \text{ eV} \]
\[ = -0.54 \text{ eV} \]
\[ E_n = \frac{-13.6 \text{ eV}}{n^2} = -0.54 \text{ eV} \]
\[ n = 5 \]

The photon is absorbed from the ground state \((1s)\). The selection rules are \(\Delta l = \pm 1\). \(\Delta l = -1\) is not possible in this case, so the final state must be \(5p\) \((n = 5, l = 1)\).

b) The idea here is to draw all of the allowed \((\Delta l = \pm 1)\) transitions assuming that the hydrogen atom starts from the state determined in part a), which is the \(5p\) state. There end up being 15 resulting possible transitions:

\[
\begin{align*}
5p &\to 4s & 5p &\to 3s & 5p &\to 2s \\
5p &\to 1s & 5p &\to 4d & 5p &\to 3d \\
4s &\to 3p & 4s &\to 2p & 4d &\to 3p \\
4d &\to 2p & 3s &\to 2p & 3p &\to 2s \\
3p &\to 1s & 3d &\to 2p & 2p &\to 1s \\
\end{align*}
\]

The resulting energy level diagram with the allowed transitions looks like:

![Energy Level Diagram]

Note: In addition to the answer above I accepted answers that did not include transitions that didn’t eventually lead to the ground state (like the \(5p \to 2s\), which cannot go to the ground state since the \(1s\) doesn’t go there.) According to the way I worded the problem, this would have been a reasonable interpretation.

Problem 7
a) The shortest wavelength photons will be those with the most energy. In this case that corresponds to the \( n = 7 \to n = 1 \) and the \( n = 7 \to n = 2 \) transitions.

\[
E_n - E_m = (Z^2)(-13.6 \text{ eV}) \left( \frac{1}{m^2} - \frac{1}{n^2} \right)
\]

\[
E_\gamma - E_1 = (26^2)(-13.6 \text{ eV}) \left( \frac{1}{7^2} - \frac{1}{1^2} \right)
= 9006 \text{ eV}
\]

\[
\lambda = \frac{hc}{E_\gamma} = \frac{1240 \text{ eV-nm}}{9006 \text{ eV}} = 0.138 \text{ nm}
\]

\[
E_\gamma - E_2 = (26^2)(-13.6 \text{ eV}) \left( \frac{1}{7^2} - \frac{1}{2^2} \right)
= 2111 \text{ eV}
\]

\[
\lambda = \frac{hc}{E_\gamma} = \frac{1240 \text{ eV-nm}}{2111 \text{ eV}} = 0.587 \text{ nm}
\]

b) The longest wavelength of the two photons from part a) is the 0.587 nm photon with an energy of 2111 eV. The ionization energy is given by:

\[
E_{ion} = E_\infty - E_1 = -E_1 = Z^2(13.6 \text{ eV})
\]

\[
2111 \text{ eV} = Z^2(13.6 \text{ eV})
\]

\[
Z = 12.46
\]

So the maximum value of \( Z \) that can be ionized is 12.

**Problem 8**

a) The energy difference between the ground and first excited state is:

\[
E_2 - E_1 = Z^2(-13.6 \text{ eV}) \left( \frac{1}{2^2} - \frac{1}{1^2} \right) = 86333 \text{ eV}
\]

Since we are told that we can use classical statistics, we can use the Maxwell-Boltzmann distribution function to get the ratio:

\[
\frac{p(E_2)}{p(E_1)} = \frac{g(E_2)e^{-E_2/kT}}{g(E_1)e^{-E_1/kT}}
\]

\[
= \frac{.05}{.95} = \frac{8e^{-E_2/kT}}{2e^{-E_1/kT}} = 4e^{-(E_2-E_1)/kT}
\]

\[
-4.3307 = \frac{(E_2 - E_1)}{kT}
\]

\[
T = 2.3 \times 10^8 \text{ K}
\]

b) The deBroglie wavelength of the atoms in the gas is given by:

\[
\lambda = \frac{h}{p} = \frac{h}{\sqrt{3mkT}} \quad \text{using} \quad \frac{p^2}{2m} = \frac{3}{2}kT
\]

\[
= \frac{hc}{\sqrt{3mc^2kT}}
= \frac{1240 \text{ eV-nm}}{\sqrt{(3)(238.1)(931.5 \text{ MeV})(8.617 \times 10^{-5} \text{ eV/K})(2.3 \times 10^8 \text{ K})}}
\]

\[
\lambda = 0.0000107 \text{ nm} \ll d = 3 \text{ nm}
\]
So classical statistics is valid since the de Broglie wavelength is much less than the interparticle spacing.

**Problem 9**

The nuclear volume is:

\[ V = \frac{4}{3} \pi R^3 = \frac{4}{3} \pi (4.3 \, \text{fm})^3 = 333 \, \text{fm}^3 \]

There are 18 protons, so the Fermi energy for the protons is:

\[
E_F = \frac{\hbar^2}{2m} \left( \frac{3}{8\pi} \frac{N}{V} \right)^{2/3} = \frac{\hbar^2 c^2}{2mc^2} \left( \frac{3}{8\pi} \frac{N}{V} \right)^{2/3} \\
= \frac{(1240 \, \text{MeV-fm}^2)^2}{2(938.3 \, \text{MeV})} \left( \frac{3}{8\pi} \right)^{2/3} \left( \frac{18}{333 \, \text{fm}^3} \right)^{2/3} = 28.4 \, \text{MeV} \\
E_m = \frac{3}{5} E_F = 17.0 \, \text{MeV} \]