RENORMALIZATION GROUP ANALYSIS OF DRIVEN DIFFUSIVE SYSTEMS

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The critical properties of statistical systems in thermal equilibrium are well understood, thanks to renormalization group analysis. For systems driven into *non-equilibrium steady states*, many surprising new features appear. For example, when the Ising lattice gas is driven with biased diffusion or coupled to *two* thermal baths, long range correlations exist at all temperatures. The second order phase transition still survives, but the associated universal properties are drastically different. After a brief overview of the phenomenology of driven lattice gases, applications of RG to the study of several specific systems will be presented.

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1 Introduction

The success of renormalization group analysis in understanding critical properties of a system undergoing a second order phase transition is well known [1]. Nearly all of the applications concern systems in thermal equilibrium, or dynamical aspects of such systems near equilibrium [2, 3] On the other hand, in the natural world, most systems, even if they appear to be relatively ‘stationary,’ are far from equilibrium. Consider, for example, a typical biological organism (from a single cell to a large animal). It is clear that such systems may appear to be quite stable, but their resemblance to equilibrium-like stationary states is illusory: When a cell is ‘isolated’ (in the sense of Boltzmann’s fundamental hypothesis for ‘isolated systems’), it will not last very long as a cell. Instead, these systems require a constant through-flux of energy of some form. They are systems in *non-equilibrium steady states*. From the point of view of statistical mechanics, these systems pose serious challenges, since their steady state distribution is not given by the Boltzmann factor in general. To address such fundamental issues is clearly beyond the scope of this short article. Instead, we give these illustrations only as a backdrop for a very small class of systems in non-equilibrium steady state, namely, driven diffusive systems.

Faced with this overwhelmingly large and unknown area of non-equilibrium steady states, one possible way forward is to study simple models, especially ones with well known equilibrium properties. Motivated by the physics of fast ionic conductors, Katz, Lebowitz and Spohn introduced such a model nearly two decades ago [4]. With the interactions and constituents of the Ising model [5] in the lattice gas language, this model differs by having an external drive on the particles. When periodic boundary conditions are imposed, the drive sets up a non-trivial current, so that the system is clearly in a non-equilibrium steady state, even though the probability distribution of each configuration is stationary. The usual second order phase transition (for $d \geq 2$) survives, but occurs at a *higher* temperature! In this next two sections, we will provide some details of driven lattice gases, some of the surprising properties, and the application of renormalization group methods for understanding critical behavior.

2 Driven Lattice Gases

Consider a familiar Ising model with nearest-neighbor interactions. In lattice gas language [6], this system consists of particles and holes on a d-dimensional square lattice, evolving in time by the particles hopping to nearest vacant sites (i.e., Kawasaki spin exchange dynamics [7]). For Monte Carlo simulations, a favorite hopping rate associated with an equilibrium Ising model is the Metropolis rate: $\min[1, e^{-\Delta H/k_B T}]$, where ΔH is the change in the internal energy $H = -4J \sum_{<i,j>} n_i n_j$, J is the nearest neighbor coupling, and $n_i = 1, 0$ is the occupation number at site i. In the thermodynamic limit, this system undergoes a second order phase transition at the Onsager critical temperature $T_O = (2.2692..)J/k_B$ [8]. A uniform drive (also referred to as DC drive) is introduced by pretending that the particles are 'charged' and the system placed in an 'electric field' E, which is typically chosen to point along one of the axes. Then the new rates are $\min[1, e^{-(\Delta H - \epsilon E)/k_B T}]$, where $\epsilon = (-1, 0, 1)$ for a particle attempting to hop (against, orthogonal to, along) the drive. As a result, particle hops are biased in the field direction. Locally, the effect of such a drive is precisely the same as that due to gravity. Indeed, if 'brick wall' boundary conditions were imposed in the direction associated with the drive, then this system will evolve towards an *equilibrium* state, similar to that of gas molecules in a room on earth, with an inhomogeneous particle density to 'balance' the bias. However, if *periodic* boundary conditions are imposed, then translational invariance is restored and the particle density is *homogeneous* (at least for all T above some finite critical T_c), but a non-zero particle current will be established. For gravity, such boundary conditions can exist only in art, as by M.C. Escher [9]. Nevertheless, it is possible to establish such conditions in physical reality, by imposing an *electric* field. For example, we can place the $d = 2$ lattice on the surface of a cylinder and apply a linearly increasing magnetic field down the cylinder axis. This is the prototype of a 'driven diffusive system.' Note that there is also a constant through-flux of energy through this system, gaining (losing) energy from (to) the drive (thermal bath). Note also that, due to the specific choice of the rates, it is possible to set $E = \infty$. All this means is that particle jumps against the field are completely suppressed, while jumps along the field are always accepted, regardless of the interaction energies. For small systems (e.g., 2×3 or 2×4), we can solve the associated master equation exactly [10] and the results show clearly that the time independent state is far from the one provided by Boltzmann-Gibbs. Instead, it is a new distribution, which characterizes a *non-equilibrium steady state*.

From the view point of physical systems, a uniform drive cannot be maintained indefinitely, since magnetic fields with linearly-increasing strength do not exist indefinitely in laboratories. Instead, AC drives are much easier to achieve. Of course, this introduces a scale into the system, which may spoil scale invariance, which typifies criticality. An alternative is to impose a *random drive*. One possibility is to chose the direction of E randomly at each update. Clearly, an overall particle current no longer exists. Nevertheless, we would expect a non-trivial energy through-flux and non-equilibrium steady states to be present. A similar model, one which we believe to belong to the same universality class as the randomly driven lattice gas, is the *two-temperature* model [11]. Here, particle jumps in one (or more) of the d directions are coupled to the usual rate with one temperature (T_1) and the rest of the jumps are coupled to a second bath (at T_2): $\min[1, e^{-\Delta H/k_B T_\alpha}]$. Needless to say, when the system settles down to a stationary state, energy will flow from the higher temperature bath to the other one, *through* our lattice. By setting $T_1 = \infty$, we can easily imagine that there is no difference (except for an overall factor) between this model and the random drive with $E = \infty$.

In all of these models, the driven system behaves as the equilibrium Ising model, but only *superficially*. Thus, there is a homogeneous phase for large temperatures, followed by a second order transition into a phase-segregated state for low T. However, with a little analysis, there appeared many surprising properties. First, from an intuitive viewpoint, the drive represents coupling of the lattice gas to an energy reservoir at, effectively a higher T. This is manifestly so for the two temperature model. As a consequence, we might guess that, in order to order the system, T has to be lowered. However, simulations show that, $T_c(E)$ appears to be monotonically *increasing* with E, saturating at about $1.41 T_c(0)$ [12] as E approaches ∞! This counter-intuitive picture — coupling a system to a much more 'energetic' source can lead to ordering — turns out to be not so rare when systems are under non-equilibrium steady states [13]. Another remarkable property of these driven systems is that, even when they are deep in the disordered phase (i.e., $T \gg T_c$), the structure factor $S(k)$ has a discontinuity singularity at the origin: $k = 0$. In terms of its Fourier transform, the two point correlation function $G(r)$, this singularity translates into power law decays at all T. It is beyond the scope of this article to delve into all the remarkable properties. Those interested in the details will find the appropriate references in [14]. Here, we will focus on critical properties, which have been investigated successfully by field theoretic methods, exploiting the renormalization group approach.

3 Renormalization group studies

To understand collective behavior in the long-time and large-scale limit, it is typically sufficient to rely on a continuum description, such as hydrodynamics and Landau-Ginzburg theories. Certainly, the φ^4 theory, augmented by renormalization group analysis, is the most successful example of such an approach. Following these lines, a continuum theory for the driven Ising lattice gas was first formulated by two groups independently [15]. The starting point is model B of critical dynamics, since it describes the Ising system with a *conserved* ordering field: $\varphi(\mathbf{x}, t)$ [3], in d spatial dimensions ($\mathbf{x} = x_1, ..., x_d$). The Langevin equation reads $\partial_t \varphi = -\nabla \cdot \mathbf{j} = \lambda \nabla^2 (\delta \mathcal{H}/\delta \varphi) - \nabla \cdot \eta$, with $\mathcal{H} = \int \{ \frac{1}{2} (\nabla \varphi)^2 + \frac{\tau}{2} \varphi^2 + \frac{u}{4} \varphi^4 \}$ being the Landau-Ginzburg Hamiltonian and η modeling thermal noise. As usual, $\tau \propto T - T_c$ and $u > 0$, while the noise is Gaussian distributed, with zero mean and positive second moment proportional to the unit matrix, i.e., $\langle \eta_i \eta_j \rangle = 2\sigma \delta_{ij} \delta(\mathbf{x} - \mathbf{x}') \delta(t - t')$, $i, j = 1, ..., d$. The study of critical properties is facilitated by transforming this description to a field theoretic formulation, via a standard route [16]. In place of the usual Langrangian (in particle physics) or Hamiltonian (in statistical physics), we consider a (bare) dynamic functional:

$$\mathcal{J}\left[\varphi,\tilde{\varphi}\right]=\int d^{d}xdt\tilde{\varphi}\left\{\partial_{t}\varphi+\lambda\left(\nabla^{4}\varphi-\tau\nabla^{2}\varphi-u\nabla^{2}\varphi^{3}\right)+\sigma\nabla^{2}\tilde{\varphi}\right\}\ , \qquad (1)$$

in the path integral $\int\mathcal{D}\tilde{\varphi}\mathcal{D}\varphi\exp\left[-\mathcal{J}\right]$. From here, all correlation and response functions can be computed, while the renormalization program is carried out in a manner entirely parallel to the static case.

3.1 A system driven by a DC field

To account for the drive, which is chosen to point along the y ($\equiv x_d$) axis, we add an "Ohmic" current, $\mathbf{j}_E \propto \hat{\mathbf{y}}$, to $\mathbf{j}$. Of course, it necessarily vanish at zero density ($\varphi = -1$). Due to the excluded volume, $\mathbf{j}_E$ must also vanish at unity density ($\varphi = 1$). The simplest choice, denoting the coarse-grained drive by $\mathcal{E}$, is $\mathbf{j}_E = \mathcal{E}(1-\varphi^2)\hat{\mathbf{y}}$. Thus, we might begin with $\partial_t\varphi = \lambda\nabla^2\left(\delta\mathcal{H}/\delta\varphi\right)+\mathcal{E}\partial\varphi^2+\nabla\cdot\eta$, where $\partial\equiv\partial/\partial y$. Terms with higher powers of $(1-\varphi^2)$ are expected, by power counting alone, to be irrelevant for universal properties. It is clear that the drive will induce serious anisotropies into the system, distinguishing the drive direction (y) from the transverse subspace. Specifically, we should (a) split $\tau\nabla^2$ into parallel ($\tau_{\|}\partial^2$) and transverse ($\tau_\perp\nabla_\perp^2$) components, with $\tau_{\|}\neq\tau_\perp$ (and similarly for ∇^4), and (b) write $\langle\eta_i\eta_j\rangle\propto\sigma_i\delta_{ij}$ with $\sigma_1 = ... = \sigma_{d-1}\equiv\sigma_\perp$ and $\sigma_d\equiv\sigma_{\|}\neq\sigma_\perp$. More crucial, due to violation of detailed balance, the "splitting" of τ and σ do not have to be the same, so that $\tau_{\|}/\sigma_{\|}\neq\tau_\perp/\sigma_\perp$ in general. Summarizing, we write down the full Langevin equation:

$$\begin{aligned}\partial_t\varphi(\mathbf{x},t) &= \lambda\{(\tau_\perp-\nabla_\perp^2)\nabla_\perp^2\varphi+(\tau_{\|}-\alpha_{\|}\partial^2)\partial^2\varphi-2\alpha_\times\partial^2\nabla_\perp^2\varphi+\\ &\quad +u(\nabla_\perp^2+\kappa\partial^2)\varphi^3+\mathcal{E}\partial\varphi^2\}-\nabla\eta(\mathbf{x},t)\ ,\end{aligned} \qquad (2)$$

with noise correlations

$$\langle\eta_i(\mathbf{x},t)\eta_j(\mathbf{x}',t')\rangle = 2\sigma_i\delta_{ij}\delta(x-x')\delta(t-t')\ . \qquad (3)$$

With the presence of *two* τ's, there are three possible "critical" systems: (i) $\tau_\perp\to 0$ but $\tau_{\|}>0$, (ii) $\tau_\perp>0$ with $\tau_{\|}\to 0$, and (iii) both $\to 0$. From simulation data, the ordered state is always observed to be phase segregated *transverse* to the drive, i.e., strips parallel to E appear when T is below $T_c(E)$. By contrast, strips in either direction can form for the equilibrium Ising model (in a square system). To model this behavior, we naturally choose case (i). Since the critical theory is given by $\tau_\perp = 0$, we must keep the $\nabla_\perp^4$ term for stability. Meanwhile, since $\tau_{\|}>0$, we can ignore all ∂'s higher than ∂^2. As a result, longitudinal and transverse momenta scale with different powers: $k_{\|}\sim k_\perp^2$, a situation similar to that in the study of Lifschitz points [17]. Expecting non-trivial renormalization, we define Δ, the anisotropy exponent, via the relation:

$$k_{\|}\sim k_\perp^{1+\Delta} \qquad (4)$$

i.e., $\Delta = 1$ at the tree level. Due to $\Delta\neq 0$, power counting must be performed carefully. Keeping only the most relevant terms in the dynamic functional and rescaling some bare parameters, we arrive at the critical

$$\mathcal{J}_c^{\mathrm{DC}} = \int\lambda\tilde{\varphi}\left\{\lambda^{-1}\partial_t\varphi+\left(\nabla_\perp^4-\rho\partial^2\right)\varphi-\mathcal{E}\partial\varphi^2+\nabla_\perp^2\tilde{\varphi}\right\}\ . \qquad (5)$$

as well as an upper critical dimension of $d_c = 5$. The effective coupling constant is found to be proportional to $\mathcal{E}^2/\rho^{3/2}$, and a non-trivial (IR stable) fixed point emerges for $d < d_c$. To add confidence to this fixed point, it is found to be stable against $u\nabla^2\varphi^3$ (from the usual φ^4 theory). By contrast, the Wilson Fisher fixed point [18] is *un*stable against the drive: $\mathcal{E}\partial\varphi^2$. We should emphasize that, due to the presence of this drive term, the fixed point functional (Eqn. 5) is associated with a true *non*equilibrium system. It is impossible to write $\mathcal{J}_c$ in the form of $\int \tilde{\varphi}\left\{\partial_t\varphi + \Gamma(\delta\mathcal{H}/\delta\varphi) - \tilde{\Gamma}\tilde{\varphi}\right\}$, even if $\Gamma$and $\tilde{\Gamma}$ were arbitrary (local) operators.

Once the fixed point is known, we can find the critical properties of our system. Thanks to the presence of an additional symmetry, all exponents can be computed *to all orders* in $d_c - d$. Referring all details to [15], we only quote the results: $\Delta = 1 + (d_c - d)/3$, while other exponents happen to assume the same value as in mean-field. We should remind the reader that the last statement can be misleading, since the two momenta scale with different powers. To be precise, when we quote $1/\nu = 2$ and $z = 4$, for example, we mean only $\tau_\perp \sim k_\perp^2$ and $\omega \sim k_\perp^4$. By contrast, if we were to write the singular behavior of, say, the structure factor $S\left(0, k_\parallel\right)$ as $k_\parallel^{-2+\eta}$, then this η is non-trivial, assuming the value of $2\Delta/(1+\Delta)$ [14]! On the simulations front, subtleties arise, especially when finite size scaling is exploited to extract exponents. If a series of system sizes were chosen without regard to $\Delta \neq 0$ (e.g., by choosing larger and larger square samples), then the underlying non-trivial anisotropic scaling can induce extra singularities in most frequently measured quantities [12]. Referring the readers to the most recent study [19] for details, we can summarize the present situation as that of "*consistency*" between the data and field theory predictions.

3.2 Random drive and two-temperature models

Let us turn next to the randomly driven Ising lattice gas, or the two-temperature model [11, 20, 21, 22, 23]. Here, there is no longer an Ohmic current. Nevertheless, the drive takes the system out of thermal equilibrium so that the steady state is characterized by two important features: (a) anisotropy, as in the case with a DC drive, so that we still have $\tau_\parallel \neq \tau_\perp$; $\sigma_\parallel \neq \sigma_\perp$ and (b) violation of detailed balance, so that we expect $\tau_\parallel/\sigma_\parallel \neq \tau_\perp/\sigma_\perp$. Consequently, we should begin with a Langevin equation like Eqn. (2), except that the $\mathcal{E}$ term would be missing. As in the previous case, however, we can expect the critical system to be given by $\tau_\perp \to 0$ with $\tau_\parallel > 0$. From the perspective of the two temperature model, it is easy to expect such a set of conditions. After all, if jumps along y are always accepted (when we set T_y to ∞), then longitudinal diffusion will always prevail, i.e., $\tau_\parallel > 0$. On the other hand, transverse phase segregation can still be modeled by *anti*diffusion in the $\mathbf{x}_\perp$ direction, i.e., $\tau_\perp$ becoming negative. Consequently, we will again have $k_\parallel \sim k_\perp^2$ ($\Delta = 1$) at the tree level. Now, in the absence of $\mathcal{E}\partial\varphi^2$, the next most relevant term turns out to be $u\nabla_\perp^2\varphi^3$, leading us to consider the critical functional

$$\mathcal{J}_c^{\mathrm{TT}} = \int \lambda\tilde{\varphi}\left\{\lambda^{-1}\partial_t\varphi + \left(\nabla_\perp^4 - \rho\partial^2\right)\varphi - u\nabla_\perp^2\varphi^3 + \nabla_\perp^2\tilde{\varphi}\right\} \ . \qquad (6)$$

A new power counting produces the upper critical dimension of $d_c = 3$, and an $\epsilon \equiv 3 - d$ expansion can be set up in the standard way [20, 21]. A non-trivial (IR stable) fixed point of $u^* \propto O(\epsilon)$, reminiscent of the Wilson-Fisher fixed point, was found. Unfortunately, the additional symmetry in the DC drive case is absent here, so that exponents must be computed order by order in ϵ. Again, referring the interested reader to the reference below, we quote only some results here. To two loops, the independent exponents are

$$\eta = \frac{4}{243}\epsilon^2 \quad \text{and} \quad \nu = \frac{1}{2} + \frac{1}{12}\epsilon + \frac{1}{36}\left[\frac{67}{54} + \ln\frac{4}{3}\right]\epsilon^2 \ . \tag{7}$$

Scaling relations lead us to the others, e.g., $z = 4 - \eta$ and $\Delta = 1 - \eta/2$. Since η is small, even for $d = 2$, simulations can rely on systems with $\Delta = 1$. The data are in good agreement with these predictions [22].

Before ending our remarks on this system, we should point out an extraordinary feature: Unlike the case for the DC drive, the fixed point functional here *is* associated with a system in equilibrium [21]! Specifically, consider uniaxial ferromagnets with dipolar interactions, evolving under Kawasaki dynamics. Once power counting is performed and all irrelevant terms are dropped, that dynamic functional is identical to (6). The reader should not be alarmed, however, that an obviously non-equilibrium system such as the two-temperature model should turn out to be an equilibrium system. Only the *leading* singularities in the thermodynamic functions of the two systems are, in the critical region, identical. Our driven lattice gas violates detailed balance, and cannot be an equilibrium system. To find the non-equilibrium aspects, we must turn to sub-leading singularities or quantities associated with dangerous irrelevant operators. A good example of the latter is the energy flux *through* our two-temperature model, easily measurable in simulations but yet to be studied theoretically. Details on all aspects (both field theoretic and Monte Carlo) can be found in [23].

3.3 The uniformly driven bilayer lattice gas

To illustrate the unexpected richness of driven systems, we turn to another generalization of the proto-model, namely, coupling two such systems transverse to the drive. Leaving the motivations and the details to references [24, 25, 26], we only focus on the critical properties here. First, let us present the model from the perspective of simulations. Take two identical $d = 2$ driven lattice gases and arrange one to be directly on top of the other, forming a bilayer system. Rescaling the in-plane interactions to unity, define the cross-plane interaction to be $J \in (-\infty, \infty)$. Setting the system at half-filling and allowing particles to jump across the planes, we can easily predict its equilibrium properties, at least qualitatively. For $J = 0$, we have two decoupled Ising systems. However, since there are interfaces associated with in-plane phase segregated states, the lower temperature state of the system will consist of both planes having *homogeneous* densities, corresponding to the two opposite values of the spontaneous magnetisation. With the fully ordered state in mind, we will refer to this as the full-empty (FE) state. Clearly, this FE state prevails for $J < 0$. By contrast, for $J > 0$, the low temperature state would consist of two identical planes with *inhomogeneous* densities, i.e., co-existing strips of high/low densities, similar to the single layer case. So, the phase diagram in the T-J plane has three regions: disordered (D) for high T, striped (S) for $J > 0, T < T_c(J)$, and FE for $J < 0, T < T_c(J)$. Using the Ising symmetry, it is clear that, in the thermodynamic limit, $T_c(J) = T_c(-J)$. [For $J = \pm\infty$, we have again a monolayer, with ± 2 for the spin values, so that $T_c(J)$ saturates at $2T_O$.] Though a symmetry relates the two ordered states, their dynamics are distinct, since the order parameters differ. In the FE/S phase, the order parameter is the *difference/sum* of the densities in the two planes. In the spin language, these are:

$$\varphi_\Delta \equiv \varphi_1 - \varphi_2 \quad \text{and} \quad \varphi_\Sigma \equiv \varphi_1 + \varphi_2 \ . \tag{8}$$

Clearly, φ_Δ is non-conserved but φ_Σ is conserved. While the static properties are identical and critical behavior is still controlled by the Wilson Fisher fixed point, the dynamics on the D-FE/D-S transition lines are those of model A/B. Finally, the FE-S phase boundary is a line of first order transitions and the three lines join at $(J,T) = (0,T_O)$, a "bicritical" point, where both fields as massless.

Remarkable phenomena appear when we add the drive (DC fields, as in the monolayer case). Though the qualitative features (3 phases: D, FE, S) in the phase diagram survive, simulations show [24, 25] that there are significant quantitative differences. First, since the drive violates the Ising symmetry, the transition line is no longer symmetric: $T_c(J,E) \neq T_c(-J,E)$. Instead of having a higher critical temperature ($T_c(0,\infty) \simeq 1.4T_O$) as in the monolayer case, we find *lower* ones at $J = \pm\infty$, e.g., $T_c(-\infty,\infty) \simeq 1.3T_O < 2T_O = T_c(-\infty,0)$. More remarkable is that the drive somehow induces particle "attraction" to the point that the S phase protrudes into the $J < 0$ half-plane! Turning to critical properties, the simplest conjecture is that the D-S transition is governed by the fixed point (5), since φ_Δ remains "massive" (non-ordering) and should turn irrelevant. Similarly, on the D-FE line, only φ_Δ is critical and, if we simply *drop* φ_Σ from the picture, experience with non-conserved ordering fields [27] leads us to expect model A (or perhaps, models C, E, or G) behavior. However, when a careful study is undertaken, such expectations once again prove to be too naive.

A simple generalization to the bilayer case [14], the "starting" equations of motion are $\partial_t\varphi_1 = \lambda\nabla^2\,(\delta\mathcal{H}/\delta\varphi_1) + \mathcal{E}\,\partial\varphi_1^2 + \lambda_\times\,(1-\varphi_1\,\varphi_2)\,[\delta\mathcal{H}/\delta\varphi_2 - \delta\mathcal{H}/\delta\varphi_1]$ and a similar one with $1 \Leftrightarrow 2$. Here, the Hamiltonian $\mathcal{H}\,[\varphi_1,\,\varphi_2]$ would contain the effects of both the in-plane and the cross-plane interactions, while $\lambda_\times$ is the rate associated with cross-plane jumps. Of course, appropriate noise terms must be added and new operators generated under renormalization (many more beyond the splitting of, e.g., ∇^2 into ∂^2 and $\nabla_\perp^2$) must be accounted for. The details are far beyond the limitations here, so we present again only the results and refer the interested reader to [26]. For the D-S transition, many of the critical properties are indeed the same as in the monolayer case, e.g., $d_c = 5$, presence of the new symmetry, a single independent exponent $\Delta = 1 + (5-d)\,/3$, etc. Nevertheless, there is a novel feature: a single non-trivial operator survives providing a "one-way coupling" of the critical φ_Σ field into the dynamics of the non-critical φ_Δ. To be specific, the critical functional is:

$$\begin{aligned}\mathcal{J}_c^{\text{D-S}} &= \int \lambda\tilde{\varphi}_\Sigma\left\{\lambda^{-1}\partial_t\varphi_\Sigma + \left(\nabla_\perp^4 - \rho\partial^2\right)\varphi_\Sigma + \mathcal{E}\partial\,\varphi_\Sigma^2 + \nabla_\perp^2\tilde{\varphi}_\Sigma\right\} + \\ &\quad + \int \tilde{\varphi}_\Delta\varphi_\Delta\left\{g'\varphi_\Sigma^2 + \mathcal{E}'\partial\,\varphi_\Sigma\right\}\ ,\end{aligned} \tag{9}$$

while the effective couplings are $\mathcal{E}^2/\rho^{3/2}$ (as above) and $g'\rho/\mathcal{E}\mathcal{E}'$. At the stable fixed point, the value of the former is unchanged, while the latter assumes -1. Before turning to the other critical line, let us remark that these features remain valid at the *bicritical* point. Though both fields are critical, the scaling dimensions remain unchanged with $d_c = 5$ dominating scaling behavior. No non-trivial renormalizations in the φ_Δ-propagators appear, so that these fields are basically Gaussian. In less technical terms, we may say that the conserved φ_Σ essentially slaves the non-conserved φ_Δ. Finally, for the D-FE transitions, only φ_Δ is critical. Unlike φ_Σ, it is not being "driven" directly, so that we can expect the usual power counting and upper critical dimension $d_c = 4$ to remain. However, the full picture is more complex. Though φ_Σ is "massive," it is a conserved field and so, slowly varying. Thus, both field appear to be "slow," having similar diffusive operators as leading terms in the Langevin equation. Of course, their dynamic coefficients are not necessarily the same, so that their dimensionless ratio enters. Similarly, the drive affects the anisotropies of the two fields differently and leads to another dimensionless ratio. In addition, renormalization splits the simple drive terms $\mathcal{E}\partial\varphi_i^2$ into all four possible combinations $(\partial\varphi_\Sigma^2, \partial\varphi_\Delta^2, \varphi_\Sigma\partial\varphi_\Delta, \varphi_\Delta\partial\varphi_\Sigma)$. Finally, the drive couples the two fields in such a way (specifically, via $\partial\varphi_\Delta^2$) that the critical φ_Δ sets up long range correlations on the φ_Σ noise and renders the $\nabla^2\tilde{\varphi}_\Sigma^2$ term irrelevant. Instead, *five* marginal operators (four drives and the ordinary Ising-like φ_Δ^3) must be kept, so that even a one-loop calculation involves 35 Feynman graphs! Together with the two ratios, these couplings lead us to RG flows in a 7-dimensional space. Many fixed points emerge, but the completely (IR) *stable* ones form a 2-dimensional subspace. For simplicity, let us present the critical functional in this class:

$$\begin{aligned}\mathcal{J}_c^{\text{D-FE}} &= \int \lambda\tilde{\varphi}_\Sigma \left\{\lambda^{-1}\partial_t\varphi_\Sigma - \left(\nabla_\perp^2 + \rho\partial^2\right)\varphi_\Sigma + \partial\left(\mathcal{E}\varphi_\Sigma^2 + \bar{\mathcal{E}}\varphi_\Delta^2\right)\right\} + \\ &+ \int \gamma\tilde{\varphi}_\Delta \left\{\gamma^{-1}\partial_t\varphi_\Delta - \left(\nabla_\perp^2 + \kappa\partial^2\right)\varphi_\Delta + g\varphi_\Delta^3 - \tilde{\varphi}_\Delta\right\} . \end{aligned} \tag{10}$$

The effective couplings of interest are $\bar{g} \equiv g/\kappa^{1/2}$ and $f \equiv \mathcal{E}\bar{\mathcal{E}}/\kappa^{3/2}$ while the dimensionless ratios are $v \equiv \rho/\kappa$ and $w \equiv \lambda/\gamma$. The fixed points are essentially in the class of model A, i.e.,

$$\bar{g}^* = 2\varepsilon + O\left(\varepsilon^2\right); \quad \varepsilon \equiv 4 - d . \tag{11}$$

Meanwhile, there is a non-trivial *domain* $\mathcal{D}$ in the v-w plane in which not only do all RG β functions vanish but the flow is stable [28]. Denoting points in $\mathcal{D}$ by (v^*, w^*), the fixed point of the second coupling is given by

$$f^* = \frac{(1+\sqrt{v^*})^2(\sqrt{1+v^*\,w^*}+\sqrt{1+w^*})^2}{3w^*\left[w^*(1+\sqrt{v^*})^2 - (\sqrt{1+v^*\,w^*}+\sqrt{1+w^*})^2\right]}\varepsilon + O\left(\varepsilon^2\right) \tag{12}$$

The presence of this coupling means that, as in the D-S case, there is a "one-way effect," by the critical field on the non-critical one. However, none of these novel aspects affect the *leading* singular behavior, namely, model A properties for φ_Δ (e.g., $\eta = 0$, $\nu^{-1} = 2 - \varepsilon/3$ and $z = 2$, up to $O(\varepsilon)$) and simple diffusion for φ_Σ. Nevertheless, we emphasize that our system is a bona-fide non-equilibrium system, so that we should expect singularities distinct from those of model A, albeit only at the sub-leading level. Their associated amplitudes may not be small and serious crossover phenomena may be present, which may be the source of the difficulties reported in [25]. In addition, there are quantities which are completely absent from model A dynamics, such as *three-point* correlations [29], which should display anomalies. We believe that there are still many surprising properties of this bilayer system which await discovery.

To end this section, let us caution the reader about "alternative field theories" for driven diffusive systems in the literature. While simulation results from different groups have not yet converged on an undisputed set of data, these "alternatives" attempt to place the randomly driven system into the same universality class as the system with DC drive, by *starting* with the same set of Langevin equations (or dynamic functional). From the point of view of symmetries, this approach does not seem to be viable to us. For the former system, the Ising and reflection symmetries ($s \Leftrightarrow -s$, $\mathbf{x} \Leftrightarrow -\mathbf{x}$) are both valid. Using the particle-hole language, these symmetries might be labeled as C (charge conjugation) and P (parity). But, for the latter, C and P are both broken by the external field, though the product, CP, is still conserved. Details of these issues may be found in [30].

4 Concluding remarks

The main goal of this short contribution is to introduce to readers the remarkable properties of a simple statistical mechanical system when driven to a far-from-equilibrium steady state. Introduced almost two decades ago, the driven lattice gas continues to present new surprises, in addition to the existing puzzles which remain to be explained. Among the properties we do understand are long range correlations above the critical temperature and, thanks to renormalization group methods, universal behavior near criticality. Without delving into the details, we illustrated the richness of these systems, by presenting only three examples. First, the prototype was the Ising lattice gas driven by a uniform external field [4]. Unlike the Ising system in equilibrium, the upper critical dimension shifts to 5, while the properties are governed by a new, non-equilibrium fixed point [15]. A natural extension is to consider random drives, where the field is reversed at random, a system which is in the same class as the two temperature model. Here, the upper critical dimension is 3, while the leading critical singularities are controlled by a fixed point which *does* correspond to an equilibrium system, namely, Ising spins with dipolar interactions. Sub-leading singularities will reveal that ours is *not* an equilibrium system, however. Finally, we present a recent study of a bilayer lattice gas, driven by a DC field. Two types of second order transitions occur in an expanded phase diagram [26]. Though there are no new *leading* singularities, novel aspects abound. Beyond these simple examples, there are many generalizations and variations, explored and to-be-explored, displaying understandable novel phenomena and yet-to-be-discovered surprises. If this short article has excited a reader to join our adventures with driven diffusive systems, it would have served its purpose.

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