The Tautochrone/Brachistochrone Problems:
How to make the Period of a Pendulum independent of its Amplitude

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I. THE TAUTOCHRONE

A. The Period of a Simple Pendulum

In introductory physics, we teach our students that a simple pendulum is a harmonic oscillator, and that its angular frequency ω and period T are given by

$$\omega = \sqrt{\frac{g}{\ell}}\,, \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}\,, \tag{1}$$

where ℓ is the length of the pendulum. This, of course, is not quite true. The period actually depends on the amplitude of the pendulum's swing.

1. *The Small-Angle Approximation*

Recall that the equation of motion for a simple pendulum is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta\,. \tag{2}$$

(Note that the equation of motion of a mass sliding frictionlessly along a semi-circular track of radius ℓ is the same. See FIG. 1.)

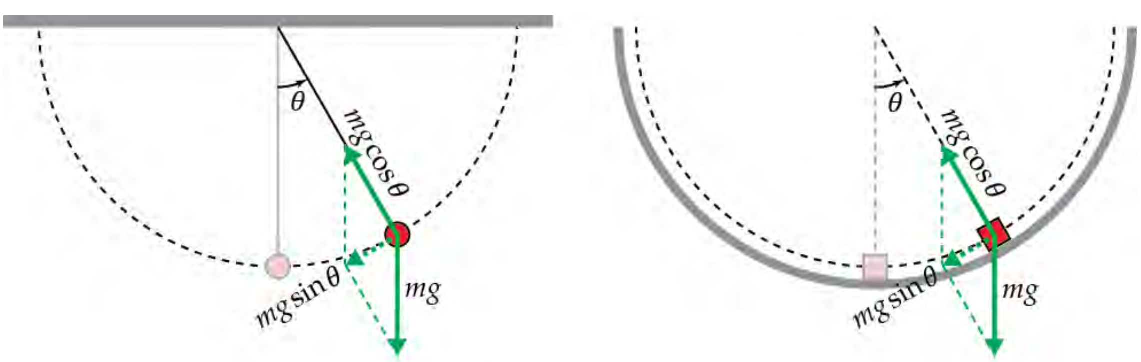

FIG. 1. The motion of the bob of a simple pendulum (left) is the same as that of a mass sliding frictionlessly along a semi-circular track (right). The tension in the string (left) is simply replaced by the normal force from the track (right).

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We need to make the small-angle approximation

$$\sin\theta \;\approx\; \theta\;, \tag{3}$$

to render the equation into harmonic oscillator form:

$$\frac{d^2\theta}{dt^2} \;\approx\; -\omega^2\theta\;,\qquad \omega \;=\; \sqrt{\frac{g}{\ell}}\;, \tag{4}$$

so that it can be solved to yield

$$\theta(t) \;\approx\; A\sin(\omega t)\;, \tag{5}$$

where we have assumed that pendulum bob is at $\theta = 0$ at time $t = 0$.

We can obtain the same result by using energy conservation. If the total energy of the pendulum is E, then we must have

$$\underbrace{mg\ell(1-\cos\theta)}_{\text{Potential Energy}} \;+\; \underbrace{\frac{1}{2}m\left(\ell\frac{d\theta}{dt}\right)^2}_{\text{Kinetic Energy}} \;=\; E\;, \tag{6}$$

where we have set the potential energy of the pendulum when the bob is at its lowest point to be zero. Since the pendulum cannot swing up further than the horizontal (unless the string is replaced by a massless rigid rod), we have

$$E \;<\; mg\ell\;, \tag{7}$$

so we can set

$$E \;=\; mg\ell(1-\cos A)\;,\qquad \text{where } 0 \le A < \pi/2\,. \tag{8}$$

A will be the amplitude of the swing in θ. We now have

$$\begin{aligned}
mg\ell(1-\cos\theta) + \frac{1}{2}m\left(\ell\frac{d\theta}{dt}\right)^2 &= mg\ell(1-\cos A) \\
&\downarrow \\
2(1-\cos\theta) + \underbrace{\frac{\ell}{g}}_{1/\omega^2}\left(\frac{d\theta}{dt}\right)^2 &= 2(1-\cos A) \\
&\downarrow \\
\left(\frac{1}{\omega}\frac{d\theta}{dt}\right)^2 &= 2(\cos\theta-\cos A) \\
&\downarrow \\
\frac{1}{\omega}\frac{d\theta}{dt} &= \sqrt{2(\cos\theta-\cos A)} \\
&\downarrow \\
\omega\,dt &= \frac{d\theta}{\sqrt{2(\cos\theta-\cos A)}}\;.
\end{aligned} \tag{9}$$

Under the small-angle approximation, we can write

$$\cos\theta \;\approx\; 1-\frac{\theta^2}{2}\;,\qquad \cos A \;\approx\; 1-\frac{A^2}{2}\;, \tag{10}$$

and our equation becomes

$$\omega\,dt \;\approx\; \frac{d\theta}{\sqrt{A^2-\theta^2}} \;=\; \frac{d\theta}{A}\frac{1}{\sqrt{1-(\theta/A)^2}}\;. \tag{11}$$

Integrating assuming $\theta = 0$ at $t = 0$, we find

$$\omega t \;=\; \omega\int_0^t dt' \;\approx\; \int_0^\theta \frac{d\theta'}{A}\frac{1}{\sqrt{1-(\theta'/A)^2}} \;=\; \arcsin\left(\frac{\theta}{A}\right)\;, \tag{12}$$

that is,

$$\theta(t) \;=\; A\sin(\omega t)\;. \tag{13}$$

2. Without the Small-Angle Approximation

If we do not make the small-angle approximation in the equation of motion, we need to solve Eq (2) which is not easy to do. Instead, let's look at energy conservation relation, Eq. (9). Using

$$\cos\theta \;=\; \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2} \;=\; 1-2\sin^2\frac{\theta}{2}\,,\qquad \cos A \;=\; \cos^2\frac{A}{2}-\sin^2\frac{A}{2} \;=\; 1-2\sin^2\frac{A}{2}\,, \tag{14}$$

we find:

$$\begin{aligned}\omega\,dt = \frac{d\theta}{\sqrt{2(\cos\theta-\cos A)}} &= \frac{d\theta}{2\sqrt{\sin^2\frac{A}{2}-\sin^2\frac{\theta}{2}}}\;. = \frac{d\theta}{2\left(\sin\frac{A}{2}\right)\sqrt{1-\dfrac{\sin^2\frac{\theta}{2}}{\sin^2\frac{A}{2}}}}\;. \\ &= \frac{d(\theta/2)}{k\sqrt{1-\dfrac{1}{k^2}\sin^2(\theta/2)}}\,,\qquad \text{where } k \;\equiv\; \sin(A/2)\;. \end{aligned}\tag{15}$$

Change variable from θ to a new one which we will call ϕ:

$$\begin{gathered}\sin\phi = \frac{1}{k}\sin(\theta/2) \\ \downarrow \\ \cos\phi\,d\phi = \frac{1}{k}\cos(\theta/2)\,d(\theta/2)\,, \\ \downarrow \\ \frac{d\phi}{\sqrt{1-k^2\sin^2(\phi)}} = \frac{d\phi}{\cos(\theta/2)} = \frac{d(\theta/2)}{k\cos\phi} = \frac{d(\theta/2)}{k\sqrt{1-\dfrac{1}{k^2}\sin^2(\theta/2)}}\end{gathered}\tag{16}$$

Note that $-A\le\theta\le A$ while $-\pi/2\le\phi\le\pi/2$. We now have

$$\omega\,dt \;=\; \frac{d\phi}{\sqrt{1-k^2\sin^2(\phi)}}\,,\qquad 0\le k^2\le\frac{1}{2}\;. \tag{17}$$

Upon integration, we obtain

$$\omega t \;=\; \omega\int_0^t dt' \;=\; \int_0^\phi \frac{d\phi'}{\sqrt{1-k^2\sin^2(\phi')}} \;=\; F(\phi,k)\,, \tag{18}$$

where the function $F(\phi,k)$ on the right-hand side is the so-called *elliptical integral of the first kind.* In *Mathematica*, it is encoded as `EllipticF[`ϕ`,`k`]`. To obtain ϕ as a function of ωt, we need the inverse function of $F(\phi,k)$.

Now, there is a function called *Jacobi's elliptical function* $\mathrm{sn}(z,k)$ which is defined as the inverse function of

$$z \;=\; \int_0^x \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} \;=\; \int_0^{\arcsin x}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \;=\; F(\arcsin x,k)\;. \tag{19}$$

Therefore, we can rewrite the relation $\omega t = F(\phi,k)$ as

$$\sin\phi(t) \;=\; \mathrm{sn}(\omega t,k) \;=\; \mathrm{sn}\left(\omega t,\;\sin\frac{A}{2}\right)\;. \tag{20}$$

Recalling the definition of ϕ, we find

$$\sin\frac{\theta(t)}{2} \;=\; k\,\sin\phi(t) \;=\; k\,\mathrm{sn}(\omega t,k) \;=\; \sin\frac{A}{2}\,\mathrm{sn}\left(\omega t,\;\sin\frac{A}{2}\right)\;. \tag{21}$$

Jacobi's elliptical function $\phi=\mathrm{sn}(z,k)$ is encoded in *Mathematica* as `JacobiSN[`z`,`k`]`. The graph of $\mathrm{sn}(\omega t,\sin(A/2))$ is shown in FIG. 2 for several values of A.

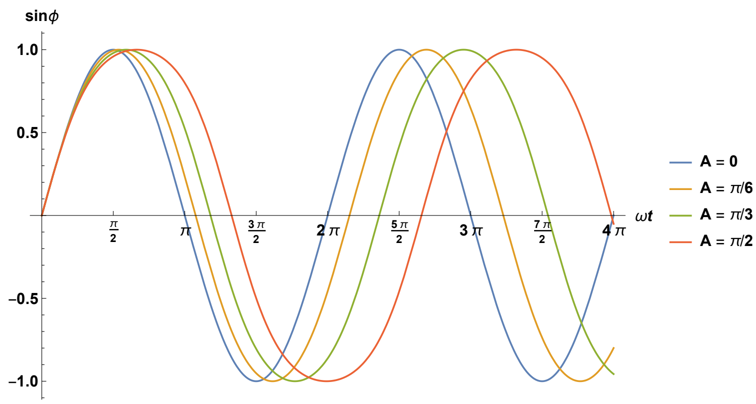


FIG. 2. The behavior of the Jacobi elliptical function $\mathrm{sn}\,(\omega t, \sin(A/2))$ for several values of the amplitude A.

Note that $\lim_{k\to 0} \mathrm{sn}(\omega t, k) = \sin(\omega t)$ with period 2π. In fact, *Mathematica* tells us that

$$\mathrm{sn}(\omega t, k) \;=\; \sin(\omega t) + \frac{k}{4}\left[-\omega t\cos(\omega t) + \sin(\omega t)\cos^2(\omega t)\right] + O(k^2)\;. \tag{22}$$

So the small-angle approximation gives us

$$\sin\frac{\theta(t)}{2} \;\approx\; \frac{\theta(t)}{2}\;,\qquad \sin\frac{A}{2} \;\approx\; \frac{A}{2}\;,\qquad \mathrm{sn}\left(\omega t, \sin\frac{A}{2}\right) \;\approx\; \sin(\omega t) + O(A)\;, \tag{23}$$

and we recover the usual result:

$$\begin{aligned} \sin\frac{\theta(t)}{2} &= \sin\frac{A}{2}\,\mathrm{sn}\left(\omega t,\ \sin\frac{A}{2}\right) \\ &\downarrow \\ \theta(t) &\approx A\sin(\omega t)\;. \end{aligned} \tag{24}$$

The dependence of the period (without the small-angle approximation) on the amplitude A can be calculated using *Mathematica*, and we obtain the graph shown in FIG. 3. As you can see, the period of the oscillation T increases as the amplitude A increases.

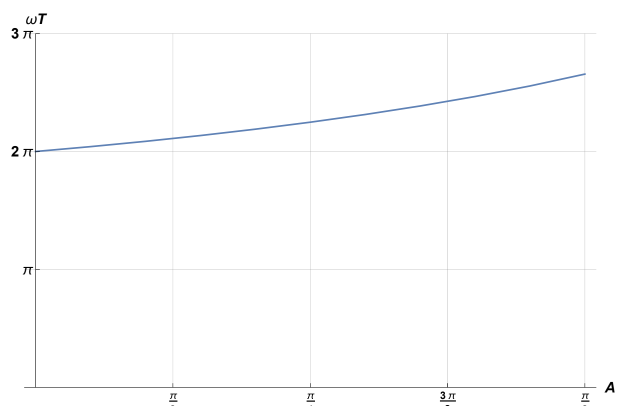


FIG. 3. Dependence of the period of oscillation T on the amplitude A.

B. The Tautochrone Problem

If you want to use the pendulum for time keeping, the fact that its period depends on the amplitude is a big problem. Now, there is not much you can do to the motion of a pendulum, but realizing that the motion of a mass sliding frictionlessly along a semi-circular track will be the same as that of the pendulum bob, we can consider replacing the semi-circular track with a track of a different shape and ask whether it would be possible to choose that shape so that the period of the oscillating mass would be independent of its amplitude. In other words, the problem is this:

> Find a curve for which the time it takes for a mass to slide frictionlessly down along it to the lowest point on the curve is independent of where the mass started to slide along the curve. Assume the mass starts sliding from rest.

This was called the *tautochrone* (Greek: *tauto+chrone* = same time) problem and was solved by Christiaan Huygens (1629–1695) in 1659, and published in his book *Horologium Oscillatorium* (The Pendulum Clock) in 1673. In it, he shows that the tautochrone curve is a *cycloid*. We will not go into his solution here, but first look at the *brachistochrone* problem which turns out to be closely related to the *tautochrone*.

II. THE BRACHISTOCHRONE

A. The Brachistochrone Problem

> Consider motion in a 2 dimensional vertical plane. Consider two points A and B, where A is higher than B but not necessarily directly above it. Let's say that an object is released at point A with initial velocity zero, and it slides along a track without friction until it reaches point B. What should the shape of the track be to minimize the time it takes for the object to slide from A to B?

This question is known as the *brachitochrone* (Greek: *brachisto+chrone* = shortest time) problem.

B. Historical Note

The Brachistochrone problem was posed by Johann Bernoulli (1667–1748) to the readers of the journal *Acta Eruditorum* in June, 1696. He also sent a letter to Newton (1642–1727) challenging him to solve the problem, which Newton received on 29 January 1697. He solved the problem overnight and had his solution published anonymously. However, Bernoulli immediately recognized the solution as Newton's saying *tanquam ex ungue leonem* (we know the lion by his claw).

Other then Newton, four mathematicians responded to the challenge: Jakob Bernoulli (1655–1705, Johann's brother), Leibniz (1646-1716), Tschirnhaus (1651–1708), and l'Hôpital (1661–1704). All five solutions were published in *Acta Eruditorum*.

C. Setting Up the Problem

The brachistochrone problem is a minimization problem which can be solved using variational calculus. It is a very nice exercise problem for students.

First, we need to figure out the quantity that we need to minimize. Call the horizontal direction x and the vertical direction y. For the sake of latter simplicity, we will take positive y to be the down direction, and $x_A = y_A = 0$. The shape of the track can be specified in many ways, *i.e.*

1. as a function $y(x)$ for $0 < x < x_B$,
2. as a function $x(y)$ for $0 < y < y_B$,
3. as two functions $x(\xi)$ and $y(\xi)$, where ξ parametrizes where the mass is along the track.

The first two methods do not allow for tracks that are multi-valued functions of x or y, but they still work as will be shown below.

The time it takes for the mass to move from point (x, y) to point $(x + dx, y + dy)$ is the distance divided by the speed

$$dt \;=\; \frac{\sqrt{dx^2 + dy^2}}{v}\;, \tag{25}$$

where the speed v is determined from energy conservation as

$$v \;=\; \sqrt{2gy}\;. \tag{26}$$

Therefore, the total time is

$$T \;=\; \int_0^T dt \;=\; \frac{1}{\sqrt{2g}} \int \sqrt{\frac{dx^2 + dy^2}{y}}\;. \tag{27}$$

(Note: if the mass were rolling down the track without slipping instead of sliding frictionlessly, we would have

$$mgy \;=\; \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \;=\; \frac{1}{2}(1+\beta)mv^2 \quad\rightarrow\quad v \;=\; \sqrt{\frac{2gy}{1+\beta}}\;, \tag{28}$$

where β is the factor that appears in the moment of inertia of the rolling mass: $I = \beta mr^2$. This will change T by a factor of $\sqrt{1+\beta}$:

$$T \;\rightarrow\; \sqrt{1+\beta}\,T\;, \tag{29}$$

but it will not change the fact that the integral of Eq. (27) be minimized.)

If the y-coordinate is specified as a function of x, then

$$T \;=\; \frac{1}{\sqrt{2g}} \int_0^{x_B} \sqrt{\frac{1+\dot{y}^2}{y}}\,dx\;, \qquad \dot{y} = \frac{\partial y}{\partial x}\;. \tag{30}$$

If the x-coordinate is specified as a function of y, then

$$T \;=\; \frac{1}{\sqrt{2g}} \int_0^{y_B} \sqrt{\frac{1+\dot{x}^2}{y}}\,dy\;, \qquad \dot{x} = \frac{\partial x}{\partial y}\;. \tag{31}$$

If the x and y-coordinates are both specified as functions of a parameter ξ, then

$$T \;=\; \frac{1}{\sqrt{2g}} \int_0^{\xi_B} \sqrt{\frac{\dot{x}^2+\dot{y}^2}{y}}\,d\xi\;, \qquad \dot{x} = \frac{\partial x}{\partial \xi}\;,\quad \dot{y} = \frac{\partial y}{\partial \xi}\;. \tag{32}$$

All we need to do now is apply the Euler-Lagrange equation to one of these expressions.

D. The Solution

1. *y as a function of x*

I will discuss the first parametrization first. The quantity we need to minimize is

$$T = \frac{1}{\sqrt{2g}} \int_0^{x_B} F(y,\dot{y})\,dx\;, \qquad F(y,\dot{y}) = \sqrt{\frac{1+\dot{y}^2}{y}}\;, \qquad \dot{y} = \frac{\partial y}{\partial x}\;. \tag{33}$$

Since

$$\frac{\partial F}{\partial y} = -\frac{1}{2}\frac{\sqrt{1+\dot{y}^2}}{\sqrt{y^3}}\;,$$

$$
\begin{aligned}
\frac{\partial F}{\partial \dot{y}} &= \frac{\dot{y}}{\sqrt{y}\sqrt{1+\dot{y}^2}}\;, \\
\frac{d}{dx}\frac{\partial F}{\partial \dot{y}} &= -\frac{1}{2}\frac{\dot{y}^2}{\sqrt{y^3}\sqrt{1+\dot{y}^2}} + \frac{\ddot{y}}{\sqrt{y}\sqrt{1+\dot{y}^2}} - \frac{\dot{y}^2\ddot{y}}{\sqrt{y}\sqrt{(1+\dot{y}^2)^3}} \\
&= -\frac{1}{2}\frac{\dot{y}^2}{\sqrt{y^3}\sqrt{1+\dot{y}^2}} + \frac{\ddot{y}}{\sqrt{y}\sqrt{1+\dot{y}^2}}\left(1-\frac{\dot{y}^2}{1+\dot{y}^2}\right) \\
&= -\frac{1}{2}\frac{\dot{y}^2}{\sqrt{y^3}\sqrt{1+\dot{y}^2}} + \frac{\ddot{y}}{\sqrt{y}\sqrt{(1+\dot{y}^2)^3}}\;,
\end{aligned}
\tag{34}
$$

the Euler-Lagrange equation is

$$
\begin{aligned}
0 &= \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial \dot{y}} \\
&= -\frac{1}{2}\frac{\sqrt{1+\dot{y}^2}}{\sqrt{y^3}} + \frac{1}{2}\frac{\dot{y}^2}{\sqrt{y^3}\sqrt{1+\dot{y}^2}} - \frac{\ddot{y}}{\sqrt{y}\sqrt{(1+\dot{y}^2)^3}} \\
&= -\frac{1}{2}\frac{1+\dot{y}^2}{\sqrt{y^3}\sqrt{1+\dot{y}^2}} + \frac{1}{2}\frac{\dot{y}^2}{\sqrt{y^3}\sqrt{1+\dot{y}^2}} - \frac{\ddot{y}}{\sqrt{y}\sqrt{(1+\dot{y}^2)^3}} \\
&= -\frac{1}{2}\frac{1}{\sqrt{y^3}\sqrt{1+\dot{y}^2}} - \frac{\ddot{y}}{\sqrt{y}\sqrt{(1+\dot{y}^2)^3}}\;,
\end{aligned}
\tag{35}
$$

or after simplification :

$$
0 \;=\; -\frac{1}{2y} - \frac{\ddot{y}}{1+\dot{y}^2}\;. \tag{36}
$$

Multiply by $2\dot{y}$:

$$
0 \;=\; -\frac{\dot{y}}{y} - \frac{2\dot{y}\ddot{y}}{1+\dot{y}^2}\;. \tag{37}
$$

Integrate :

$$
\text{constant} \;=\; -\ln y - \ln(1+\dot{y}^2) \;=\; -\ln y(1+\dot{y}^2)\;. \tag{38}
$$

Therefore,

$$
y(1+\dot{y}^2) \;=\; \text{constant} \;\equiv\; 2r\;. \tag{39}
$$

Solve for $\dot{y}$:

$$
\dot{y} \;=\; \frac{dy}{dx} \;=\; \sqrt{\frac{2r}{y}-1}\;. \tag{40}
$$

This can be rewritten as

$$
dx \;=\; \frac{dy}{\sqrt{\dfrac{2r}{y}-1}} \;=\; \frac{y\,dy}{\sqrt{2ry-y^2}} \;=\; \frac{y\,dy}{\sqrt{r^2-(y-r)^2}}\;. \tag{41}
$$

Change variable to

$$
\xi \;=\; y-r\;. \tag{42}
$$

Then

$$
dx \;=\; \frac{(\xi+r)\,d\xi}{\sqrt{r^2-\xi^2}} \;=\; r\,\frac{\left(1+\dfrac{\xi}{r}\right)\dfrac{d\xi}{r}}{\sqrt{1-\dfrac{\xi^2}{r^2}}}\;. \tag{43}
$$

Change variable again to

$$\zeta \;=\; \frac{\xi}{r}\,. \tag{44}$$

Then,

$$\frac{dx}{r} \;=\; \frac{(1+\zeta)\,d\zeta}{\sqrt{1-\zeta^2}} \;=\; \left(\frac{1}{\sqrt{1-\zeta^2}}+\frac{\zeta}{\sqrt{1-\zeta^2}}\right)d\zeta\,. \tag{45}$$

This can be integrated to yield

$$\frac{x}{r} \;=\; \frac{\pi}{2}+\sin^{-1}\zeta-\sqrt{1-\zeta^2}\,, \tag{46}$$

where the integration constant has been chosen so that $x=0$ when $y=0$ ($\xi=-r$, $\zeta=-1$). Define

$$\theta \;\equiv\; \frac{\pi}{2}+\sin^{-1}\zeta\,. \tag{47}$$

Then,

$$\zeta \;=\; \sin\left(\theta-\frac{\pi}{2}\right) \;=\; -\cos\theta\,. \tag{48}$$

Therefore,

$$\begin{aligned}\frac{x}{r} &\;=\; \theta-\sqrt{1-\cos^2\theta} \;=\; \theta-\sin\theta\,,\\ \frac{y}{r} &\;=\; 1+\zeta \;=\; 1-\cos\theta\,,\end{aligned} \tag{49}$$

i.e.

$$\begin{cases} x \;=\; r(\theta-\sin\theta) \\ y \;=\; r(1-\cos\theta) \end{cases} \tag{50}$$

which is the equation for a *cycloid*. See FIG. 4.

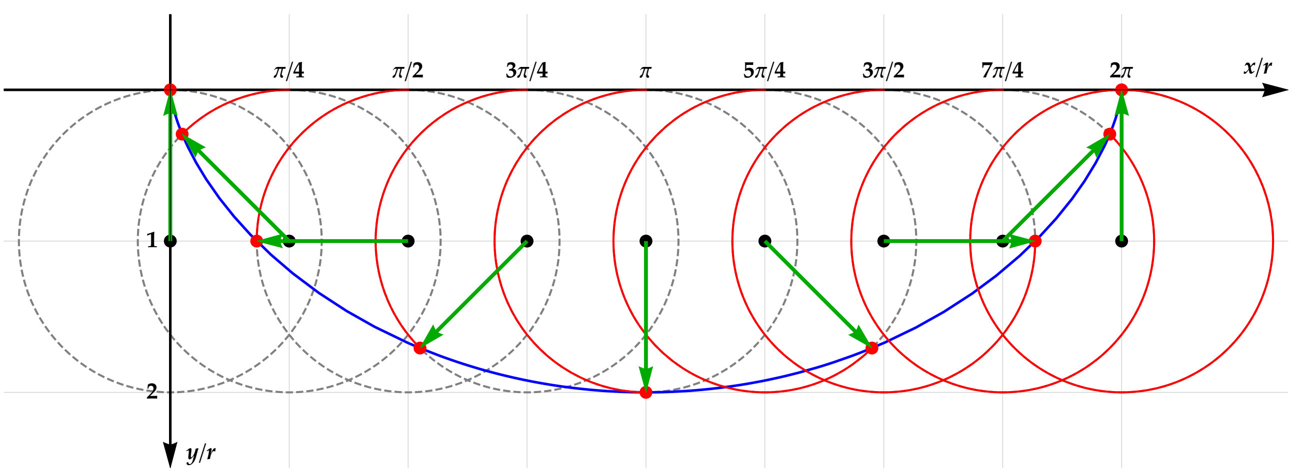


FIG. 4. A *cycloid*, shown above in blue, is the curve that is traced out by a point on the circumference of a circle (shown above in red) when the circle rolls without slipping along a straight horizontal line.

2. *x as a function of y*

The quantity we need to minimize is

$$T = \frac{1}{\sqrt{2g}}\int_0^{y_B} F(x,\dot{x})\,dy\;, \qquad F(y,\dot{y}) = \sqrt{\frac{1+\dot{x}^2}{y}}\;, \qquad \dot{x} = \frac{\partial x}{\partial y}\;. \tag{51}$$

Since

$$\frac{\partial F}{\partial x} \;=\; 0\;, \qquad \frac{\partial F}{\partial \dot{x}} \;=\; \frac{\dot{x}}{\sqrt{y}\sqrt{1+\dot{x}^2}}\;, \tag{52}$$

the Euler-Lagrange equation is

$$\begin{aligned}
\frac{\partial F}{\partial x} - \frac{d}{dy}\frac{\partial F}{\partial \dot{x}} &= 0 \\
&\downarrow \\
-\frac{d}{dy}\left[\frac{\dot{x}}{\sqrt{y}\sqrt{1+\dot{x}^2}}\right] &= 0 \\
&\downarrow \\
\frac{\dot{x}}{\sqrt{y}\sqrt{1+\dot{x}^2}} &= \text{constant} \;\equiv\; \frac{1}{\sqrt{2r}} \\
&\downarrow \\
\frac{\dot{x}^2}{(1+\dot{x}^2)} &= \frac{y}{2r} \\
&\downarrow \\
\dot{x} &= \pm\sqrt{\frac{y}{2r-y}}\;.
\end{aligned} \tag{53}$$

Note that we must have $0 \le y < 2r$ for the square-root to remain real. Change variable from y to:

$$y \;=\; r(1-\cos\theta)\;, \qquad dy \;=\; r\sin\theta\,d\theta\;. \tag{54}$$

When θ runs from 0 to π, y runs from 0 to $2r$. We now have

$$\begin{aligned}
dx \;=\; \pm\sqrt{\frac{y}{2r-y}}\,dy &= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,r\sin\theta\,d\theta \\
&= \pm\sqrt{\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)}}\,2r\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta \\
&= \pm 2r\sin^2\frac{\theta}{2}\,d\theta \\
&= \pm r(1-\cos\theta)\,d\theta\;,
\end{aligned} \tag{55}$$

which integrates to

$$x \;=\; \pm r(\theta - \sin\theta) + \text{constant}\;. \tag{56}$$

When θ runs from 0 to π, the two possible signs give two branches of the cycloid: the left-half and the right-half which meet in the middle:

$$\left\{\begin{array}{lcl} x &=& r(\theta-\sin\theta) \\ y &=& r(1-\cos\theta) \end{array}\right. \qquad\qquad \left\{\begin{array}{lcl} x &=& 2\pi - r(\theta-\sin\theta) \\ y &=& r(1-\cos\theta) \end{array}\right. \tag{57}$$

3. *x and y as functions of t*

Consider the parametric representation of the track. The quantity we need to minimize is

$$T \;=\; \int_{\xi_A}^{\xi_B} F(x,\dot{x},y,\dot{y})\,d\xi\;, \qquad F(x,\dot{x},y,\dot{y}) \;\equiv\; \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{v}\;, \qquad \dot{x} = \frac{\partial x}{\partial \xi}\;, \quad \dot{y} = \frac{\partial y}{\partial \xi}\;. \tag{58}$$

The Euler-Lagrange equations are

$$\begin{aligned} \frac{\partial F}{\partial x} - \frac{d}{d\xi}\frac{\partial F}{\partial \dot{x}} &= 0\;, \\ \frac{\partial F}{\partial y} - \frac{d}{d\xi}\frac{\partial F}{\partial \dot{y}} &= 0\;, \end{aligned} \tag{59}$$

but these equations cannot be solved unless we specify how ξ parametrizes the track. This is because there are an infinite numbers of ways to parametrize the same track. Choosing a particular parametrization is called *gauge fixing* (due to historical reasons).

So instead of a generic parameter ξ, let's use the time t itself that it takes for the object to travel along the curve from point A as the parameter. Then, the dot will designate differentiation with respect to t, and the Euler-Lagrange equations become

$$\begin{aligned} \frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} &= 0\;, \\ \frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial \dot{y}} &= 0\;, \end{aligned} \tag{60}$$

Note that this choice of parameter lets us set $F(x,\dot{x},y,\dot{y}) = 1$, or

$$v \;=\; \sqrt{\dot{x}^2+\dot{y}^2}\;. \tag{61}$$

along the original curve from which t was defined. (F is not equal to 1 away from the original curve around which you are allowing the path to vary so you must not use this relation until *after* you have taken all the functional derivatives.) Since

$$\begin{aligned} \frac{\partial F}{\partial x} &= 0\;, & \frac{\partial F}{\partial \dot{x}} &= \frac{\dot{x}}{v\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{\dot{x}}{v^2}\;, \\ \frac{\partial F}{\partial y} &= -\frac{\sqrt{\dot{x}^2+\dot{y}^2}}{v^2}\frac{\partial v}{\partial y} = -\frac{g}{v^2}\;, & \frac{\partial F}{\partial \dot{y}} &= \frac{\dot{y}}{v\sqrt{\dot{x}^2+\dot{y}^2}} = \frac{\dot{y}}{v^2}\;, \end{aligned} \tag{62}$$

we find

$$\begin{aligned} \frac{d}{dt}\left\{\frac{\dot{x}}{v^2}\right\} &= 0\;, \\ \frac{d}{dt}\left\{\frac{\dot{y}}{v^2}\right\} &= -\frac{g}{v^2}\;. \end{aligned} \tag{63}$$

Notice that since $v^2 = \dot{x}^2+\dot{y}^2$, the above equations involve only $\dot{x}$ and $\dot{y}$. Define

$$\dot{x} = v\sin\theta\;, \qquad \dot{y} = v\cos\theta\;. \tag{64}$$

Then Eq. (63) become

$$\begin{aligned} \frac{d}{dt}\left\{\frac{\sin\theta}{v}\right\} &= 0\;, \\ \frac{d}{dt}\left\{\frac{\cos\theta}{v}\right\} &= -\frac{g}{v^2}\;. \end{aligned} \tag{65}$$

The first equation tells us that

$$\frac{\sin\theta}{v} \;=\; \text{constant} \equiv \frac{1}{2u}\;, \tag{66}$$

or $v = 2u\sin\theta$. (The factor of 2 is for latter convenience.) Substituting into the second equation gives us

$$\frac{d}{dt}\{\cot\theta\} \;=\; -\frac{g}{2u\sin^2\theta}\;. \tag{67}$$

The left-hand side is

$$\frac{d}{dt}\{\cot\theta\} \;=\; -\frac{\dot{\theta}}{\sin^2\theta}\;, \tag{68}$$

so the equation simplifies to

$$\dot{\theta} \;=\; \frac{g}{2u}\;. \tag{69}$$

The solution is

$$\theta(t) \;=\; \frac{\omega}{2}\,t\;, \qquad \omega \;\equiv\; \frac{g}{u}\;. \tag{70}$$

Note that the integration constant has been chosen so that $v(0) = 2u\sin\theta(0) = 0$. Therefore,

$$\begin{aligned} \dot{x} = v\sin\theta \;&=\; 2u\sin^2\theta \;=\; u\,(1-\cos 2\theta) \;=\; u\,(1-\cos\omega t)\;, \\ \dot{y} = v\cos\theta \;&=\; 2u\sin\theta\cos\theta \;=\; u\,\sin 2\theta \;=\; u\,\sin\omega t\;. \end{aligned} \tag{71}$$

These equations can be integrated easily to yield

$$\begin{aligned} x(t) &= r(\omega\,t - \sin\omega t)\;, \\ y(t) &= r(1-\cos\omega t)\;, \end{aligned} \tag{72}$$

where $r \equiv u/\omega = u^2/g$, and the integration constants have been chosen so that $x(0) = y(0) = 0$. This is an equation for a *cycloid*. It is the trajectory of a point on the circumference of a circle which rolls horizontally without slipping. Note that there is only one free paramter, u, in this solution ($r = u^2/g$, $\omega = g/u$). This parameter must be adjusted so that the track goes through point B. So we can tell that the brachistochrone problem has a unique solution. Note also that the circle that is rolling to generate the cycloid is rolling at constant velocity

$$u \;=\; r\omega \;=\; \sqrt{gr} \tag{73}$$

in the x-direction. We can also tell that the time it takes for the mass to roll from the origin (Point A) down to the bottom of the cycloid (which corresponds to $\omega t = \pi$) is

$$T(0\to\pi) \;=\; \frac{\pi}{\omega} \;=\; \pi\sqrt{\frac{r}{g}}\;. \tag{74}$$

E. Tautochrone Property of the Cycloid

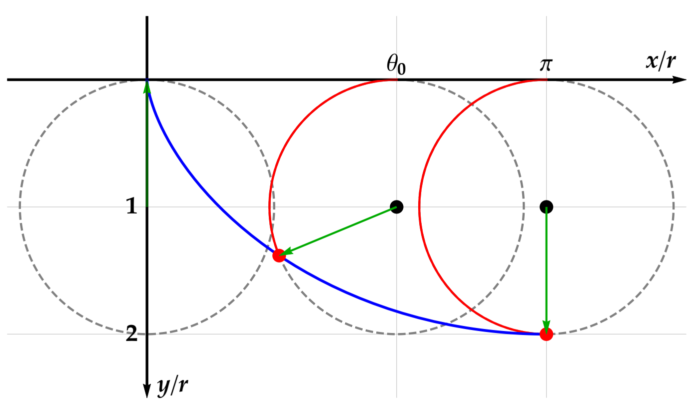


FIG. 5. The time it takes for an object to slide down from any point along the cycloid to the bottom is always the same.

The calculation above shows that if an object slides along the cycloid from the origin ($\theta = 0$) to the lowest point ($\theta = \pi$), the time it takes is $\pi\sqrt{r/g}$. What if the object started sliding from somewhere else along the cycloid, say, from the point which corresponds to $\theta = \theta_0$? (See FIG. 5.)

The time it takes in that case will be:

$$T(\theta_0 \to \pi) \;=\; \frac{1}{\sqrt{2g}} \int_{\theta_0}^{\pi} \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{\sqrt{y-y_0}}\, d\theta\;. \tag{75}$$

Since

$$\dot{x} \;=\; \frac{dx}{d\theta} \;=\; r(1-\cos\theta)\;, \qquad \dot{y} \;=\; \frac{dy}{d\theta} \;=\; r\sin\theta\;, \tag{76}$$

we obtain

$$\sqrt{\dot{x}^2+\dot{y}^2} \;=\; r\sqrt{2(1-\cos\theta)} \;=\; r\sqrt{4\sin^2\frac{\theta}{2}} \;=\; 2r\sin\frac{\theta}{2}\;. \tag{77}$$

And from

$$y \;=\; r(1-\cos\theta)\;, \qquad y_0 \;=\; r(1-\cos\theta_0)\;, \tag{78}$$

we find

$$\sqrt{y-y_0} \;=\; \sqrt{r}\sqrt{(1-\cos\theta)-(1-\cos\theta_0)} \;=\; \sqrt{2r}\sqrt{\cos^2\frac{\theta_0}{2}-\cos^2\frac{\theta}{2}}\;. \tag{79}$$

Therefore,

$$T(\theta_0 \to \pi) \;=\; \sqrt{\frac{r}{g}} \int_{\theta_0}^{\pi} \frac{\sin\dfrac{\theta}{2}}{\sqrt{\cos^2\dfrac{\theta_0}{2}-\cos^2\dfrac{\theta}{2}}}\, d\theta\;. \tag{80}$$

Change variable to

$$z \;=\; \frac{\cos\dfrac{\theta}{2}}{\cos\dfrac{\theta_0}{2}}\;, \qquad dz \;=\; -\frac{\sin\dfrac{\theta}{2}}{\cos\dfrac{\theta_0}{2}}\frac{d\theta}{2}\;. \tag{81}$$

Then

$$T(\theta_0 \to \pi) \;=\; 2\sqrt{\frac{r}{g}} \int_0^1 \frac{dz}{\sqrt{1-z^2}} \;=\; 2\sqrt{\frac{r}{g}}\Big[\arcsin z\Big]_0^1 \;=\; \pi\sqrt{\frac{r}{g}}\;, \tag{82}$$

which is completely independent of θ_0. So no matter where the object starts sliding from, the time it takes to reach the bottom of the cycloid will always be the same. So the cycloid is not only the solution to the barchistochrone problem, it is also the solution to the tautochrone problem!

F. Tautochrone Pendulum

Now, would it be possible to construct a pendulum such that its bob follows a cycloid instead of a circle? It that were possible, we would have a pendulum whose period is independent of the amplitude. The answer happens to be YES!

Consider the arc-length of the cycloid from the origin O ($\theta = 0$) to the bottom of the trajectory C ($\theta = \pi$). See FIG. 6. It is not difficult to see that this length is

$$L_{\mathrm{OC}} \;=\; \int_0^{\pi} \sqrt{\dot{x}^2+\dot{y}^2}\, d\theta \;=\; 2r\int_0^{\pi}\sin\frac{\theta}{2}\, d\theta \;=\; 4r\left[-\cos\frac{\theta}{2}\right]_0^{\pi} \;=\; 4r\;. \tag{83}$$

So if we suspend a pendulum of length $4r$ from the origin O, and wrap the string around the cycloid (think of it as a wall) then the pendulum bob will end up at C. Now, let's find out the trajectory of the bob if it is released from

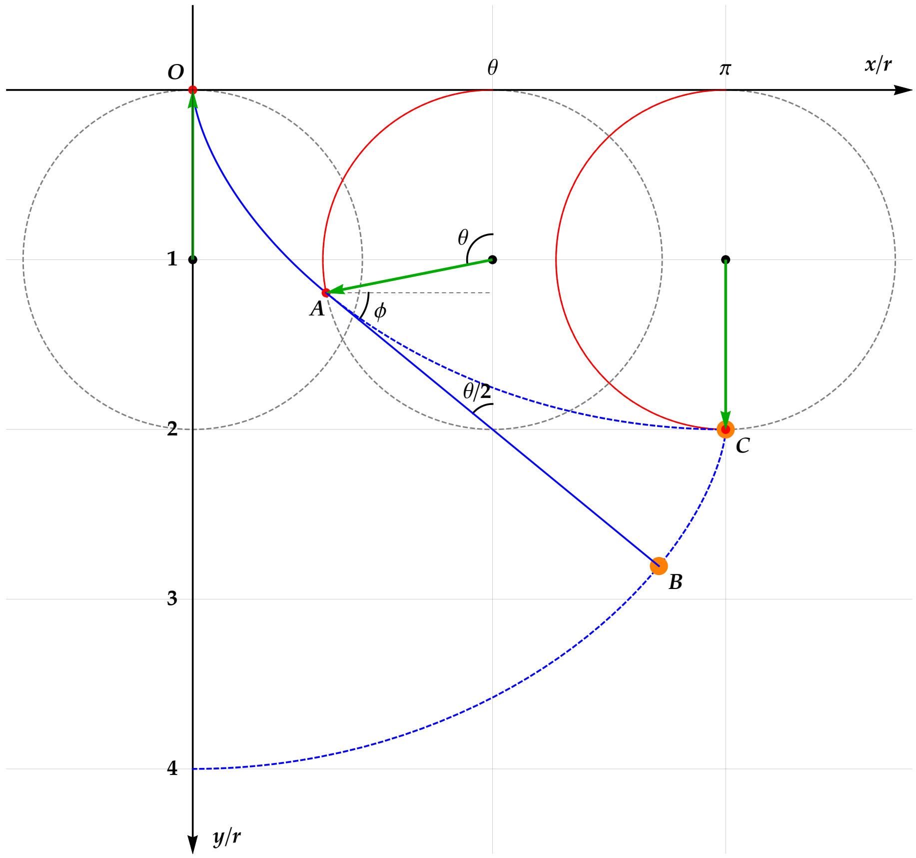


FIG. 6. The arc-length of the cycloid from O to C is exactly $4r$. If a pendulum of length $4r$ is suspended from O, and the string wraps around the cycloid, then the pendulum bob will be at C. If the pendulum bob is released, it follows another cycloid shown in dashed blue above.

C. As the bob swings downward, the string will gradually lose contact with the cycloid wall. Let's say that when the bob is at point B, the string is in contact with the cycloid wall between the origin O and point A. The length of the string between O and A is

$$L_{OA} \;=\; \int_0^\theta \sqrt{\dot{x}^2+\dot{y}^2}\,d\theta \;=\; 2r\int_0^\theta \sin\frac{\theta}{2}\,d\theta \;=\; 4r\left[-\cos\frac{\theta}{2}\right]_0^\theta \;=\; 4r\left(1-\cos\frac{\theta}{2}\right)\;. \tag{84}$$

So the length of the string between points A and B is

$$L_{AB} \;=\; 4r - L_{OA} \;=\; 4r\cos\frac{\theta}{2}\;. \tag{85}$$

The slope of AB is

$$\tan\phi \;=\; \frac{\dot{y}}{\dot{x}} \;=\; \frac{\sin\theta}{1-\cos\theta} \;=\; \frac{\cos\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}} \;=\; \cot\frac{\theta}{2}\;. \tag{86}$$

So the coordinates of point B are

$$\begin{aligned}
x_B &= x_A + L_{AB}\cos\phi \\
&= r(\theta-\sin\theta) + 4r\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\
&= r(\theta-\sin\theta) + 2r\sin\theta \\
&= r(\theta+\sin\theta)\;,
\end{aligned}$$

$$\begin{aligned} y_B &= y_B + L_{AB}\sin\phi \\ &= r(1-\cos\theta) + 4r\cos\frac{\theta}{2}\cos\frac{\theta}{2} \\ &= r(1-\cos\theta) + 2r(1+\cos\theta) \\ &= r(3+\cos\theta)\;. \end{aligned} \tag{87}$$

It is not difficult to see that as point A moves along the cycloid wall, point B moves along another cycloid as shown in FIG. 7. So if a pendulum of length $\ell = 4r$ is suspended from the origin, and the motion of its string restricted by a cycloid shaped wall, then the trajectory of the pendulum bob will be another cycloid. Due to the tautochrone property of the cycloid, this means that this pendulum will have a period equal to

$$T \;=\; 4T(\theta_0 \to \pi) \;=\; 4\pi\sqrt{\frac{r}{g}} \;=\; 2\pi\sqrt{\frac{\ell}{g}}\;, \tag{88}$$

which is completely independent of the amplitude.

Recall that in the case of the simple pendulum of length ℓ, we needed to use the small angle approximation to obtain simple harmonic motion and the period was only approximately equal to

$$T \;\approx\; 2\pi\sqrt{\frac{\ell}{g}}\;. \tag{89}$$

In the case of the tautochrone pendulum, the period is exactly equal to $2\pi\sqrt{\ell/g}$. This mechanism can be used in pendulum clocks so that they keep the correct time regardless of the amplitude of the pendulum.

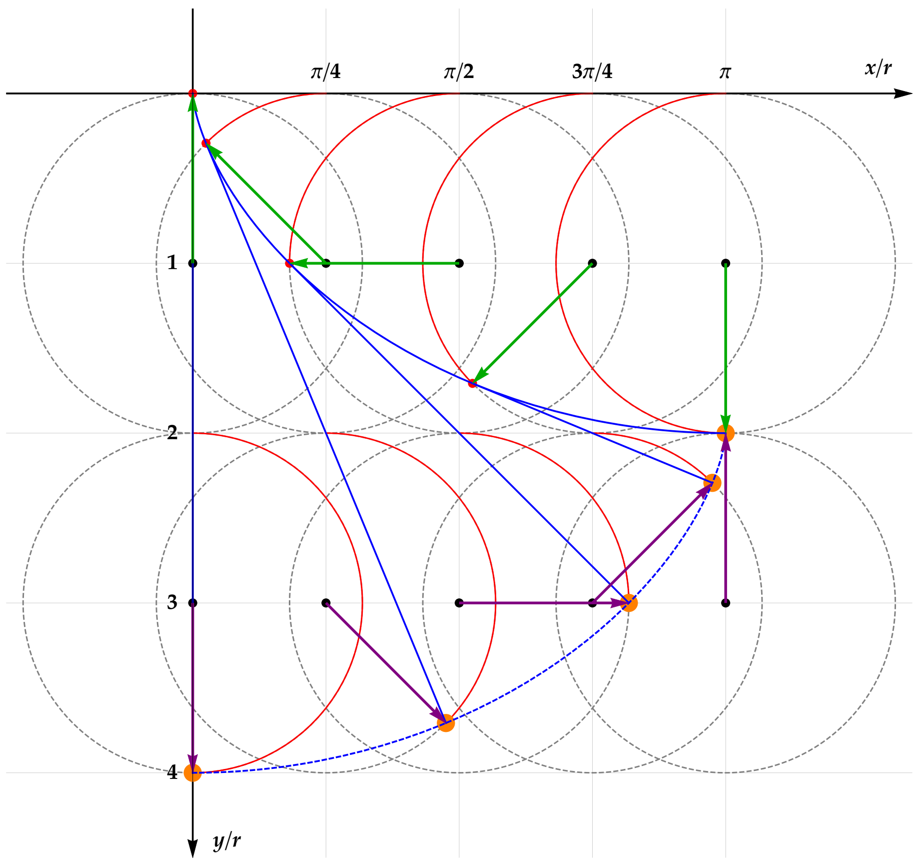


FIG. 7. If a pendulum of length $\ell = 4r$ is suspended from the origin, and the motion of its string is restricted by a cycloid shaped wall, then the trajectory of the pendulum bob will be another cycloid. Due to the tautochrone property of the cycloid, this means that this pendulum will have a period which is completely independent of the amplitude, in contradistinction to the simple pendulum.

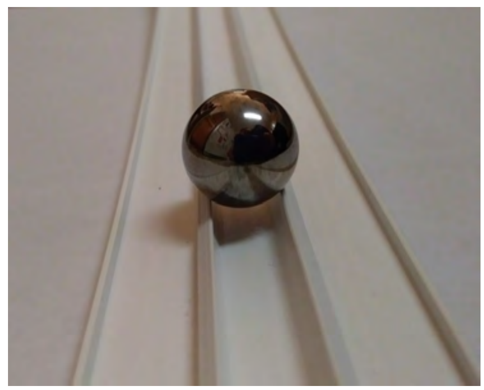
III. CLASSROOM DEMONSTRATIONS

A. Cycloid Track Demo

1. Materials

1. **5/8 inch diameter** marbles or metal bearing balls.

2. Screen Tight® **1-1/2 inch** Porch Screening System Cap (1.5 in.× 8 ft.):



   This is a strip of plastic with a groove running down the middle of one side which provides a perfect track for the **5/8 inch diameter** marbles/metal bearing balls. Available at:

   Home Depot® for $3.75 + tax:
   https://www.homedepot.com/p/Screen-Tight-1-1-2-in-White-Porch-Screening-System-Cap-WCAP18/100074953

   Lowes® for $3.78 + tax:
   https://www.lowes.com/pd/Screen-Tight-Vinyl-Frame-Connector/3024713

3. **40 mm × 40 mm** L-shaped corner braces.

   Available at Amazon for about $10 for 16∼20 pieces. Any manufacturer's product will do.

4. **10 mm diameter** $\times$ **2 mm thick** round refrigerator magnets. 2 to 3 per corner bracket.

   I purchased the following set of 120 from Amazon for $9.99:
   https://www.amazon.com/Adhesive-Magnets-Permanent-Building-Scientific/dp/B07KMXSLRB

   This set comes with 30 round double-sided adhesive mounting tapes which allows you to attach 2 magnets each onto 15 corner braces.

5. Foam mounting double-sided adhesive tape (if your magnets did not come with any). Get a strong one.
6. Packaging string. (About 8 feet.)
7. Packaging tape.

2. Preparation

1. Attach 2 (or 3) refrigerator magnets to each corner brace with the foam mounting tapes. 15 corner braces would suffice.

2. Puncture a hole at each end of the plastic strip large enough to thread the packaging string through. Thread one end of the packaging string through one of the holes and tie firmly to the plastic strip. Bend the plastic strip a bit, track side in, and thread the other end of the string through the other hole and tie loosely to the plastic strip. The plastic strip and string should now look like a strung archery bow.

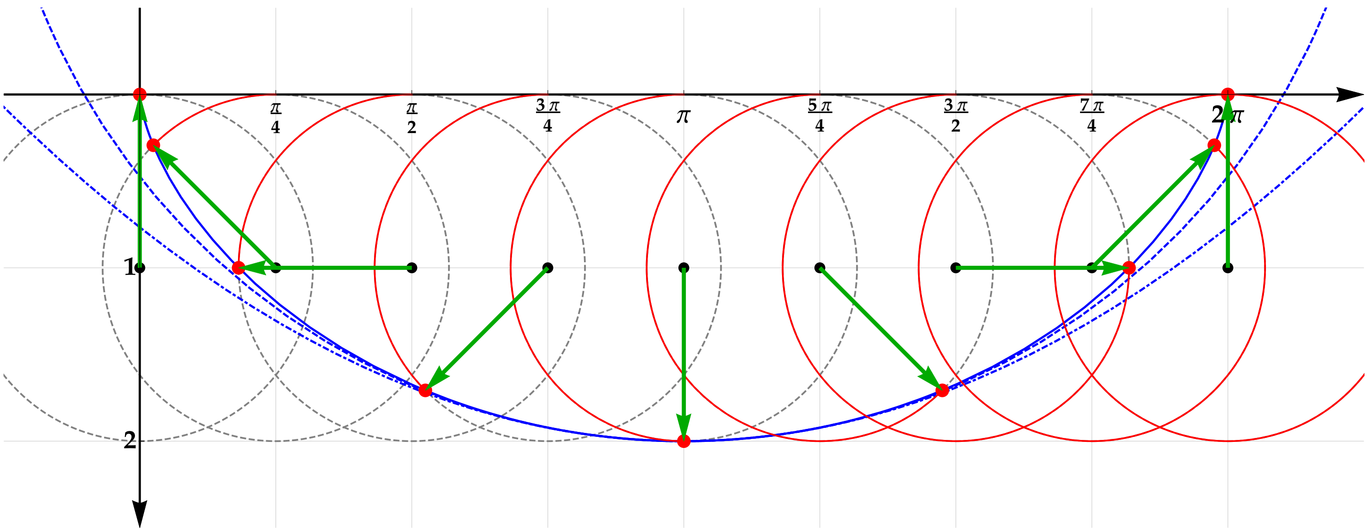
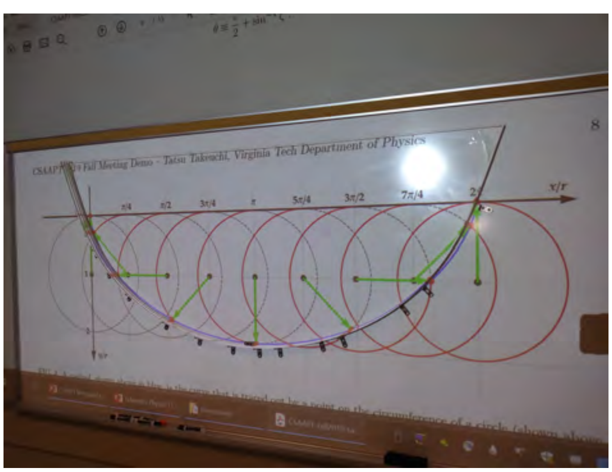
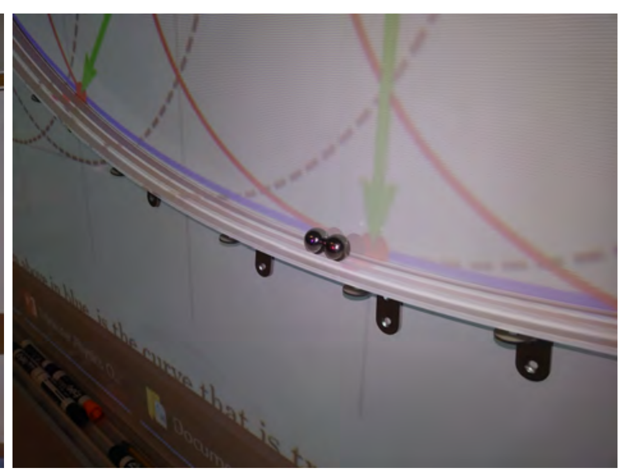


FIG. 8. The Cycloid (blue solid), Semi-Circle (blue dashed), and Parabola (blue dot-dashed). The have all been scaled to have the same curvature at $(x, y) = (\pi, 2)$. The differences only become notable as you move away from the center.

3. Tautochrone Demo

1. Project FIG. 8 onto a large magnetic blackboard or whiteboard.
2. Attach the corner braces onto the black/whiteboard along the cycloid. These will support the plastic strip.
3. Place the plastic strip on top of the corner braces, track-side up, and adjust the positions of the braces to nudge the plastic strip into the shape of a cycloid.
4. Untie and then tie the string at the end where it is only loosely attached to the plastic strip to adjust the length and tension of the string to control the strip's shape toward the ends. As you can see from FIG. 8, the shape of the curve toward the ends is important. Ideally, the tangent of the cycloid at both ends should be vertical.
5. If the plastic strip is unstable, use packaging tape to fix it to the black/whiteboard and/or the corner braces.
6. Release two marbles/bearing balls simultaneously from arbitrary heights on both sides and see if they meet in the middle.
7. Adjust the shape of the plastic strip to the semi-circle, and then the parabola, and repeat the experiment. Try other shapes as well. Make sure the left-right symmetry of the shape is maintained.





4. Brachistochrone Demo

1. In addition to the cycloid track you prepared in the previous demo, use a second screening system cap to prepare another track of a different shape, e.g. a straight one supported by a 1.5 in. $\times$ 0.5 in $\times$ 8 ft. piece of wood (or by the hands of multiple students).
2. Race two marbles down the two tracks and see which one wins. Make sure to start from the origin of FIG. 8.
3. Have the students try out various shapes.

B. Tautochrone Pendulum

This could be a nice student project.

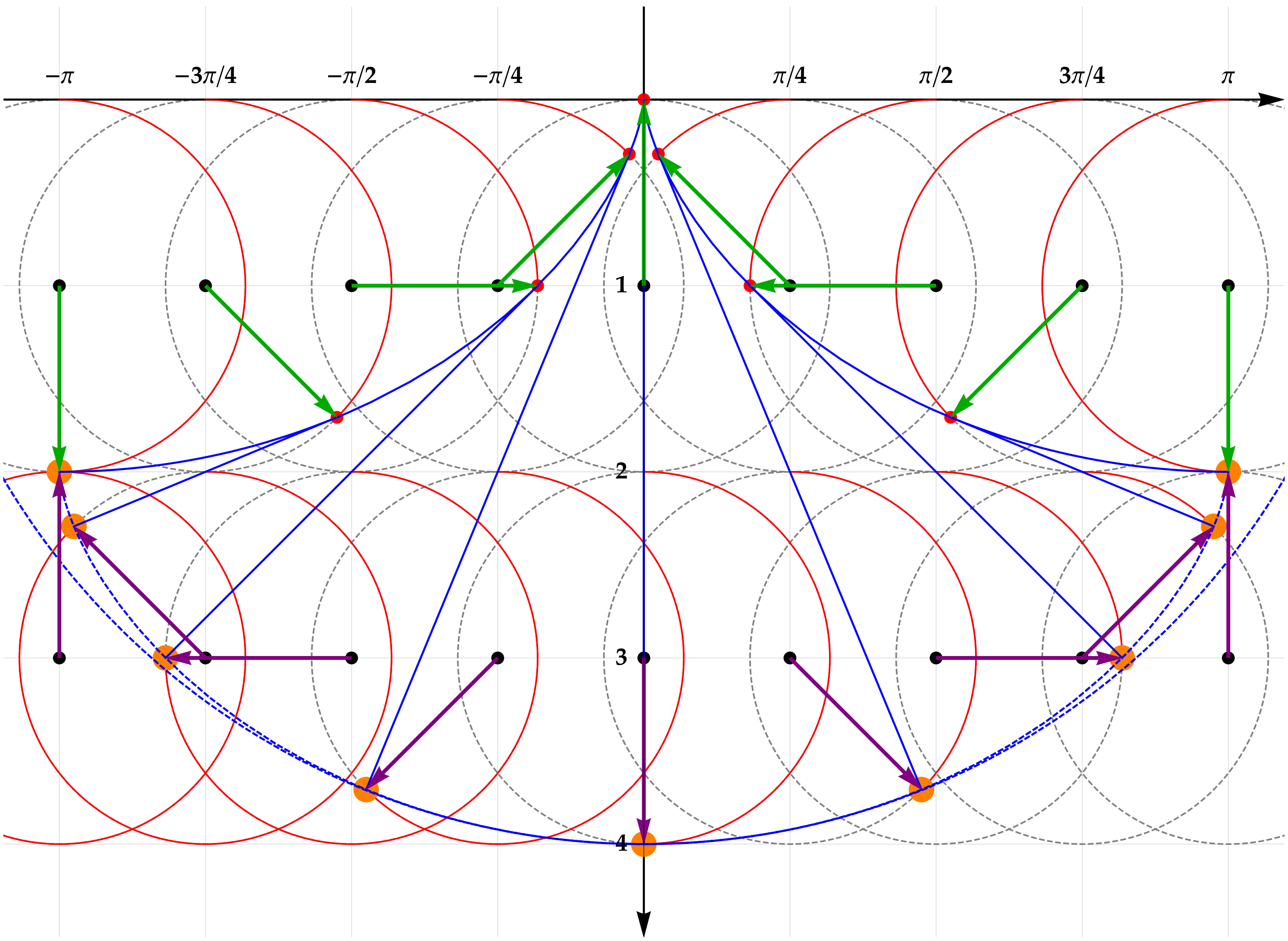


FIG. 9. The Tautochrone Pendulum.

1. Print out FIG. 9 as a large poster and glue it onto some rigid flat material such as plywood. Do not use foam boards for science fair presentations since they will warp.
2. Construct walls along the two arcs of the cycloid on the top. (Use any material you like.)
3. Suspend a pendulum of length $4r$ in the middle and study how its period depends on the amplitude.
4. Compare with a simple pendulum.