Field Theory Approach to Equilibrium Critical Phenomena

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Lecture 1: Critical Scaling: Mean-Field Theory, Real-Space RG

- Ising model: mean-field theory
- Real-space renormalization group
- Landau theory for continuous phase transitions
- Scaling theory

Lecture 2: Momentum Shell Renormalization Group

- Landau–Ginzburg–Wilson Hamiltonian
- Gaussian approximation
- Wilson's momentum shell renormalization group
- Dimensional expansion and critical exponents

Lecture 3: Field Theory Approach to Critical Phenomena

- Perturbation expansion and Feynman diagrams
- Ultraviolet and infrared divergences, renormalization
- Renormalization group equation and critical exponents
- Recent developments

Lecture 1: Critical Scaling: Mean-Field Theory, Re

Ferromagnetic Ising model

Principal task of statistical mechanics: understand *macroscopic* properties of matter (interacting many-particle systems):
$\rightarrow$ thermodynamic *phases* and *phase transitions*

Phase transitions at temperature $T > 0$ driven by competition between energy E minimization and entropy S maximization:
minimize *free energy* $F = E - T\,S$

Example: *Ising model* for N "spin" variables $\sigma_i = \pm 1$ with ferromagnetic exchange couplings $J_{ij} > 0$ in external field h:

$$H(\{\sigma_i\}) = -\frac{1}{2} \sum_{i,j=1}^{N} J_{ij}\, \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i$$

Goal: *partition function* $Z(T, h, N) = \sum_{\{\sigma_i = \pm 1\}} e^{-H(\{\sigma_i\})/k_{\mathrm{B}} T}$,
free energy $F(T, h, N) = -k_{\mathrm{B}} T \ln Z(T, h, N)$, *thermal averages*:

$$\langle A(\{\sigma_i\}) \rangle = \frac{1}{Z(T, h, N)} \sum_{\{\sigma_i = \pm 1\}} A(\{\sigma_i\})\, e^{-H(\{\sigma_i\})/k_{\mathrm{B}} T}$$

Curie–Weiss mean-field theory

Mean-field approximation: replace effective local field with average:

$$h_{\mathrm{eff},i} = -\frac{\partial H}{\partial \sigma_i} = h + \sum_j J_{ij}\sigma_j \;\to\; h + \widetilde{J} m\,,\; \widetilde{J} = \sum_i J(x_i)\,,\; m = \langle \sigma_i \rangle$$

More precisely: $\sigma_i = m + (\sigma_i - \langle \sigma_i \rangle)$

$\to \sigma_i\,\sigma_j = m^2 + m\,(\sigma_i - \langle \sigma_i \rangle + \sigma_j - \langle \sigma_j \rangle) + (\sigma_i - \langle \sigma_i \rangle)(\sigma_j - \langle \sigma_j \rangle)$

Neglect fluctuations / spatial correlations $\to$

$$H \approx \frac{N m^2 \widetilde{J}}{2} - \left(h + \widetilde{J} m\right) \sum_{i=1}^{N} \sigma_i\,,\; Z \approx e^{-N m^2 \widetilde{J}/2 k_{\mathrm{B}} T} \left(2 \cosh \frac{h + \widetilde{J} m}{k_{\mathrm{B}} T}\right)^N$$

yields *Curie–Weiss equation of state*

$$m(T,h) = -\frac{1}{N} \left(\frac{\partial F_{\mathrm{mf}}}{\partial h}\right)_{T,N} = \tanh \frac{h + \widetilde{J}\, m(T,h)}{k_{\mathrm{B}} T}$$

- Solution for large T: *disordered, paramagnetic phase* $m = 0$
- $T < T_c = \widetilde{J}/k_{\mathrm{B}}$: *ordered, ferromagnetic phase* $m \neq 0$
- *Spontaneous symmetry breaking* at *critical point* $T_c, h = 0$

Mean-field critical power laws

Expand equation of state near T_c:

$$|\tau| = \frac{|T-T_c|}{T_c} \ll 1 \text{ and } h \ll \widetilde{J} \to |m| \ll 1:$$

$$\to \quad \frac{h}{k_\mathrm{B}\, T_c} \approx \tau m + \frac{m^3}{3}$$

- *critical isotherm*: $T = T_c: \;\; h \approx \frac{k_\mathrm{B}\, T_c}{3}\, m^3$
- *coexistence curve*: $h = 0\,, \;\; T < T_c: \;\; m \approx \pm(-3\tau)^{1/2}$
- *isothermal susceptibility*:

$$\chi_T = N\left(\frac{\partial m}{\partial h}\right)_T \approx \frac{N}{k_\mathrm{B}\, T_c}\, \frac{1}{\tau + m^2} \approx \frac{N}{k_\mathrm{B}\, T_c} \begin{cases} 1/\tau^1 & \tau > 0 \\ 1/2|\tau|^1 & \tau < 0 \end{cases}$$

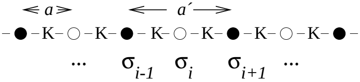
$\to$ *Power law singularities* in the vicinity of the critical point

Deficiencies of mean-field approximation:

- predicts transition in any spatial dimension d, but Ising model does not display long-range order at $d = 1$ for $T > 0$
- experimental *critical exponents differ* from mean-field values
- origin: *diverging* susceptibility indicates *strong fluctuations*

Real-space renormalization group: Ising chain

Partition sum for $h = 0$, $K = J/k_\mathrm{B}T$:

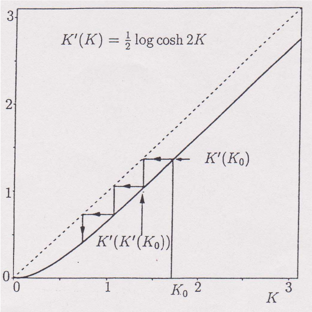


$$Z(K,N) = \sum_{\{\sigma_i=\pm1\}} e^{K\sum_{i=1}^{N}\sigma_i\sigma_{i+1}}$$

"*decimation*" of $\sigma_i, \sigma_{i+2}, \ldots$

$$\sum_{\sigma_i=\pm1} e^{K\sigma_i(\sigma_{i-1}+\sigma_{i+1})} = \begin{Bmatrix} 2\cosh 2K & \sigma_{i-1}\sigma_{i+1} = +1 \\ 2 & \sigma_{i-1}\sigma_{i+1} = -1 \end{Bmatrix} = e^{2g+K'\sigma_{i-1}\sigma_{i+1}}$$

$$\to Z(K,N) = Z\left(K' = \frac{1}{2}\ln\cosh 2K, \frac{N}{2}\right)$$



ℓ decimations: $N^{(\ell)} = N/2^\ell$, $a^{(\ell)} = 2^\ell a$,

RG recursion: $K^{(\ell)} = \frac{1}{2}\ln\cosh 2K^{(\ell-1)}$

Fixed points $\to$ phases, phase transition:

- $K^* = 0$ stable $\to T = \infty$, disordered
- $K^* = \infty$ unstable $\to T = 0$, ordered

$T \to 0$: expand $K'^{-1} \approx K^{-1}\left(1 + \frac{\ln 2}{2K}\right) \to \frac{dK^{-1}(\ell)}{d\ell} \approx \frac{\ln 2}{2} K^{-1}(\ell)^2$

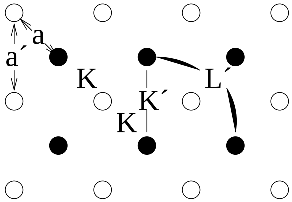
Correlation length: $K\left(\ell = \frac{\ln(2\xi/a)}{\ln 2}\right) \approx 0 \to \xi(T) \approx \frac{a}{2}\, e^{2J/k_\mathrm{B}T}$

Real-space RG for the Ising square lattice

$$-\beta H(\{\sigma_i\}) = K \sum_{n.n.\,(i,j)} \sigma_i \sigma_j$$

$$\rightarrow \; -\beta H'(\{\sigma_i\}) = A' + K' \sum_{n.n.\,(i,j)} \sigma_i \sigma_j$$

$$+L' \sum_{n.n.n.\,(i,j)} \sigma_i \sigma_j + M' \sum_{\square\,(i,j,k,l)} \sigma_i \sigma_j \sigma_k \sigma_l$$



$$2\cosh K(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) =$$
$$= e^{A' + \frac{1}{2}K'(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1) + L'(\sigma_1\sigma_3 + \sigma_2\sigma_4) + M'\sigma_1\sigma_2\sigma_3\sigma_4}$$

List possible configurations for four nearest neighbors of given spin:

σ_1	σ_2	σ_3	σ_4	
+	+	+	+	$\rightarrow \; 2\cosh 4K = e^{A' + 2K' + 2L' + M'}$
+	+	+	−	$\rightarrow \; 2\cosh 2K = e^{A' - M'}$
+	+	−	−	$\rightarrow \; 2 = e^{A' - 2L' + M'}$
+	−	+	−	$\rightarrow \; 2 = e^{A' - 2K' + 2L' + M'}$

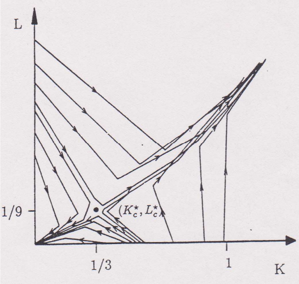
RG recursion relations

$$K' = \frac{1}{4}\ln\cosh 4K \approx 2K^2 + O(K^4)$$

$$L' = \frac{K'}{2} = \frac{1}{8}\ln\cosh 4K \approx K^2$$

$$A' = L' + \frac{1}{2}\ln 4\cosh 2K \approx \ln 2 + 2K^2$$

$$M' = A' - \ln 2\cosh 2K \approx 0 \ \rightarrow \ \text{drop}$$



$$\rightarrow \ a^{(\ell)} = 2^{\ell/2}a,\ K^{(\ell)} \approx 2\left[K^{(\ell-1)}\right]^2 + L^{(\ell-1)},\ L^{(\ell)} \approx \left[K^{(\ell-1)}\right]^2$$

- $K^* = 0 = L^*$ stable $\rightarrow T = \infty$: disordered paramagnet
- $K^* = \infty = L^*$ stable $\rightarrow T = 0$: ordered ferromagnet
- $K_c^* = 1/3,\ L_c^* = 1/9$ unstable: *critical fixed point*

Linearize RG flow: $\begin{pmatrix} \delta K^{(\ell)} = K^{(\ell)} - K_c^* \\ \delta L^{(\ell)} = L^{(\ell)} - L_c^* \end{pmatrix} = \begin{pmatrix} 4/3 & 1 \\ 2/3 & 0 \end{pmatrix} \begin{pmatrix} \delta K^{(\ell-1)} \\ \delta L^{(\ell-1)} \end{pmatrix}$

with eigenvalues $\lambda_{1/2} = \frac{1}{3}\left(2 \pm \sqrt{10}\right)$ and associated eigenvectors:

$$\rightarrow \ \begin{pmatrix} K^{(\ell)} \\ L^{(\ell)} \end{pmatrix} \approx \begin{pmatrix} 1/3 \\ 1/9 \end{pmatrix} + c_1\,\lambda_1^\ell \begin{pmatrix} 3 \\ \sqrt{10} - 2 \end{pmatrix} + c_2\,\lambda_2^\ell \begin{pmatrix} -3 \\ \sqrt{10} + 2 \end{pmatrix}$$

Critical point scaling

Utilize linearized RG flow to analyze critical behavior:

- $\lambda_1 > 1 \rightarrow$ *relevant* direction; $|\lambda_2| < 1 \rightarrow$ *irrelevant* direction
- *Critical line*: $c_1 = 0$, set $L_c = 0$ (n.n. Ising model), $\ell = 0$

$$\begin{pmatrix} K_c \\ 0 \end{pmatrix} \approx \begin{pmatrix} 1/3 \\ 1/9 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ \sqrt{10}+2 \end{pmatrix} \;\rightarrow\; c_2 = \frac{-1}{9(\sqrt{10}+2)}$$

$\rightarrow\; K_c \approx 0.3979$; mean-field: $K_c = 0.25$; exact: $K_c = 0.4406$

- Relevant eigenvalue determines *critical exponent*:

$\ell \gg 1$: $\lambda_2^\ell \rightarrow 0$, $\delta K^{(\ell)} \approx e^{\ell \ln \lambda_1}(K - K_c)$

correlations: $\xi^{(\ell)} = 2^{-\ell/2}\xi \;\rightarrow\; \xi = \xi^{(\ell)} \left| \frac{\delta K^{(\ell)}}{K - K_c} \right|^{\ln 2 / 2 \ln \lambda_1}$

$$\xi^{(\ell)} \approx a \;\rightarrow\; \xi(T) \propto |T - T_c|^{-\nu}\,,\; \nu = \frac{\ln 2}{2 \ln \frac{2+\sqrt{10}}{3}} \approx 0.6385$$

compare mean-field theory: $\nu = \frac{1}{2}$; exact (L. Onsager): $\nu = 1$

Real-space renormalization group approach:

- difficult to improve systematically, no small parameter
- successful applications to *critical disordered systems*

General mean-field theory: Landau expansion

Expand free energy (density) in terms of order parameter (scalar field) ϕ near a *continuous (second-order) phase transition* at T_c:

$$f(\phi) = \frac{r}{2}\,\phi^2 + \frac{u}{4!}\,\phi^4 + \ldots - h\,\phi$$

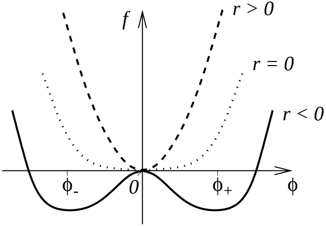
$r = a(T - T_c)$, $u > 0$; conjugate field h breaks $Z(2)$ symmetry $\phi \to -\phi$

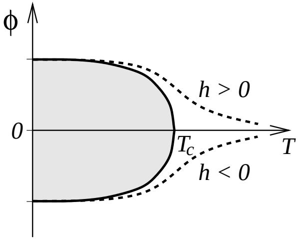
$f'(\phi) = 0 \;\to\;$ *equation of state*:

$$h(T,\phi) = r(T)\,\phi + \frac{u}{6}\,\phi^3$$

Stability: $f''(\phi) = r + \frac{u}{2}\,\phi^2 > 0$

- *Critical isotherm* at $T = T_c$: $h(T_c,\phi) = \frac{u}{6}\,\phi^3$
- *Spontaneous order parameter* for $r < 0$: $\phi_\pm = \pm(6|r|/u)^{1/2}$



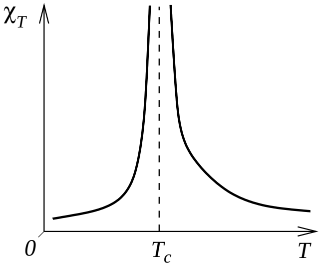


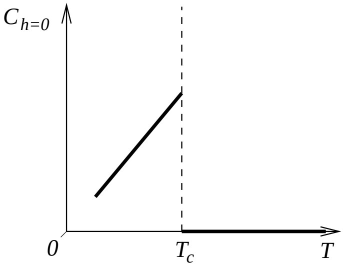
Thermodynamic singularities at critical point

- Isothermal order parameter *susceptibility*:

$$V\chi_T^{-1} = \left(\frac{\partial h}{\partial \phi}\right)_T = r + \frac{u}{2}\,\phi^2 \;\rightarrow\; \frac{\chi_T}{V} = \begin{cases} 1/r^{\color{red}1} & r > 0 \\ 1/2|r|^{\color{red}1} & r < 0 \end{cases}$$

$\rightarrow$ *divergence* at T_c, *amplitude ratio* 2





- *Free energy* and *specific heat* vanish for $T \geq T_c$; for $T < T_c$:

$$f(\phi_\pm) = \frac{r}{4}\,\phi_\pm^2 = -\frac{3r^2}{2u}\,,\;\; C_{h=0} = -VT\left(\frac{\partial^2 f}{\partial T^2}\right)_{h=0} = VT\,\frac{3a^2}{u}$$

$\rightarrow$ *discontinuity* at T_c

Scaling hypothesis for free energy

Postulate: (sing.) free energy generalized *homogeneous function*:

$$f_{\text{sing}}(\tau, h) = |\tau|^{2-\alpha}\, \hat{f}_\pm \left(\frac{h}{|\tau|^\Delta} \right),\ \tau = \frac{T - T_c}{T_c}$$

two-parameter scaling, with *scaling functions* $\hat{f}_\pm$, $\hat{f}_\pm(0) = \text{const.}$

Landau theory: *critical exponents* $\alpha = 0$, $\Delta = \frac{3}{2}$

- *Specific heat*:

$$C_{h=0} = -\frac{VT}{T_c^2} \left(\frac{\partial^2 f_{\text{sing}}}{\partial \tau^2} \right)_{h=0} = C_\pm\, |\tau|^{-\alpha}$$

- *Equation of state*:

$$\phi(\tau, h) = -\left(\frac{\partial f_{\text{sing}}}{\partial h} \right)_\tau = -|\tau|^{2-\alpha-\Delta}\ \hat{f}'_\pm \left(\frac{h}{|\tau|^\Delta} \right)$$

- *Coexistence line* $h = 0$, $\tau < 0$:

$$\phi(\tau, 0) = -|\tau|^{2-\alpha-\Delta}\ \hat{f}'_-(0) \propto |\tau|^\beta,\ \beta = 2 - \alpha - \Delta$$

Scaling relations

- *Critical isotherm*: τ dependence in $\hat{f}'_{\pm}$ must cancel prefactor, as $x \to \infty$: $\hat{f}'_{\pm}(x) \propto x^{(2-\alpha-\Delta)/\Delta}$

$$\to \; \phi(0,h) \propto h^{(2-\alpha-\Delta)/\Delta} = h^{1/\delta}\,, \; \delta = \frac{\Delta}{\beta}$$

- Isothermal *susceptibility*:

$$\frac{\chi_\tau}{V} = \left(\frac{\partial \phi}{\partial h}\right)_{\tau,\, h=0} = \chi_{\pm}\, |\tau|^{-\gamma}\,, \; \gamma = \alpha + 2(\Delta - 1)$$

Eliminate Δ $\to$ *scaling relations*:

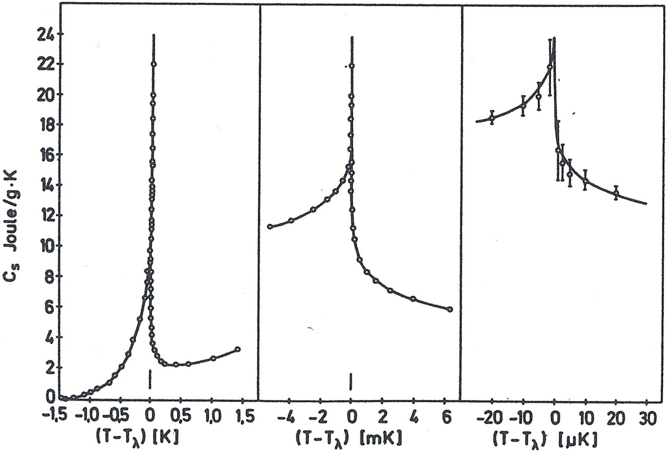
$$\Delta = \beta\,\delta\,, \; \alpha + \beta(1+\delta) = 2 = \alpha + 2\beta + \gamma\,, \; \gamma = \beta(\delta - 1)$$

$\to$ only *two independent* (static) critical exponents

Mean-field: $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = 1$, $\delta = 3$, $\Delta = \frac{3}{2}$ (dim. analysis)

Experimental exponent values different, but still *universal*: depend only on symmetry, dimension ..., *not* microscopic details

Thermodynamic self-similarity in the vicinity of T_c



Temperature dependence of the *specific heat* near the *normal- to superfluid transition* of He 4, shown in successively reduced scales

From: M.J. Buckingham and W.M. Fairbank, in: Progress in low temperature physics, *Vol. III, ed. C.J. Gorter, 80–112, North-Holland (Amsterdam, 1961).*

Selected literature:

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Some exercises

1. *Ising model in one dimension.*
(a) Evaluate the partition sum $Z(T, N)$ for a one-dimensional open *Ising chain* with N spins $\sigma_i = \pm 1$,
$$H_N(\{\sigma_i\}) = -\sum_{i=1}^{N-1} J_i \sigma_i \sigma_{i+1} \ .$$
Hint: Derive the recursion $Z(T, N) = 2Z(T, N-1)\cosh(J_{N-1}/k_{\mathrm{B}}T)$.
(b) Compute the two-spin correlation function
$$G_{i,n} = \langle \sigma_i \sigma_{i+n} \rangle = \prod_{k=i}^{i+n-1} \tanh(J_k/k_{\mathrm{B}}T) \ .$$
For uniform $J_i = J$, define ξ via $G_n = e^{-n/\xi}$. Show that this correlation length diverges *exponentially* as $T \to 0$.
(c) Calculate the isothermal magnetic susceptibility
$$\chi_T = \frac{1}{k_{\mathrm{B}}T} \sum_{i,j=1}^{N} \langle \sigma_i \sigma_j \rangle = \frac{N}{k_{\mathrm{B}}T} \left(\frac{1+\alpha}{1-\alpha} - \frac{2\alpha}{N} \, \frac{1-\alpha^N}{(1-\alpha)^2} \right)$$
for uniform exchange couplings, where $\alpha = \tanh(J/k_{\mathrm{B}}T)$. Show that $\chi_T/N \propto \xi$ as $T \to 0$ and $N \to \infty$.
Hint: Count the number of terms with $|i-j| = n$ in the sums.

2. *Landau theory for the ϕ^6 model.*
Consider the following*effective free energy*, where $r = a(T - T_0)$, $v > 0$, and h denotes an external field:
$$f(\phi) = \frac{r}{2}\,\phi^2 + \frac{u}{4!}\,\phi^4 + \frac{v}{6!}\,\phi^6 - h\,\phi \ .$$
(a) Show that for $u > 0$, there is a *second-order* phase transition at $h = 0$ and $T = T_0$ with the usual mean-field critical exponents β, γ, δ, and α. Why can v be neglected near the critical point ?
(b) Compute $\beta_t, \gamma_t, \delta_t$, and α_t at the *tricritical point* $u = 0$.
(c) Now assume $u = -|u| < 0$ and $h = 0$. Show that there is a *first-order* transition at $r_d = 5u^2/8v$, and calculate the jump in the order parameter and the associated free-energy barrier.
(d) For $h \neq 0$ and $u < 0$, find parametric equations $r_c(|u|, v)$ and $h_c(|u|, v)$ for two additional *second-order* transition lines, with all three continuous phase boundaries merging at the tricritical point $u = 0$, $h = 0$.

Lecture 2: Momentum Shell Renormalization Grou

Landau–Ginzburg–Wilson Hamiltonian

Coarse-grained Hamiltonian, *order parameter field* $S(x)$:

$$\mathcal{H}[S] = \int d^d x \left[\frac{r}{2}\, S(x)^2 + \frac{1}{2}\, [\nabla S(x)]^2 + \frac{u}{4!}\, S(x)^4 - h(x)\, S(x) \right]$$

$r = a(T - T_c^0)$, $u > 0$, $h(x)$ local external field;
gradient term $\sim [\nabla S(x)]^2$ suppresses spatial inhomogeneities

Probability density for configuration $S(x)$: *Boltzmann factor*

$$\mathcal{P}_s[S] = \exp(-\mathcal{H}[S]/k_{\mathrm{B}} T)/\mathcal{Z}[h]$$

canonical *partition function* and moments $\to$ functional integrals:

$$\mathcal{Z}[h] = \int \mathcal{D}[S]\; e^{-\mathcal{H}[S]/k_{\mathrm{B}} T}\,,\ \phi = \langle S(x) \rangle = \int \mathcal{D}[S]\; S(x)\, \mathcal{P}_s[S]$$

- Integral measure: discretize $x \to x_i$, $\to$ $\mathcal{D}[S] = \prod_i dS(x_i)$
- or employ Fourier transform: $S(x) = \int \frac{d^d q}{(2\pi)^d}\, S(q)\, e^{iq\cdot x}$

$$\to\ \mathcal{D}[S] = \prod_q \frac{dS(q)}{V} = \prod_{q,q_1>0} \frac{d\,\mathrm{Re}\, S(q)\ d\,\mathrm{Im}\, S(q)}{V}$$

Landau–Ginzburg approximation

Most likely configuration $\rightarrow$ *Ginzburg–Landau equation*:

$$0 = \frac{\delta \mathcal{H}[S]}{\delta S(x)} = \left[r - \nabla^2 + \frac{u}{6} S(x)^2 \right] S(x) - h(x)$$

Linearize $S(x) = \phi + \delta S(x) \ \rightarrow \ \delta h(x) \approx \left(r - \nabla^2 + \frac{u}{2} \phi^2 \right) \delta S(x)$

Fourier transform $\rightarrow$ *Ornstein–Zernicke susceptibility*:

$$\chi_0(q) = \frac{1}{r + \frac{u}{2} \phi^2 + q^2} = \frac{1}{\xi^{-2} + q^2} \, , \ \xi = \begin{cases} 1/r^{1/2} & r > 0 \\ 1/|2r|^{1/2} & r < 0 \end{cases}$$

Zero-field two-point *correlation function* (cumulant):

$$C(x - x') = \langle S(x) \, S(x') \rangle - \langle S(x) \rangle^2 = (k_{\rm B} T)^2 \frac{\delta^2 \ln \mathcal{Z}[h]}{\delta h(x) \, \delta h(x')} \bigg|_{h=0}$$

Fourier transform $C(x) = \int \frac{d^d q}{(2\pi)^d} \, C(q) \, e^{iq \cdot x}$

$\rightarrow$ *fluctuation–response theorem*: $C(q) = k_{\rm B} T \, \chi(q)$

Scaling hypothesis for correlation function

Scaling ansatz, defines *Fisher exponent* η and *correlation length* ξ:

$$C(\tau, q) = |q|^{-2+\eta}\,\hat{C}_{\pm}(q\xi)\,,\ \xi = \xi_{\pm}\,|\tau|^{-\nu}$$

- Thermodynamic *susceptibility*:

$$\chi(\tau, q = 0) \propto \xi^{2-\eta} \propto |\tau|^{-\nu(2-\eta)} = |\tau|^{-\gamma}\,,\ \gamma = \nu(2-\eta)$$

- Spatial *correlations* for $x \to \infty$:

$$C(\tau, x) = |x|^{-(d-2+\eta)}\,\widetilde{C}_{\pm}(x/\xi) \propto \xi^{-(d-2+\eta)} \propto |\tau|^{\nu(d-2+\eta)}$$

$\langle S(x)S(0)\rangle \to \langle S\rangle^2 = \phi^2 \propto (-\tau)^{2\beta} \ \to$ *hyperscaling relations*:

$$\beta = \frac{\nu}{2}\,(d-2+\eta)\,,\ 2-\alpha = d\nu$$

Mean-field values: $\nu = \frac{1}{2}$, $\eta = 0$ (Ornstein–Zernicke)

Diverging spatial correlations induce thermodynamic singularities !

Gaussian approximation

High-temperature phase, $T > T_c$: neglect nonlinear contributions:

$$\mathcal{H}_0[S] = \int \frac{d^d q}{(2\pi)^d} \left[\frac{1}{2} \left(r + q^2 \right) |S(q)|^2 - h(q) S(-q) \right]$$

Linear transformation $\widetilde{S}(q) = S(q) - \frac{h(q)}{r+q^2}$, $\int_q \ldots = \int \frac{d^d q}{(2\pi)^d}$ and Gaussian integral:

$$\mathcal{Z}_0[h] = \int \mathcal{D}[S] \; \exp(-\mathcal{H}_0[S]/k_{\mathrm{B}} T) =$$

$$= \exp\left(\frac{1}{2 k_{\mathrm{B}} T} \int_q \frac{|h(q)|^2}{r + q^2} \right) \int \mathcal{D}[\widetilde{S}] \, \exp\left(- \int_q \frac{r + q^2}{2 k_{\mathrm{B}} T} \, |\widetilde{S}(q)|^2 \right)$$

$$\to \; \left\langle S(q) S(q') \right\rangle_0 = \frac{(k_{\mathrm{B}} T)^2}{\mathcal{Z}_0[h]} \frac{(2\pi)^{2d} \, \delta^2 \mathcal{Z}_0[h]}{\delta h(-q) \, \delta h(-q')} \bigg|_{h=0}$$

$$= C_0(q) \, (2\pi)^d \delta(q + q') \, , \;\; C_0(q) = \frac{k_{\mathrm{B}} T}{r + q^2}$$

Gaussian model: free energy and specific heat

$$F_0[h] = -k_\mathrm{B} T \ln \mathcal{Z}_0[h] = -\frac{1}{2} \int_q \left(\frac{|h(q)|^2}{r + q^2} + k_\mathrm{B} T \, V \, \ln \frac{2\pi \, k_\mathrm{B} T}{r + q^2} \right) \; .$$

Leading singularity in *specific heat*:

$$C_{h=0} = -T \left(\frac{\partial^2 F_0}{\partial T^2} \right)_{h=0} \approx \frac{V k_\mathrm{B} \, (a T_c^0)^2}{2} \int_q \frac{1}{(r + q^2)^2} \; .$$

- $d > 4$: integral UV-divergent; regularized by cutoff Λ (Brillouin zone boundary) $\rightarrow$ $\alpha = 0$ as in mean-field theory
- $d = d_c = 4$: integral diverges logarithmically:

$$\int_0^{\Lambda\xi} \frac{k^3}{(1 + k^2)^2} \, dk \sim \ln(\Lambda\xi)$$

- $d < 4$: with $k = q/\sqrt{r} = q\xi$, surface area $K_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$:

$$C_\mathrm{sing} \approx \frac{V k_\mathrm{B} (a T_c^0)^2 \, \xi^{4-d}}{2^d \pi^{d/2} \, \Gamma(d/2)} \int_0^\infty \frac{k^{d-1}}{(1 + k^2)^2} \, dk \propto |T - T_c^0|^{-\frac{4-d}{2}}$$

$\rightarrow$ diverges; *stronger singularity* than in mean-field theory

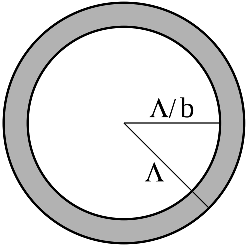
Renormalization group program in statistical physics

- Goal: *critical* (IR) singularities; perturbatively inaccessible.
- Exploit fundamental new symmetry:
  divergent correlation length induces *scale invariance*.
- Analyze theory in ultraviolet regime: integrate out short-wavelength modes / renormalize UV divergences.
- Rescale onto original Hamiltonian, obtain recursion relations for effective, now scale-dependent *running couplings*.
- Under such RG transformations:
  - $\rightarrow$ *Relevant* parameters grow: set to 0: *critical surface*.
  - $\rightarrow$ Certain couplings approach *IR-stable fixed point*:
    scale-invariant behavior.
  - $\rightarrow$ *Irrelevant* couplings vanish: origin of *universality*.
- Scale invariance at critical fixed point $\rightarrow$ infer correct IR scaling behavior from (approximative) analysis of UV regime $\rightarrow$ *derivation of scaling laws*.
- Dimensional expansion: $\epsilon = d_c - d$ small parameter, permits perturbational treatment $\rightarrow$ *computation of critical exponents*.

Wilson's momentum shell renormalization group

RG transformation steps:

(1) Carry out the partition integral over all Fourier components $S(q)$ with wave vectors $\Lambda/b \leq |q| \leq \Lambda$, where $b > 1$: *eliminates short-wavelength modes*



(2) *Scale transformation* with the same scale parameter $b > 1$:
$x \to x' = x/b$, $q \to q' = b\,q$

Accordingly, we also need to *rescale the fields*:

$$S(x) \to S'(x') = b^{\zeta} S(x)\,,\ S(q) \to S'(q') = b^{\zeta - d} S(q)$$

Proper choice of ζ $\to$ rescaled Hamiltonian assumes original form
$\to$ *scale-dependent effective couplings*, analyze dependence on b

Notice *semi-group* character: RG transformation has no inverse

Momentum shell RG: Gaussian model

$$\mathcal{H}_0[S_<] + \mathcal{H}_0[S_>] = \left(\int_q^< + \int_q^> \right) \left[\frac{r+q^2}{2} \, |S(q)|^2 - h(q)\, S(-q) \right]$$

where $\int_q^< \ldots = \int_{|q|<\Lambda/b} \frac{d^d q}{(2\pi)^d} \ldots$, $\int_q^> \ldots = \int_{\Lambda/b \le |q| \le \Lambda} \frac{d^d q}{(2\pi)^d} \ldots$

Choose $\zeta = \frac{d-2}{2} \;\to\; r \to r' = b^2 r$,

$$h(q) \to h'(q') = b^{-\zeta} h(q)\,, \; h(x) \to h'(x') = b^{d-\zeta} h(x)$$

r, h both *relevant* $\to$ *critical surface*: $r = 0 = h$

- *Correlation length*: $\xi \to \xi' = \xi/b \;\to\; \xi \propto r^{-1/2}$: $\nu = \frac{1}{2}$
- *Correlation function*: $C'(x') = b^{2\zeta}\, C(x) \;\to\; \eta = 0$

Add other couplings:

- $c \int d^d x\, (\nabla^2 S)^2$: $c \to c' = b^{d-4-2\zeta} c = b^{-2} c$, *irrelevant*
- $u \int d^d x\, S(x)^4$: $u \to u' = b^{d-4\zeta} u = b^{4-d} u$; *relevant* for $d < 4$, (dangerously) *irrelevant* for $d > 4$, *marginal* at $d = d_c = 4$
- $v \int d^d x\, S(x)^6$: $v \to v' = b^{6-2d} v$, marginal for $d = 3$; *irrelevant* near $d_c = 4$: $v' = b^{-2} v$

Momentum shell RG: general structure

General choice: $\zeta = \frac{d-2+\eta}{2} \;\rightarrow\; \tau' = b^{1/\nu}\tau,\ h' = b^{(d+2-\eta)/2}h$

- Only *two relevant* parameters τ and h
- Few *marginal* couplings $u_i \rightarrow u_i' = u_i^* + b^{-x_i}u_i,\ x_i > 0$
- Other couplings *irrelevant*: $v_i \rightarrow v_i' = b^{-y_i}v_i,\ y_i > 0$

After single RG transformation:

$$f_{\rm sing}(\tau, h, \{u_i\}, \{v_i\}) = b^{-d} f_{\rm sing}\left(b^{1/\nu}\tau, b^{d-\zeta}h, \left\{u_i^* + \frac{u_i}{b^{x_i}}\right\}, \left\{\frac{v_i}{b^{y_i}}\right\}\right)$$

After sufficiently many $\ell \gg 1$ RG transformations:

$$f_{\rm sing}(\tau, h, \{u_i\}, \{v_i\}) = b^{-\ell d} f_{\rm sing}\left(b^{\ell/\nu}\tau, b^{\ell(d+2-\eta)/2}h, \{u_i^*\}, \{0\}\right)$$

Choose *matching condition* $b^\ell|\tau|^\nu = 1 \;\rightarrow\;$ *scaling form*:

$$f_{\rm sing}(\tau, h) = |\tau|^{d\nu}\, \hat{f}_\pm\left(h/|\tau|^{\nu(d+2-\eta)/2}\right)$$

Correlation function scaling law: use $b^\ell = \xi/\xi_\pm \;\rightarrow$

$$C(\tau, x, \{u_i\}, \{v_i\}) = b^{-2\ell\zeta}\, C\left(b^{\ell/\nu}\tau, \frac{x}{b^\ell}, \{u_i^*\}, \{0\}\right) \rightarrow \frac{\widetilde{C}_\pm(x/\xi)}{|x|^{d-2+\eta}}$$

Perturbation expansion

Nonlinear interaction term:

$$\mathcal{H}_{\rm int}[S] = \frac{u}{4!} \int_{|q_i|<\Lambda} S(q_1) S(q_2) S(q_3) S(-q_1 - q_2 - q_3)$$

Rewrite *partition function* and N-point *correlation functions*:

$$\mathcal{Z}[h] = \mathcal{Z}_0[h] \left\langle e^{-\mathcal{H}_{\rm int}[S]} \right\rangle_0 , \ \left\langle \prod_i S(q_i) \right\rangle = \frac{\left\langle \prod_i S(q_i)\, e^{-\mathcal{H}_{\rm int}[S]} \right\rangle_0}{\left\langle e^{-\mathcal{H}_{\rm int}[S]} \right\rangle_0}$$

contraction: $\underbrace{S(q)S}(q') = \langle S(q) S(q') \rangle_0 = C_0(q)\,(2\pi)^d \delta(q + q')$

$\rightarrow$ *Wick's theorem*:

$$\langle S(q_1) S(q_2) \dots S(q_{N-1}) S(q_N) \rangle_0 = \\ = \sum_{\substack{\text{permutations} \\ i_1(1) \dots i_N(N)}} \underbrace{S(q_{i_1(1)}) S}(q_{i_2(2)}) \dots \underbrace{S(q_{i_{N-1}(N-1)}) S}(q_{i_N(N)})$$

$\rightarrow$ compute all expectation values in the *Gaussian ensemble*

First-order correction to two-point function

Consider $\langle S(q)S(q')\rangle = C(q)\,(2\pi)^d\delta(q+q')$ for $h=0$; to $O(u)$:

$$\left\langle S(q)S(q')\left[1-\frac{u}{4!}\int_{|q_i|<\Lambda} S(q_1)S(q_2)S(q_3)S(-q_1-q_2-q_3)\right]\right\rangle_0$$

- Contractions of external legs $\underbrace{S(q)S}(q')$:
  terms cancel with denominator, leaving $\langle S(q)S(q')\rangle_0$
- The remaining twelve contributions are of the form
  $\int_{|q_i|<\Lambda} \underbrace{S(q)S}(q_1)\,\underbrace{S(q_2)S}(q_3)\,\underbrace{S(-q_1-q_2-q_3)S}(q') =$
  $= C_0(q)^2\,(2\pi)^d\delta(q+q')\int_{|p|<\Lambda} C_0(p)$

$$\rightarrow\ C(q) = C_0(q)\left[1-\frac{u}{2}\,C_0(q)\int_{|p|<\Lambda} C_0(p) + O(u^2)\right]$$

re-interpret as *first-order self-energy in Dyson's equation*:

$$C(q)^{-1} = r + q^2 + \frac{u}{2}\int_{|p|<\Lambda}\frac{1}{r+p^2} + O(u^2)$$

Notice: to first order in u, there is *only "mass" renormalization*,
no change in momentum dependence of $C(q)$

Wilson RG procedure: first-order recursion relations

Split field variables in outer ($S_>$) / inner ($S_<$) momentum shell:

- simply re-exponentiate terms $\sim u \int S_<^4 \, e^{-\mathcal{H}_0[S]}$
- contributions such as $u \int S_<^3 \, S_> e^{-\mathcal{H}_0[S]}$ *vanish*
- terms $\sim u \int S_>^4 \, e^{-\mathcal{H}_0[S]} \to \mathrm{const.}$, contribute to *free energy*
- contributions $\sim u \int S_<^2 \, S_>^2 \, e^{-\mathcal{H}_0}$: Gaussian integral over $S_>$

With $S_d = K_d/(2\pi)^d = 1/2^{d-1}\pi^{d/2}\Gamma(d/2)$ and $\eta = 0$ to $O(u)$:

$$r' = b^2\left[r + \frac{u}{2}\,A(r)\right] = b^2\left[r + \frac{u}{2}\,S_d \int_{\Lambda/b}^{\Lambda} \frac{p^{d-1}}{r + p^2}\,dp\right]$$

$$u' = b^{4-d}u\left[1 - \frac{3u}{2}\,B(r)\right] = b^{4-d}u\left[1 - \frac{3u}{2} S_d \int_{\Lambda/b}^{\Lambda} \frac{p^{d-1}\,dp}{(r + p^2)^2}\right]$$

- $r \gg 1$: fluctuation contributions disappear, Gaussian theory
- $r \ll 1$: expand

$$A(r) = S_d \Lambda^{d-2} \frac{1 - b^{2-d}}{d-2} - r\,S_d \Lambda^{d-4}\,\frac{1 - b^{4-d}}{d-4} + O(r^2)$$

$$B(r) = S_d \Lambda^{d-4}\,\frac{1 - b^{4-d}}{d-4} + O(r)$$

Differential RG flow, fixed points, dimensional expansion

Differential RG flow: set $b = e^{\delta\ell}$ with $\delta\ell \to 0$:

$$\frac{d\tilde{r}(\ell)}{d\ell} = 2\tilde{r}(\ell) + \frac{\tilde{u}(\ell)}{2} S_d \Lambda^{d-2} - \frac{\tilde{r}(\ell)\tilde{u}(\ell)}{2} S_d \Lambda^{d-4} + O(\tilde{u}\tilde{r}^2, \tilde{u}^2)$$

$$\frac{d\tilde{u}(\ell)}{d\ell} = (4-d)\tilde{u}(\ell) - \frac{3}{2}\,\tilde{u}(\ell)^2 S_d \Lambda^{d-4} + O(\tilde{u}\tilde{r}, \tilde{u}^2)$$

Renormalization group *fixed points*: $d\tilde{r}(\ell)/d\ell = 0 = d\tilde{u}(\ell)/d\ell$

- *Gauss*: $u_0^* = 0 \;\leftrightarrow\;$ *Ising*: $u_{\rm I}^* S_d = \frac{2}{3}\,(4-d)\Lambda^{4-d}$, $d < 4$
- *Linearize* $\delta\tilde{u}(\ell) = \tilde{u}(\ell) - u_{\rm I}^*$: $\frac{d}{d\ell}\,\delta\tilde{u}(\ell) \approx (d-4)\delta\tilde{u}(\ell)$
  $\to$ u_0^* stable for $d > 4$, $u_{\rm I}^*$ stable for $d < 4$
- *Small expansion parameter*: $\epsilon = 4 - d = d_c - d$
  $u_{\rm I}^*$ emerges continuously from $u_0^* = 0$
- Insert: $r_{\rm I}^* = -\frac{1}{4}\,u_{\rm I}^* S_d \Lambda^{d-2} = -\frac{1}{6}\,\epsilon\Lambda^2$: non-universal, describes *fluctuation-induced downward T_c-shift*
- RG procedure *generates new terms* $\sim S^6$, $\nabla^2 S^4$, etc; to $O(\epsilon^3)$, feedback into recursion relations can be neglected

Critical exponents

Deviation from true T_c: $\tau = r - r_{\rm I}^* \propto T - T_c$
Recursion relation for this (relevant) *running coupling*:

$$\frac{d\tilde{\tau}(\ell)}{d\ell} = \tilde{\tau}(\ell) \left[2 - \frac{\tilde{u}(\ell)}{2} \, S_d \Lambda^{d-4} \right]$$

Solve near Ising fixed point: $\tilde{\tau}(\ell) = \tilde{\tau}(0) \exp\left[\left(2 - \frac{\epsilon}{3}\right) \ell\right]$
Compare with $\widetilde{\xi}(\ell) = \xi(0)\, e^{-\ell} \;\rightarrow\; \nu^{-1} = 2 - \frac{\epsilon}{3}$
Consistently to order $\epsilon = 4 - d$:

$$\nu = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2)\,, \;\; \eta = 0 + O(\epsilon^2)$$

Note at $d = d_c = 4$: $\tilde{u}(\ell) = \tilde{u}(0)/[1 + 3\,\tilde{u}(0)\,\ell/16\pi^2]$

$\rightarrow$ *logarithmic corrections* to mean-field exponents

Renormalization group procedure:

- Derive scaling laws.
- Two relevant couplings $\rightarrow$ independent critical exponents.
- Compute scaling exponents via power series in $\epsilon = d_c - d$.

Selected literature:

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- U.C. Täuber, *Critical dynamics — A field theory approach to equilibrium and non-equilibrium scaling behavior*, Cambridge University Press (Cambridge, 2014), Chap. 1.
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Some exercises

1. *Gaussian approximation for the Heisenberg model.*
Isotropic magnets with continuous rotational spin symmetry are described by the *Heisenberg model*. The corresponding effective Landau–Ginzburg–Wilson Hamiltonian reads

$$\mathcal{H}[S] = \int d^d x \sum_{\alpha=1}^{n} \left(\frac{r}{2} [S^\alpha(x)]^2 + \frac{1}{2} [\nabla S^\alpha(x)]^2 + \frac{u}{4!} \sum_{\beta=1}^{n} [S^\alpha(x)]^2 [S^\beta(x)]^2 - h^\alpha(x) S^\alpha(x) \right) ,$$

where $S^\alpha(x)$ is an n-component order parameter vector field.
(a) Determine the two-point correlation functions in the high- and low-temperature phases in harmonic (Gaussian) approximation.
Notice: For $T < T_c$, it is useful to expand about the spontaneous magnetization: e.g., $S^\alpha(x) = \pi^\alpha(x)$ for $\alpha = 1, \dots, n-1$, and $S^n(x) = \phi + \sigma(x)$; then $\langle \pi^\alpha \rangle = 0 = \langle \sigma \rangle$. The components along and perpendicular to ϕ must be carefully distinguished.
(b) For $d < d_c = 4$, compute the specific heat in Gaussian approximation on both sides of the phase transition, and show that $C_{h=0} = C_\pm |\tau|^{-(4-d)/2}$. Compute the *universal amplitude ratio* $C_+ / C_- = 2^{-d/2} n$.

2. *First-order recursion relations for the Heisenberg model.*
For the n-component *Heisenberg model* above, derive the renormalization group recursion relations

$$r' = b^2 \left[r + \frac{n+2}{6} \, u \, A(r) \right] , \quad u' = b^{4-d} u \left[1 - \frac{n+8}{6} \, u \, B(r) \right] .$$

Determine the associated RG fixed points and discuss their stability. Compute the critical exponent ν to first order in $\epsilon = 4 - d$.

3. *RG flow equations for the n-vector model with cubic anisotropy.*
The $O(n)$ rotational invariance of the Hamiltonian in the previous problems is broken by additional quartic terms with cubic symmetry,

$$\Delta \mathcal{H}[S] = \int d^d x \sum_{\alpha=1}^{n} \frac{v}{4!} [S^\alpha(x)]^4 .$$

(a) Derive the differential RG flow equations for the running couplings $\tilde{r}(\ell)$, $\tilde{u}(\ell)$, and $\tilde{v}(\ell)$.
(b) Discuss the ensuing RG fixed points and their stability as function of the number n of order parameter components, and compute the associated correlation length critical exponents ν.

Lecture 3: Field Theory Approach to Critical Phen

Perturbation expansion

O(n)-symmetric Hamiltonian (henceforth set $k_{\mathrm{B}}T = 1$):

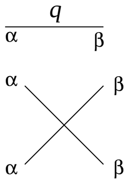
$$\mathcal{H}[S] = \int d^d x \sum_{\alpha=1}^{n} \left[\frac{r}{2} S^\alpha(x)^2 + \frac{1}{2} [\nabla S^\alpha(x)]^2 + \frac{u}{4!} \sum_{\beta=1}^{n} S^\alpha(x)^2 S^\beta(x)^2 \right]$$

Construct *perturbation expansion* for $\left\langle \prod_{ij} S^{\alpha_i} S^{\alpha_j} \right\rangle$:

$$\frac{\left\langle \prod_{ij} S^{\alpha_i} S^{\alpha_j} e^{-\mathcal{H}_{\rm int}[S]} \right\rangle_0}{\left\langle e^{-\mathcal{H}_{\rm int}[S]} \right\rangle_0} = \frac{\left\langle \prod_{ij} S^{\alpha_i} S^{\alpha_j} \sum_{l=0}^{\infty} \frac{(-\mathcal{H}_{\rm int}[S])^l}{l!} \right\rangle_0}{\left\langle \sum_{l=0}^{\infty} \frac{(-\mathcal{H}_{\rm int}[S])^l}{l!} \right\rangle_0}$$

Diagrammatic representation:

- *Propagator* $C_0(q) = \frac{1}{r+q^2}$
- *Vertex* $-\frac{u}{6}$



$= C_0(q)\, \delta^{\alpha\beta}$

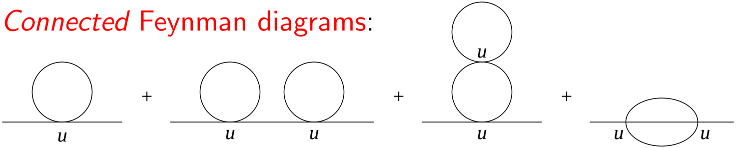
$= -\frac{u}{6}$

Generating functional for correlation functions (cumulants):

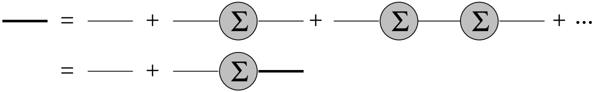
$$\mathcal{Z}[h] = \left\langle \exp \int d^d x \sum_\alpha h^\alpha S^\alpha \right\rangle, \quad \left\langle \prod_i S^{\alpha_i} \right\rangle_{(c)} = \prod_i \frac{\delta (\ln) \mathcal{Z}[h]}{\delta h^{\alpha_i}} \Big|_{h=0}$$

Vertex functions

Connected Feynman diagrams:



Dyson equation:



$\rightarrow$ propagator self-energy: $C(q)^{-1} = C_0(q)^{-1} - \Sigma(q)$

Generating functional for *vertex functions*, $\Phi^\alpha = \delta \ln \mathcal{Z}[h] / \delta h^\alpha$:

$$\Gamma[\Phi] = -\ln \mathcal{Z}[h] + \int d^d x \sum_\alpha h^\alpha \, \Phi^\alpha \, , \;\; \Gamma^{(N)}_{\{\alpha_i\}} = \prod_i^N \frac{\delta \Gamma[\Phi]}{\delta \Phi^{\alpha_i}} \Big|_{h=0}$$

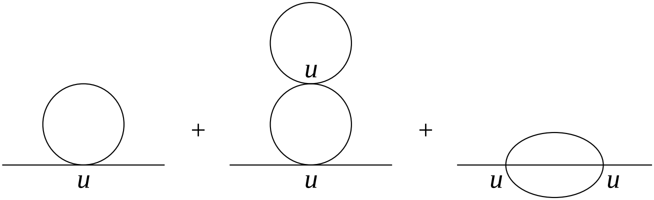
$$\rightarrow \;\; \Gamma^{(2)}(q) = C(q)^{-1} \, , \;\; \Big\langle \prod_{i=1}^4 S(q_i) \Big\rangle_c = -\prod_{i=1}^4 C(q_i) \, \Gamma^{(4)}(\{q_i\})$$

$\rightarrow$ *one-particle irreducible Feynman graphs*

Perturbation series in nonlinear coupling u $\leftrightarrow$ *loop expansion*

Explicit results

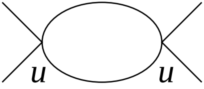
Two-point vertex function to two-loop order:



$$\Gamma^{(2)}(q) = r + q^2 + \frac{n+2}{6}\, u \int_k \frac{1}{r+k^2} - \left(\frac{n+2}{6}\, u\right)^2 \int_k \frac{1}{r+k^2} \int_{k'} \frac{1}{(r+k'^2)^2} - \frac{n+2}{18}\, u^2 \int_k \frac{1}{r+k^2} \int_{k'} \frac{1}{r+k'^2}\, \frac{1}{r+(q-k-k')^2}$$

four-point vertex function to one-loop order:

$$\Gamma^{(4)}(\{q_i = 0\}) = u - \frac{n+8}{6}\, u^2 \int_k \frac{1}{(r+k^2)^2}$$



Ultraviolet and infrared divergences

Fluctuation correction to four-point vertex function:

$$d < 4: \; u \int \frac{d^d k}{(2\pi)^d} \frac{1}{(r + k^2)^2} = \frac{u \, r^{-2+d/2}}{2^{d-1}\pi^{d/2}\Gamma(d/2)} \int_0^\infty \frac{x^{d-1}}{(1+x^2)^2} \, dx$$

effective coupling $u \, r^{(d-4)/2} \to \infty$ as $r \to 0$: *infrared divergence*

$\rightarrow$ fluctuation corrections singular, modify critical power laws

$$\int_0^\Lambda \frac{k^{d-1}}{(r+k^2)^2} \, dk \sim \left\{ \begin{array}{cc} \ln(\Lambda^2/r) & d = 4 \\ \Lambda^{d-4} & d > 4 \end{array} \right\} \to \infty \quad \text{as } \Lambda \to \infty$$

ultraviolet divergences for $d > d_c = 4$: *upper critical dimension*

Power counting in terms of arbitrary momentum scale μ:

- $[x] = \mu^{-1}$, $[q] = \mu$, $[S^\alpha(x)] = \mu^{-1+d/2}$;
- $[r] = \mu^2 \to$ *relevant*, $[u] = \mu^{4-d}$ *marginal* at $d_c = 4$
- only divergent vertex functions: $\Gamma^{(2)}(q)$, $\Gamma^{(4)}(\{q_i = 0\})$
- field dimensionless at *lower critical dimension* $d_{lc} = 2$

Dimension regimes and dimensional regularization

dimension interval	perturbation series	$O(n)$-symmetric Φ^4 field theory	critical behavior
$d \leq d_{lc} = 2$	IR-singular UV-convergent	ill-defined u relevant	no long-range order $(n \geq 2)$
$2 < d < 4$	IR-singular UV-convergent	super-renormalizable u relevant	non-classical exponents
$d = d_c = 4$	logarithmic IR-/ UV-divergence	renormalizable u marginal	logarithmic corrections
$d > 4$	IR-regular UV-divergent	non-renormalizable u irrelevant	mean-field exponents

Integrals in *dimensional regularization*: even for non-integer d, σ:

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^{2\sigma}}{(\tau + k^2)^s} = \frac{\Gamma(\sigma + d/2)\,\Gamma(s - \sigma - d/2)}{2^d\,\pi^{d/2}\,\Gamma(d/2)\,\Gamma(s)}\ \tau^{\sigma - s + d/2}$$

- in effect: discard divergent surface integrals
- UV singularities $\rightarrow$ *dimensional poles* in Euler Γ functions

Renormalization

Susceptibility $\chi^{-1} = C(q=0)^{-1} = \Gamma^{(2)}(q=0) = \tau = r - r_c$

$$\rightarrow \; r_c = -\frac{n+2}{6}\, u \int_k \frac{1}{r_c + k^2} + O(u^2) = -\frac{n+2}{6} \frac{u\, K_d}{(2\pi)^d} \frac{\Lambda^{d-2}}{d-2}$$

(non-universal) T_c-shift: *additive renormalization*

$$\Rightarrow \; \chi(q)^{-1} = q^2 + \tau \left[1 - \frac{n+2}{6}\, u \int_k \frac{1}{k^2(\tau + k^2)} \right]$$

Multiplicative renormalization:

absorb UV poles at $\epsilon = 0$ into *renormalized* fields and parameters:

$$S_R^\alpha = Z_S^{1/2}\, S^\alpha \; \rightarrow \; \Gamma_R^{(N)} = Z_S^{-N/2}\, \Gamma^{(N)}$$

$$\tau_R = Z_\tau\, \tau\, \mu^{-2}\,, \; u_R = Z_u\, u\, A_d\, \mu^{d-4}\,, \; A_d = \frac{\Gamma(3 - d/2)}{2^{d-1}\, \pi^{d/2}}$$

Normalization point outside IR regime, $\tau_R = 1$ or $q = \mu$:

$$O(u_R): \; Z_\tau = 1 - \frac{n+2}{6} \frac{u_R}{\epsilon}\,, \; Z_u = 1 - \frac{n+8}{6} \frac{u_R}{\epsilon}$$

$$O(u_R^2): \; Z_S = 1 + \frac{n+2}{144} \frac{u_R^2}{\epsilon}$$

Renormalization group equation

Unrenormalized quantities *cannot* depend on arbitrary scale μ:

$$0 = \mu \frac{d}{d\mu} \Gamma^{(N)}(\tau, u) = \mu \frac{d}{d\mu} \left[Z_S^{N/2} \, \Gamma_R^{(N)}(\mu, \tau_R, u_R) \right]$$

$\rightarrow$ *renormalization group* equation:

$$\left[\mu \frac{\partial}{\partial \mu} + \frac{N}{2} \gamma_S + \gamma_\tau \, \tau_R \frac{\partial}{\partial \tau_R} + \beta_u \frac{\partial}{\partial u_R} \right] \Gamma_R^{(N)}(\mu, \tau_R, u_R) = 0$$

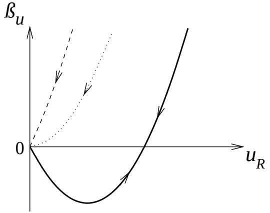
with *Wilson's flow* and *RG beta functions*:

$$\gamma_S = \mu \frac{\partial}{\partial \mu}\Big|_0 \ln Z_S = -\frac{n+2}{72} u_R^2 + O(u_R^3)$$

$$\gamma_\tau = \mu \frac{\partial}{\partial \mu}\Big|_0 \ln \frac{\tau_R}{\tau} = -2 + \frac{n+2}{6} u_R + O(u_R^2)$$

$$\beta_u = \mu \frac{\partial}{\partial \mu}\Big|_0 u_R = u_R \left[d - 4 + \mu \frac{\partial}{\partial \mu}\Big|_0 \ln Z_u \right]$$

$$= u_R \left[-\epsilon + \frac{n+8}{6} u_R + O(u_R^2) \right]$$



Method of characteristics

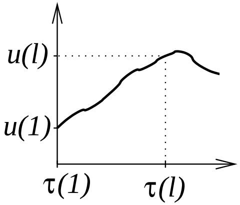
Susceptibility $\chi(q) = \Gamma^{(2)}(q)^{-1}$:

$$\chi_R(\mu, \tau_R, u_R, q)^{-1} = \mu^2 \, \hat{\chi}_R\Big(\tau_R, u_R, \frac{q}{\mu}\Big)^{-1}$$

solve RG equation: *method of characteristics*

$$\mu \to \mu(\ell) = \mu \, \ell$$

$$\chi_R(\ell)^{-1} = \chi_R(1)^{-1} \, \ell^2 \, \exp\left[\int_1^\ell \gamma_S(\ell') \, \frac{d\ell'}{\ell'}\right]$$



with *running couplings*, initial values $\tilde{\tau}(1) = \tau_R$, $\tilde{u}(1) = u_R$:

$$\ell \, \frac{d\tilde{\tau}(\ell)}{d\ell} = \tilde{\tau}(\ell) \, \gamma_\tau(\ell) \, , \; \ell \, \frac{d\tilde{u}(\ell)}{d\ell} = \beta_u(\ell)$$

Near *infrared-stable RG fixed point*: $\beta_u(u^*) = 0$, $\beta_u'(u^*) > 0$

$$\tilde{\tau}(\ell) \approx \tau_R \, \ell^{\gamma_\tau^*} \, , \; \chi_R(\tau_R, q)^{-1} \approx \mu^2 \, \ell^{2+\gamma_S^*} \, \hat{\chi}_R\Big(\tau_R \, \ell^{\gamma_\tau^*}, u^*, \frac{q}{\mu \, \ell}\Big)^{-1}$$

matching $\ell = |q|/\mu \;\to\;$ scaling form with $\eta = -\gamma_S^*$, $\nu = -1/\gamma_\tau^*$

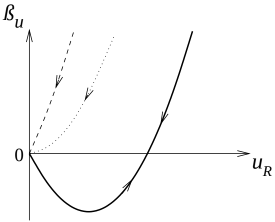
Critical exponents

Systematic $\epsilon = 4 - d$ expansion:

$$\beta_u = u_R \left[-\epsilon + \frac{n+8}{6} u_R + O(u_R^2) \right]$$

$$\rightarrow \; u_0^* = 0 \, , \; u_H^* = \frac{6\,\epsilon}{n+8} + O(\epsilon^2)$$

IR stability: $\beta_u'(u^*) > 0$



- *$d > 4$: Gaussian fixed point* u_0^* $\Rightarrow$ $\eta = 0$, $\nu = \frac{1}{2}$ (mean-field)
- *$d < 4$: Heisenberg fixed point* u_H^* stable

$$\rightarrow \; \eta = \frac{n+2}{2\,(n+8)^2}\,\epsilon^2 + O(\epsilon^3) \, , \; \nu^{-1} = 2 - \frac{n+2}{n+8}\,\epsilon + O(\epsilon^2)$$

- *$d = d_c = 4$: logarithmic corrections*:

$$\tilde{u}(\ell) = \frac{u_R}{1 - \frac{n+8}{6}\, u_R \ln \ell} \, , \; \tilde{\tau}(\ell) \sim \frac{\tau_R}{\ell^2 (\ln |\ell|)^{(n+2)/(n+8)}}$$

$$\rightarrow \; \xi \propto \tau_R^{-1/2} \, (\ln \tau_R)^{(n+2)/2(n+8)}$$

- Accurate exponent values: Monte Carlo simulations; or: Borel resummation; non-perturbative "exact" (numerical) RG

Non-perturbative RG, critical dynamics

- *Non-perturbative RG*: numerically solve exact RG flow equation for *effective potential* $\Gamma = \Gamma_{k \to 0}$
$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr} \int_q \left[\Gamma_k^{(2)}(q) + R_k(q) \right]^{-1} \partial_t R_k(q)$$
with appropriately chosen *regulator* R_k, $t = \ln(k/\Lambda)$
- *Critical dynamics*: relaxation time $t_c(\tau) \sim \xi(\tau)^z \sim |\tau|^{-z\nu}$ with *dynamic critical exponent z*; time scale separation $\to$ *Langevin equations* for order parameter and conserved fields:
$$\partial_t S^\alpha(x,t) = F^\alpha[S](x,t) + \zeta^\alpha(x,t) \, , \; \langle \zeta^\alpha(x,t) \rangle = 0$$
$$\langle \zeta^\alpha(x,t) \zeta^\beta(x',t') \rangle = 2L^\alpha \, \delta(x - x') \, \delta(t - t') \, \delta^{\alpha\beta}$$
map onto *Janssen–De Dominicis response functional*:
$$\langle A[S] \rangle_\zeta = \int \mathcal{D}[S] \, A[S] \, \mathcal{P}[S] \, , \; \mathcal{P}[S] \propto \int \mathcal{D}[i\widetilde{S}] \, e^{-\mathcal{A}[\widetilde{S},S]}$$
$$\mathcal{A}[\widetilde{S},S] = \int d^d x \int_0^{t_f} dt \sum_\alpha \left[\widetilde{S}^\alpha \Big(\partial_t \, S^\alpha - F^\alpha[S] \Big) - \widetilde{S}^\alpha L^\alpha \widetilde{S}^\alpha \right]$$

Non-equilibrium dynamic scaling

Field theory representations for non-equilibrium dynamical systems:

- Coarse-grained effective Langevin description:
  $\rightarrow$ *Janssen–De Dominicis functional*
- Interacting / reacting particle systems:
  $\rightarrow$ *Doi–Peliti field theory* from stochastic master equation
- Non-equilibium quantum dynamics:
  $\rightarrow$ *Keldysh–Baym–Kadanoff Green function formalism*

All contain *additional field* encoding non-equilibrium dynamics
anisotropic $(d+1)$-dimensional field theory: *dynamic exponent(s)*
RG fixed points $\rightarrow$ dynamic scaling properties, characterize:

- non-equilibrium *stationary states / phases*
- universality classes for non-equilibrium *phase transitions*
- non-equilibrium *relaxation and aging scaling* features
- properties of systems displaying *generic scale invariance*

Selected literature:

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- M. Le Bellac, *Quantum and statistical field theory*, Oxford University Press (Oxford, 1991).
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- U.C. Täuber, *Critical dynamics — A field theory approach to equilibrium and non-equilibrium scaling behavior*, Cambridge University Press (Cambridge, 2014).
- A.N. Vasil'ev, *The field theoretic renormalization group in critical behavior theory and stochastic dynamics*, Chapman & Hall / CRC (Boca Raton, 2004).
- J. Zinn-Justin, *Quantum field theory and critical phenomena*, Clarendon Press (Oxford, 1993).

Some exercises

1. *Relationship between cumulants and vertex functions.*
   By means of appropriate derivatives of the generating functional for the vertex functions, establish the relations

$$\Gamma^{(2)}(q) = C(q)^{-1} \ , \ \Big\langle \prod_{i=1}^{4} S(q_i) \Big\rangle_c = - \prod_{i=1}^{4} C(q_i) \, \Gamma^{(4)}(\{q_i\})$$

   between the two- and four-point vertex functions and cumulants.

2. *Explicit two-loop perturbation theory for the vertex functions.*
   Confirm the explicit two-loop result for $\Gamma^{(2)}(q)$ and the one-loop expression for $\Gamma^{(4)}(\{q_i = 0\})$.

3. *Singular contribution to the two-loop propagator self-energy.*
   Employ Feynman parametrization

$$\frac{1}{A^r \, B^s} = \frac{\Gamma(r+s)}{\Gamma(r)\,\Gamma(s)} \int_0^1 \frac{x^{r-1}\,(1-x)^{s-1}}{[x\,A + (1-x)\,B]^{r+s}} \, dx$$

   to extract the UV-singular part of the two-loop integral

$$D(q) = \int_k \frac{1}{\tau + k^2} \int_{k'} \frac{1}{\tau + k'^2} \, \frac{1}{\tau + (q-k-k')^2} \ ,$$

$$\Rightarrow \ \frac{\partial D(q)}{\partial q^2}\bigg|_{q=0}^{\rm sing.} = - \, \frac{A_d^2 \, \tau^{-\epsilon}}{8 \, \epsilon} \ .$$