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Renormalization Group: Applications in Statistical Physics

Uwe C. Täuber

Department of Physics, Virginia Tech, Blacksburg, VA 24061-0435, USA

Abstract

These notes aim to provide a concise pedagogical introduction to some important applications of the renormalization group in statistical physics. After briefly reviewing the scaling approach and Ginzburg–Landau theory for critical phenomena near continuous phase transitions in thermal equilibrium, Wilson's momentum shell renormalization group method is presented, and the critical exponents for the scalar Φ^4 model are determined to first order in a dimensional ϵ expansion about the upper critical dimension $d_c = 4$. Subsequently, the physically equivalent but technically more versatile field-theoretic formulation of the perturbational renormalization group for static critical phenomena is described. It is explained how the emergence of scale invariance connects ultraviolet divergences to infrared singularities, and the renormalization group equation is employed to compute the critical exponents for the $O(n)$-symmetric Landau–Ginzburg–Wilson theory to lowest non-trivial order in the ϵ expansion. The second part of this overview is devoted to field theory representations of non-linear stochastic dynamical systems, and the application of renormalization group tools to critical dynamics. Dynamic critical phenomena in systems near equilibrium are efficiently captured through Langevin stochastic equations of motion, and their mapping onto the Janssen–De Dominicis response functional, as exemplified by the field-theoretic treatment of purely relaxational models with non-conserved (model A) and conserved order parameter (model B). As examples for other universality classes, the Langevin description and scaling exponents for isotropic ferromagnets at the critical point (model J) and for driven diffusive non-equilibrium systems are discussed. Finally, an outlook is presented to scale-invariant phenomena and non-equilibrium phase transitions in interacting particle systems. It is shown how the stochastic master equation associated with chemical reactions or population dynamics models can be mapped onto imaginary-time, non-Hermitian "quantum" mechanics. In the continuum limit, this Doi–Peliti Hamiltonian is in turn represented through a coherent-state path integral action, which allows an efficient and powerful renormalization group analysis of, e.g., diffusion-limited annihilation processes, and of phase transitions from active to inactive, absorbing states.

Keywords:
renormalization group, critical phenomena, critical dynamics, driven diffusive systems, diffusion-limited chemical reactions, non-equilibrium phase transitions
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1. Introduction

Since Ken Wilson's seminal work in the early 1970s [1], based also on the groundbreaking foundations laid by Leo Kadanoff, Ben Widom, Michael Fisher [2], and others in the preceding decade, the renormalization group (RG) has had a profound impact on modern statistical physics. Not only do renormalization group methods provide a powerful tool to analytically describe and quantitatively capture both static and dynamic critical phenomena near continuous phase transitions that are governed by strong interactions, fluctuations, and correlations, they also allow us to address physical prop-

Email address: `tauber@vt.edu` (Uwe C. Täuber)

erties associated with the emerging generic scale invariance in certain entire thermodynamic phases, many non-equilibrium steady states, and in relaxation phenomena towards either equilibrium or non-equilibrium stationary states. In fact, the renormalization group presents us with a conceptual framework and mathematical language that has become ubiquitous in the theoretical description of many complex interacting many-particle systems encountered in nature. One may even argue that the fundamental RG notions of universality and relevance or irrelevance of interactions and perturbations, and the accompanying systematic coarse-graining procedures are of crucial importance for any attempt at capturing natural phenomena in terms of only a few meso- or macroscopic degrees of freedom, and thus also form the essential philosophical basis for any computational modeling, including Monte Carlo simulations.

In these lecture notes, I aim to give a pedagogical introduction and concise overview of first the classic applications of renormalization group methods to equilibrium critical phenomena, and subsequently to the study of critical dynamics, both near and far away from thermal equilibrium. The second half of this article will specifically explain how the stochastic dynamics of interacting many-particle systems, mathematically described either through (coupled) non-linear Langevin or more "microscopic" master equations, can be mapped onto dynamical field theory representations, and then analyzed by means of RG-improved perturbative expansions. In addition, it will be demonstrated how exploiting the general structure of the RG flow equations, fixed point conditions, and prevalent symmetries yields certain exact statements. Other authors contributing to this volume will discuss additional applications of renormalization group tools to a broad variety of physical systems and problems, and also cover more recently developed efficient non-perturbative approaches.

2. Critical Phenomena

We begin with a quick review of Landau's generic mean-field treatment of continuous phase transitions in thermal equilibrium, define the critical exponents that characterize thermodynamic singularities, and then venture to an even more general description of critical phenomena by means of scaling theory. Next we generalize to spatially inhomogeneous configurations, investigate critical infrared singularities in the two-point correlation function, and analyze the Gaussian fluctuations for the ensuing Landau–Ginzburg–Wilson Hamiltonian (scalar Euclidean Φ^4 field theory). This allows us to identify $d_c = 4$ as the upper critical dimension below which fluctuations crucially impact the critical power laws. Finally, we introduce Wilson's momentum shell renormalization group approach, reconsider the Gaussian model, discuss the general emerging structure, and at last perturbatively compute the fluctuation corrections to the critical exponents to first order in the dimensional expansion parameter $\epsilon = d_c - d$. Far more detailed expositions of the contents of this chapter can be found in the excellent textbooks [3]–[8] and in chap. 1 of Ref. [9].

2.1. Continuous phase transitions

Different thermodynamic phases are characterized by certain macroscopic, usually extensive state variables called order parameters; examples are the magnetization in ferromagnetic systems, polarization in ferroelectrics, and the macroscopically occupied ground-state wave function for superfluids and superconductors. We shall henceforth set our order parameter to vanish in the high-temperature disordered phase, and to assume a finite value in the low-temperature ordered phase. Landau's basic construction of a general mean-field description for phase transitions relies on an expansion of the free energy (density) in terms of the order parameter, naturally constrained by the symmetries of the physical system under consideration. For example, consider a scalar order parameter ϕ with discrete inversion or Z_2 symmetry that in the ordered phase may take either of two degenerate values $\phi_\pm = \pm|\phi_0|$. We shall see that the following generic expansion (with real coefficients) indeed describes a *continuous* or second-order phase transition:

$$f(\phi) = \frac{r}{2}\,\phi^2 + \frac{u}{4!}\,\phi^4 + \ldots - h\,\phi\,, \qquad (1)$$

if the temperature-dependent parameter r changes sign at T_c. For simplicity, and again in the spirit of a regular Taylor expansion, we let $r = a(T - T_c^0)$, where T_c^0 denotes the *mean-field critical temperature*. Stability requires that $u > 0$ (otherwise more expansion terms need to be added); near the critical point we can simply take u to be a constant. Note that the external field h, thermodynamically conjugate to the order parameter, *explicitly* breaks the assumed Z_2 symmetry $\phi \to -\phi$.

Minimizing the free energy with respect to ϕ then yields the thermodynamic ground state. Thus, from $f'(\phi) = 0$ we immediately infer the *equation of state*

$$h(T, \phi) = r(T)\,\phi + \frac{u}{6}\,\phi^3\,, \qquad (2)$$

and the minimization or stability condition reads $0 < f''(\phi) = r + \frac{u}{2}\,\phi^2$. At $T = T_c^0$, (2) reduces to the *critical isotherm* $h(T_c^0, \phi) = \frac{u}{6}\,\phi^3$. For $r > 0$, the *spontaneous* *order parameter* at zero external field $h = 0$ vanishes; for $r < 0$, one obtains $\phi_\pm = \pm\phi_0$, where

$$\phi_0 = (6|r|/u)^{1/2} . \tag{3}$$

Note the emergence of characteristic power laws in the thermodynamic functions that describe the properties near the critical point located at $T = T_c^0$, $h = 0$.

The continuous, but non-analytic onset of spontaneous ordering is the hallmark of a second-order phase transition, and induces additional thermodynamic singularities at the critical point: The isothermal order parameter *susceptibility* becomes $V\chi_T^{-1} = (\partial h/\partial\phi)_T = r + \frac{u}{2}\phi_0^2$, whence

$$\frac{\chi_T}{V} = \begin{cases} 1/r & r > 0 \\ 1/2|r| & r < 0 \end{cases} , \tag{4}$$

diverging as $|T - T_c^0|^{-1}$ on both sides of the phase transition, with *amplitude ratio* $\chi_T(T \downarrow T_c^0)/\chi_T(T \uparrow T_c^0) = 2$. Inserting (3) into the Landau *free energy* (1) one finds for $T < T_c^0$ and $h = 0$

$$f(\phi_\pm) = \frac{r}{4}\phi_0^2 = -\frac{3r^2}{2u} , \tag{5}$$

and consequently for the *specific heat*

$$C_{h=0} = -VT\left(\frac{\partial^2 f}{\partial T^2}\right)_{h=0} = VT\,\frac{3a^2}{u} , \tag{6}$$

whereas per construction $f(0) = 0$ and $C_{h=0} = 0$ in the disordered phase. Thus, Landau's mean-field theory predicts a critical point *discontinuity* $\Delta C_{h=0} = VT_c^0\,\frac{3a^2}{u}$ for the specific heat. Experimentally, one indeed observes singularities in thermodynamic observables and power laws at continuous phase transitions, but often with critical exponents that differ from the above mean-field predictions. Indeed, the divergence of the order parameter susceptibility (4) indicates violent fluctuations, inconsistent with any mean-field description that entirely neglects such fluctuations and correlations.

2.2. Scaling theory

The emergence of scale-free power laws suggests the following general scaling hypothesis for the free energy, namely that its singular contributions near a critical point ($T = T_c$, $h = 0$) can be written as a generalized *homogeneous* function

$$f_{\rm sing}(\tau, h) = |\tau|^{2-\alpha}\,\hat{f}_\pm\left(h/|\tau|^\Delta\right) , \tag{7}$$

where $\tau = \frac{T-T_c}{T_c}$ measures the deviation from the (true) critical temperature T_c. Thus, the free energy near criticality is not an independent function of the two intensive control parameters T or τ and h, but satisfies a remarkable two-parameter scaling law, with analytic *scaling functions* $\hat{f}_\pm(x)$ respectively for $T > T_c$ and $T < T_c$ that only depend on the ratio $x = h/|\tau|^\Delta$, and satisfy $\hat{f}_\pm(0) = $ const. In Landau theory, the corresponding *critical exponents* are $\alpha = 0$, compare (5), and $\Delta = 3/2$, as can be inferred by combining Eqs. (2) and (3). The associated *specific heat* singularity follows again via

$$C_{h=0} = -\frac{VT}{T_c^2}\left(\frac{\partial^2 f_{\rm sing}}{\partial\tau^2}\right)_{h=0} = C_\pm\,|\tau|^{-\alpha} , \tag{8}$$

indicating a divergence if $\alpha > 0$, and a cusp singularity for $\alpha < 0$. Similarly, one obtains the *equation of state*

$$\phi(\tau, h) = -\left(\frac{\partial f_{\rm sing}}{\partial h}\right)_\tau = -|\tau|^{2-\alpha-\Delta}\,\hat{f}'_\pm\left(h/|\tau|^\Delta\right) , \tag{9}$$

and therefrom the *coexistence line* at $h = 0$, $\tau < 0$

$$\phi(\tau, 0) = -|\tau|^{2-\alpha-\Delta}\,\hat{f}'_-(0) \propto |\tau|^\beta , \tag{10}$$

where we have identified $\beta = 2 - \alpha - \Delta$.

Additional *scaling relations*, namely identities that relate different critical exponents, can be easily derived; for example, on the *critical isotherm* at $\tau = 0$, the τ-dependence in $\hat{f}'_\pm$ on the r.h.s. of (9) must cancel the singular prefactor, i.e., $\hat{f}'_\pm(x \to \infty) \sim x^{(2-\alpha-\Delta)/\Delta}$, and

$$\phi(0, h) \propto h^{(2-\alpha-\Delta)/\Delta} = h^{1/\delta} , \text{ with } \delta = \Delta/\beta . \tag{11}$$

Finally, the isothermal *susceptibility* becomes

$$\frac{\chi_\tau}{V} = \left(\frac{\partial\phi}{\partial h}\right)_{\tau, h=0} = \chi_\pm\,|\tau|^{-\gamma} , \ \gamma = \alpha + 2(\Delta - 1) , \tag{12}$$

and upon eliminating $\Delta = \beta\delta$, one arrives at the following set of scaling relations

$$\alpha + \beta(1 + \delta) = 2 = \alpha + 2\beta + \gamma , \ \gamma = \beta(\delta - 1) . \tag{13}$$

Clearly, as consequence of the two-parameter scaling hypothesis (7), there can only be *two independent* thermodynamic critical exponents. In the framework of Landau's mean-field approximation, the set of critical exponents reads $\alpha = 0, \beta = \frac{1}{2}, \gamma = 1, \delta = 3$, and $\Delta = \frac{3}{2}$; note that these integer or rational numbers really just follow from straightforward dimensional analysis. In both computer and real experiments, one typically measures different critical exponent values, yet these still turn out to be *universal* in the sense that at least for short-range interaction forces they depend only on basic symmetry properties of the order parameter and the spatial dimensionality d, but *not* on microscopic details such as lattice structure, nature and strength of interaction potentials, etc. Indeed, the Ising ferromagnetic and the liquid-gas critical points, both characterized by a scalar real order parameter, are governed by identical power laws, as is the critical behavior for planar magnets with a two-component vector order parameter and the normal- to superfluid transition in helium 4, with a complex scalar or equivalently, a real two-component order parameter. The striking emergence of thermodynamic self-similarity in the vicinity of T_c has been spectacularly demonstrated in the latter system, with the Λ-like shape of the specific heat curve appearing identical on milli- and micro-Kelvin temperature scales.

2.3. Landau–Ginzburg–Wilson Hamiltonian

In order to properly include the effects of fluctuations, we need to generalize the Landau expansion (1) to spatially varying order parameter configurations $S(x)$, which leads us to the *coarse-grained* effective *Landau–Ginzburg–Wilson (LGW) Hamiltonian*

$$\mathcal{H}[S] = \int d^d x \left[\frac{r}{2} S(x)^2 + \frac{1}{2} [\nabla S(x)]^2 + \frac{u}{4!} S(x)^4 - h(x) S(x) \right], \quad (14)$$

where $r = a(T - T_c^0)$ and $u > 0$ as before, and $h(x)$ now represents a local external field. Under the natural assumption that spatial inhomogeneities are energetically unfavorable, the gradient term $\sim [\nabla S(x)]^2$ comes with a positive coefficient that has been absorbed into the scalar order parameter field. Within the canonical framework of statistical mechanics, the *probability density* for a configuration $S(x)$ is given by the *Boltzmann factor* $\mathcal{P}_s[S] = \exp(-\mathcal{H}[S]/k_B T)/\mathcal{Z}[h]$. Here, the *partition function* $\mathcal{Z}[h]$ and expectation values of observables $A[S]$ are represented through functional integrals:

$$\mathcal{Z}[h] = \int \mathcal{D}[S]\, e^{-\mathcal{H}[S]/k_B T} , \quad (15)$$

$$\langle A[S] \rangle = \int \mathcal{D}[S]\, A[S(x)]\, \mathcal{P}_s[S] . \quad (16)$$

At $h = 0$, for example, the kth order parameter moments follow via functional derivatives

$$\left\langle \prod_{j=1}^{k} S(x_j) \right\rangle = (k_B T)^k \prod_{j=1}^{k} \frac{\delta}{\delta h(x_j)} \mathcal{Z}[h] \Big|_{h=0}, \quad (17)$$

and similarly the associated cumulants can be obtained from functional derivatives of $\ln \mathcal{Z}[h]$; the partition function thus also serves as a *generating function*. For explicit calculations, one requires the integral measure in (16), e.g., through discretizing $x \to x_i$ on, say, a d-dimensional cubic hyperlattice, whence simply $\mathcal{D}[S] = \prod_i dS(x_i)$. Alternatively, one may employ the Fourier transform $S(x) = \int \frac{d^d q}{(2\pi)^d} S(q)\, e^{iq\cdot x}$; noting that $S(-q) = S(q)^*$ since $S(x)$ is real, and consequently the real and imaginary parts of $S(q)$ are not independent, one only needs to integrate over wave vector half-space,

$$\mathcal{D}[S] = \prod_{q, q_1 > 0} \frac{d \operatorname{Re} S(q)\, d \operatorname{Im} S(q)}{V} . \quad (18)$$

In the *Ginzburg–Landau approximation*, one considers only the most likely configuration $S(x)$, which is readily found by the method of steepest descent for the path integrals in (16), leading to the classical field or Ginzburg–Landau equation

$$0 = \frac{\delta \mathcal{H}[S]}{\delta S(x)} = \left[r - \nabla^2 + \frac{u}{6} S(x)^2 \right] S(x) - h(x). \quad (19)$$

In the spatially homogeneous case, (19) reduces to the mean-field equation of state (2). Let us next expand in the fluctuations $\delta S(x) = S(x) - \phi$ about the mean order parameter $\phi = \langle S \rangle$ and linearize, which yields $\delta h(x) \approx \left(r - \nabla^2 + \frac{u}{2} \phi^2 \right) \delta S(x)$. Through Fourier transform one then immediately obtains the order parameter response function in the mean-field approximation, also known as *Ornstein–Zernicke susceptibility*

$$\chi_0(q) = \frac{\partial S(q)}{\partial h(q)} \Big|_{h=0} = \frac{1}{\xi^{-2} + q^2} , \quad (20)$$

where we have introduced the characteristic *correlation length* $\xi = (r + \frac{u}{2} \phi_0^2)^{-1/2}$, i.e.,

$$\xi = \begin{cases} 1/r^{1/2} & r > 0 \\ 1/|2r|^{1/2} & r < 0 \end{cases} . \quad (21)$$

On the other hand, consider the connected zero-field two-point *correlation function* (cumulant)

$$C(x - x') = \langle S(x)\, S(x') \rangle - \langle S \rangle^2 = (k_B T)^2 \frac{\delta^2 \ln \mathcal{Z}[h]}{\delta h(x)\, \delta h(x')} \Big|_{h=0} ; \quad (22)$$

in a spatially translation-invariant system, we may define its Fourier transform as $C(x) = \int \frac{d^d q}{(2\pi)^d} C(q)\, e^{iq\cdot x}$, and through comparison with the definition of the susceptibility in (20) arrive at the *fluctuation-response theorem* $C(q) = k_B T\, \chi(q)$, valid in thermal equilibrium.

Generalizing the Ginzburg–Landau mean-field result (20), we may formulate the *scaling hypothesis* for the two-point correlation function in terms of the following scaling ansatz, which defines both the *Fisher exponent* η and the critical exponent ν that describes the divergence of the correlation length ξ at T_c:

$$C(\tau, q) = |q|^{-2+\eta}\, \hat{C}_\pm(q\xi) , \quad \xi = \xi_\pm\, |\tau|^{-\nu} . \quad (23)$$

The thermodynamic susceptibility then becomes

$$\chi(\tau, q = 0) \propto \xi^{2-\eta} \propto |\tau|^{-\gamma} \,, \text{ with } \gamma = \nu(2-\eta) \,, \tag{24}$$

providing us with yet another scaling relation that connects the thermodynamic critical exponent γ with η and ν. Consequently, we see that the thermodynamic critical point singularities are induced by the diverging spatial correlations. Fourier back-transform gives

$$C(\tau, x) = |x|^{-(d-2+\eta)} \, \widetilde{C}_\pm(x/\xi) \propto \xi^{-(d-2+\eta)} \tag{25}$$

at large distances $|x| \to \infty$. In this limit, one expects $\langle S(x) \, S(0) \rangle \to \phi^2 \propto (-\tau)^{2\beta}$, and comparison with (25) therefore implies the *hyperscaling* relations

$$\beta = \frac{\nu}{2} (d - 2 + \eta) \text{ and } 2 - \alpha = d\nu \,. \tag{26}$$

The Ornstein–Zernicke function (20) satisfies the scaling law (23) with the mean-field values $\nu = \frac{1}{2}$ and $\eta = 0$. Notice that the set of mean-field critical exponents obeys (26) only in $d = 4$ dimensions.

2.4. Gaussian approximation

We now proceed to analyze the LGW Hamiltonian (14) in the Gaussian approximation, where non-linear fluctuation contributions are neglected. In the high-temperature phase, we have $\phi = 0$, and thus simply omit the terms $\sim u \, S(x)^4$, leaving the Gaussian Hamiltonian

$$\mathcal{H}_0[S] = \int_q \left[\frac{r + q^2}{2} \, |S(q)|^2 - h(q) S(-q) \right] \,, \tag{27}$$

with the abbreviation $\int_q = \int \frac{d^d q}{(2\pi)^d}$. The associated Gaussian partition function is readily computed by completing the square in (27), or the linear field transformation $\widetilde{S}(q) = S(q) - h(q)/(r + q^2)$,

$$\begin{aligned} \mathcal{Z}_0[h] &= \int \mathcal{D}[S] \, e^{-\mathcal{H}_0[S]/k_B T} \\ &= \exp\left(\frac{1}{2k_B T} \int_q \frac{|h(q)|^2}{r + q^2} \right) \mathcal{Z}_0[h = 0] \,, \end{aligned} \tag{28}$$

which yields the Gaussian two-point correlator

$$\begin{aligned} \langle S(q) \, S(q') \rangle_0 &= \frac{(k_B T)^2}{\mathcal{Z}_0[h]} \, \frac{(2\pi)^{2d} \, \delta^2 \mathcal{Z}_0[h]}{\delta h(-q) \, \delta h(-q')} \bigg|_{h=0} \\ &= C_0(q) \, (2\pi)^d \delta(q + q') \,, \; C_0(q) = \frac{k_B T}{r + q^2} \,. \end{aligned} \tag{29}$$

Gaussian integrations give the free energy $F_0[h] = -k_B T \ln \mathcal{Z}_0[h]$ of the model (27),

$$F_0[h] = -\frac{1}{2} \int_q \left(\frac{|h(q)|^2}{r + q^2} + k_B T V \, \ln \frac{2\pi \, k_B T}{r + q^2} \right) . \tag{30}$$

Let us explore the leading singularity near T_c^0 in the *specific heat* $C_{h=0} = -T (\partial^2 F_0 / \partial T^2)_{h=0}$ that originates from derivatives with respect to the control parameter r,

$$\frac{C_{h=0}}{V} \approx \frac{k_B}{2} \, (a T_c^0)^2 \int_q \frac{1}{(r + q^2)^2} \,. \tag{31}$$

- In high dimensions $d > 4$, the r.h.s. integral is UV-divergent, but can be regularized by a Brillouin zone boundary cutoff $\Lambda \sim 2\pi/a_0$ stemming from the original underlying lattice. Consequently the fluctuation contribution (31) is finite as $r \to 0$ and $\alpha = 0$ as in mean-field theory.

- In low dimensions $d < 4$, we set $k = q/\sqrt{r} = q\xi$ to render the fluctuation integral, which is UV-finite, dimensionless. With the d-dimensional unit sphere surface area $K_d = 2\pi^{d/2}/\Gamma(d/2)$, one finds:

  $$\frac{C_{h=0}}{V} \approx \frac{k_B (a T_c^0)^2 \, \xi^{4-d}}{2^d \pi^{d/2} \, \Gamma(d/2)} \int_0^\infty \frac{k^{d-1}}{(1 + k^2)^2} \, dk \,. \tag{32}$$

  As the critical temperature is approached, the correlation length prefactor diverges $\propto |T - T_c^0|^{-\frac{4-d}{2}}$; already the lowest-order fluctuation contribution contains a strong infrared (IR) singularity that dominates over the mean-field power law.

- At $d = d_c = 4$, the integral diverges logarithmically as either Λ or $\xi \to \infty$:

  $$\int_0^{\Lambda\xi} \frac{k^3}{(1 + k^2)^2} \, dk \sim \ln(\Lambda\xi) \,; \tag{33}$$

  note that at this *upper critical dimension*, ultraviolet and infrared divergences are intimately coupled.

Above the critical dimension, we thus expect the mean-field scaling exponents to correctly describe the critical power laws. In dimensions $d \leq d_c$, however, fluctuation contributions become prevalent and modify the mean-field scaling laws. Given the strongly fluctuating and correlated nature of critical systems, standard theoretical approaches such as perturbation, cluster, or high- and low-temperature expansions typically fail to yield reliable approximations. Fortunately, the renormalization group provides a powerful method to tackle interacting many-particle systems dominated by fluctuations and correlations, especially in a scale-invariant regime.

2.5. Wilson's momentum shell renormalization group

The *renormalization group program in statistical physics* can be summarized as follows: The goal is to establish a mathematical framework that can properly capture the infrared (IR) singularities appearing in thermodynamic properties as well as correlation functions near a continuous phase transitions and in related situations that are *not* perturbatively accessible. To this end, one exploits a fundamental new symmetry that emerges at a critical point, namely *scale invariance*, induced by the divergence of the dominant characteristic correlation length scale ξ. Approximation schemes need to carefully avoid the region where the physical IR singularities become manifest; instead, one analyzes the theory in the ultraviolet (UV) regime, by means of either of various equivalent methods: In Wilson's momentum shell RG approach, one integrates out short-wavelength modes; in the field-theoretic version of the RG, one explicitly renormalizes the UV divergences. Either method quantifies the weight of fluctuation contributions to certain coarse-grained or "renormalized" physical parameters and couplings. One then maps the resulting system back to the original theory given in terms of some "effective" Hamiltonian, which in Wilson's scheme entails a rescaling of both control parameters and field degrees of freedom. Thus one obtains recursion relations for *effective*, now scale-dependent *running couplings*. Subject to a recursive sequence of such renormalization group transformations, these effective couplings will

- either grow, and ultimately tend to infinity: to access a scale-invariant regime, one therefore has to set these *relevant* parameters to zero at the outset, which defines the *critical surface* of the problem;
- or diminish, and eventually approach zero: these *irrelevant* couplings consequently do not affect the asymptotic critical scaling properties;
- certain *marginal* parameters may also approach an *infrared-stable fixed point*, provided their initial value is located in the fixed point's basin of attraction: clearly, scale-invariant behavior thus emerges near an IR-stable fixed point, and the independence from a wide range of initial conditions along with the automatic disappearance of the irrelevant couplings constitute the origin of *universality*.

The central idea now is to take advantage of the emerging scale invariance at a critical fixed point as a means to infer the proper infrared scaling behavior from an at least approximative analysis of the ultraviolet regime, where, e.g., perturbation theory is feasible. Thus one may establish a solid theoretical foundation for scaling laws such as (7) and (23), thereby derive scaling relations, and also construct a systematic approximation scheme to compute critical exponents and even scaling functions. We shall soon see that an appropriate small parameter for a perturbational expansion is given through a *dimensional expansion* in terms of the deviation from the upper critical dimension $\epsilon = d_c - d$.

Wilson's momentum shell renormalization group approach consists of two RG transformation steps:

(1) Carry out the partition integral over all Fourier components $S(q)$ with wave vectors residing in the spherical momentum shell $\Lambda/b \leq |q| \leq \Lambda$, where $b > 1$: this effectively *eliminates* the short-wavelength modes.

(2) Perform a *scale transformation* with the same scale parameter $b > 1$: $x \to x' = x/b$, $q \to q' = b\,q$. Accordingly, one also needs to rescale the fields:

$$S(x) \to S'(x') = b^{\zeta} S(x) \,, \quad S(q) \to S'(q') = b^{\zeta-d} S(q) \,, \tag{34}$$

with a proper choice of ζ ensuring that the rescaled residual Hamiltonian assumes the original form.

Subsequent iterations of this procedure yield *scale-dependent effective couplings*, and the task will be to analyze their dependence on the scale parameter b. Notice the *semi-group* character of the above RG transformations: there obviously exists no unique inverse, since the elimination step (1) discards detailed information about fluctuations in the UV regime.

The mechanism and efficacy of the momentum shell RG is best illuminated by first considering the exactly tractable Gaussian model. Introducing the short-hand notations $\int_q^< = \int_{|q|<\Lambda/b} \frac{d^dq}{(2\pi)^d}$ and $\int_q^> = \int_{\Lambda/b\leq|q|\leq\Lambda} \frac{d^dq}{(2\pi)^d}$, one may readily decompose the Hamiltonian (27) into distinct additive Fourier mode contributions

$$\mathcal{H}_0[S] = \left(\int_q^< + \int_q^> \right) \left[\frac{r+q^2}{2} \, |S(q)|^2 - h(q)\, S(-q) \right] .$$

Integrating out the momentum shell fluctuations then just gives a constant contribution to the free energy. We now wish to achieve that $\mathcal{H}_0[S^<] \to \mathcal{H}_0[S']$ under the scale transformations in step (2). For the term $\sim q^2 \, |S(q)|^2$, this is accomplished through the choice $\zeta = \frac{d-2}{2}$ in (34); the other contributions then immediately result in the following recursion relations for the control parameters r and h: $r \to r' = b^2 r$, and $h(q) \to h'(q') = b^{-\zeta} h(q)$, whence $h(x) \to h'(x') = b^{d-\zeta} h(x)$. Both the temperature variable r and the external field thus constitute *relevant* parameters, and the *critical surface* in parameter space is given by $r = 0 = h$. As any other length scale, the correlation length scales according to $\xi \to \xi' = \xi/b$; eliminating the scale parameter b one arrives at the relation $\xi \propto r^{-1/2}$, or $\nu = \frac{1}{2}$. Likewise, for the rescaled correlation function one finds $C'(x') = b^{2\zeta} C(x)$, whence $\eta = 0$: for the Gaussian theory, we recover the mean-field scaling exponents.

We can gain additional non-trivial information by considering further couplings; e.g., imagine adding contributions of the form $c_s \int d^d x (\nabla^s S)^2$ to the Hamiltonian (27) that represent higher-order terms in a gradient expansion for spatial order parameter fluctuations, subject to preserving the Z_2 and spatial inversion symmetries. One readily confirms that $c_s \to c_s' = b^{d-2s-2\zeta} c_s = b^{-2(s-1)} c_s$, which implies that all these additional couplings c_s are *irrelevant* for $s > 1$ and scale to zero under repeated scale transformations. The inversion symmetry $S(x) \to -S(x)$ permits general local non-linearities of the from $u_p \int d^d x\, S(x)^{2p}$. Under Gaussian model RG transformation, these scale as $u_p \to u_p' = b^{d-2p\zeta} u_p = b^{2p-(p-1)d} u_p$; these couplings are consequently *marginal* at $d_c(p) = 2p/(p-1)$, *relevant* for $d < d_c(p)$, and *irrelevant* for $d > d_c(p)$. The upper critical dimension decreases monotonously for $p \geq 2$, with the asymptote $d_c(\infty) = 2$. For the quartic coupling in the LGW Hamiltonian (14), this confirms $d_c(2) = 4$, while $d_c(3) = 3$ for a sixth-order term $v \int d^d x\, S(x)^6$: $v \to v' = b^{6-2d} v$, which becomes *irrelevant* near the upper critical dimension of the quartic term: $v' = b^{-2} v$ at $d_c(2) = 4$. In general, the coupling ratio $\frac{u_{p+1}'}{u_p'} = b^{2-d} \frac{u_{p+1}}{u_p}$ renormalizes to zero in dimensions $d > 2$. At two dimensions, the fields $S(x)$ become dimensionless, $\zeta = 0$, and consequently all these non-linearities scale identically. The LGW Hamiltonian thus does not represent the correct asymptotic field theory, and one must resort to other effective descriptions (e.g., the non-linear sigma model).

The above considerations already allow a discussion of the general structure of the momentum shell RG procedure. According to (25), the general field rescaling (34) should contain Fisher's exponent η, $\zeta = \frac{d-2+\eta}{2}$, whence $h'(x) = b^{(d+2-\eta)/2} h(x)$, and (23) implies $\tau' = b^{1/\nu} \tau$ with the correlation length critical exponent ν. In the simplest scenario, there are thus only *two* relevant parameters τ and h. Let us further assume the presence of (a few) *marginal* perturbations $u_i \to u_i' = u_i^* + b^{-x_i} u_i$, while other couplings are *irrelevant*: $v_i \to v_i' = b^{-y_i} v_i$, with both $x_i > 0$ and $y_i > 0$. After a single RG transformation step, the free energy density becomes

$$f_{\rm sing}(\tau, h, \{u_i\}, \{v_i\}) = b^{-d} f_{\rm sing}\Big(b^{1/\nu} \tau, b^{d-\zeta} h, \Big\{u_i^* + \frac{u_i}{b^{x_i}}\Big\}, \Big\{\frac{v_i}{b^{y_i}}\Big\}\Big) . \tag{35}$$

After sufficiently many ($\ell \gg 1$) RG transformations, the marginal couplings have reached their fixed point values u_i^*, whereas the irrelevant perturbations have scaled to zero,

$$f_{\rm sing}(\tau, h, \{u_i\}, \{v_i\}) = b^{-\ell d} f_{\rm sing}\Big(b^{\ell/\nu} \tau, b^{\ell(d+2-\eta)/2} h, \{u_i^*\}, \{0\}\Big) . \tag{36}$$

Upon choosing $b^\ell |\tau|^\nu = 1$ for the scale parameter b^ℓ, one arrives at the *scaling form*

$$f_{\rm sing}(\tau, h) = |\tau|^{d\nu}\, \hat{f}_\pm \left(h / |\tau|^{\nu(d+2-\eta)/2} \right) , \tag{37}$$

with $\hat{f}_\pm(x) = f_{\rm sing}(\pm 1, x, \{u_i^*\}, \{0\})$. With the exponent identities (13) and (26), this is equivalent to the scaling hypothesis (7). In a similar manner, one readily derives the correlation function scaling law (25), employing the *matching condition* $b^\ell = \xi / \xi_\pm$ for

$$C(\tau, x, \{u_i\}, \{v_i\}) = b^{-2\ell\zeta}\, C\Big(b^{\ell/\nu} \tau, \frac{x}{b^\ell}, \{u_i^*\}, \{0\}\Big) . \tag{38}$$

2.6. Dimensional expansion and critical exponents

We are now ready to treat the non-linear fluctuation corrections by means of a systematic *perturbation expansion*. The quartic contribution to the Hamiltonian reads in Fourier space

$$\mathcal{H}_{\rm int} = \frac{u}{4!} \int_{|q_i| < \Lambda} S(q_1) S(q_2) S(q_3) S(-q_1 - q_2 - q_3) . \tag{39}$$

Both the full partition function for the Hamiltonian (14) and any associated N-point correlation functions can then be rewritten in terms of expectation values in the *Gaussian ensemble* (we shall henceforth set $k_{\rm B} T = 1$)

$$\mathcal{Z}[h] = \mathcal{Z}_0[h] \left\langle e^{-\mathcal{H}_{\rm int}[S]} \right\rangle_0 , \quad \left\langle \prod_i S(q_i) \right\rangle = \frac{\left\langle \prod_i S(q_i)\, e^{-\mathcal{H}_{\rm int}[S]} \right\rangle_0}{\left\langle e^{-\mathcal{H}_{\rm int}[S]} \right\rangle_0} . \tag{40}$$

Note that all expectation values of an odd number of fields $S(q_i)$ obviously vanish in the symmetric high-temperature phase, when the external field $h = 0$. Defining the *contraction* of two fields as the Gaussian two-point function or *propagator* in Fourier space,

$$\underbrace{S(q) S}(q') = \langle S(q) S(q') \rangle_0 = C_0(q)\, (2\pi)^d \delta(q + q') ,$$

we may write down *Wick's theorem* for Gaussian correlators containing an even number of fields, here a straightforward property of Gaussian integrations:

$$\langle S(q_1) S(q_2) \ldots S(q_{N-1}) S(q_N) \rangle_0 = \sum_{\substack{\text{permutations} \\ i_1(1) \ldots i_N(N)}} \underbrace{S(q_{i_1(1)}) S(q_{i_2(2)})} \ldots \underbrace{S(q_{i_{N-1}(N-1)}) S(q_{i_N(N)})} . \tag{41}$$

Consequently, any arbitrary expectation value (40) can now be perturbatively evaluated via a series expansion with respect to the non-linear coupling u, and by means of (41) expressed through sums and integrals of products of Gaussian propagators $C_0(q_i)$.

As an example, we consider the first-order fluctuation correction to the zero-field two-point function $\langle S(q)S(q')\rangle = C(q)\,(2\pi)^d\delta(q+q')$: $\Big\langle S(q)S(q')\Big[1 - \frac{u}{4!}\int_{|q_i|<\Lambda} S(q_1)S(q_2)S(q_3)S(-q_1-q_2-q_3)\Big]\Big\rangle_0$. According to Wick's theorem (41), there are two types of contributions: (i) Contractions of external legs $S(q)S(q')$ yield terms that precisely cancel with the denominator in (40), leaving just the Gaussian propagator $\langle S(q)S(q')\rangle_0$. (ii) The twelve remaining contributions are all of the form $\int_{|q_i|<\Lambda} \underbrace{S(q)\,S(q_1)}\,\underbrace{S(q_2)\,S(q_3)}\,\underbrace{S(-q_1-q_2-q_3)\,S(q')} = (2\pi)^d\delta(q+q')\,C_0(q)^2\int_{|p|<\Lambda} C_0(p)$. Collecting all terms, one obtains

$$C(q) = C_0(q)\left[1 - \frac{u}{2}\,C_0(q)\int_{|p|<\Lambda} C_0(p) + O(u^2)\right]; \quad (42)$$

interpreting the bracket as the lowest-order contribution in Dyson's equation (see Chap.3.1 below), the integral turns out to be the associated self-energy to $O(u)$, and (42) can be recast in the form

$$C(q)^{-1} = r + q^2 + \frac{u}{2}\int_{|p|<\Lambda} \frac{1}{r+p^2} + O(u^2)\,. \quad (43)$$

Notice that to order u, fluctuations here merely renormalize the "mass" r, but there is no modification of the momentum dependence in the two-point correlation function $C(q)$, implying that η will remain zero in this approximation. In a similar manner, one readily finds the first-order fluctuation correction to the four-point function at vanishing external wave vectors, i.e., the non-linear coupling u to be $-\frac{3}{2}u^2\int_{|p|<\Lambda} C_0(p)^2$.

It is now a straightforward task to translate these perturbation theory results into first-order recursion relations for the couplings r and u by means of Wilson's RG procedure. To this end, we split the field variables in outer ($S_>$: $S(q)$ with $\Lambda/b \leq |q| \leq \Lambda$) and inner ($S_<$: $S(q)$ with $|q| < \Lambda/b$) momentum shell contributions; we then realize that there are four types of contributions:

- terms involving merely inner shell fields that are not integrated, e.g. $\sim u\int^> S_<^4 e^{-\mathcal{H}_0[S]}$ just need to be re-exponentiated;
- integrals such as $u\int^> S_<^3\,S_> e^{-\mathcal{H}_0[S]}$ vanish;
- contributions $\sim u\int^> S_>^4 e^{-\mathcal{H}_0[S]}$ that contain only outer shell fields become constants that directly contribute to the free energy;
- for terms of the form $\sim u\int^> S_<^2\,S_>^2 e^{-\mathcal{H}_0}$, one has to perform Gaussian integrations over the outer shell fields $S_>$, yielding corrections to the propagator for the inner shell modes.

Employing (43), using $\eta = 0$, and introducing $S_d = K_d/(2\pi)^d = 1/2^{d-1}\pi^{d/2}\Gamma(d/2)$, one thus finds to $O(u)$:

$$\begin{aligned} r' &= b^2\left[r + \frac{u}{2}\,A(r)\right] \\ &= b^2\left[r + \frac{u}{2}\,S_d\int_{\Lambda/b}^{\Lambda} \frac{p^{d-1}}{r+p^2}\,dp\right], \end{aligned} \quad (44)$$

$$\begin{aligned} u' &= b^{4-d}u\left[1 - \frac{3u}{2}\,B(r)\right] \\ &= b^{4-d}u\left[1 - \frac{3u}{2}S_d\int_{\Lambda/b}^{\Lambda} \frac{p^{d-1}\,dp}{(r+p^2)^2}\right]. \end{aligned} \quad (45)$$

For $T \gg T_c$, or $r \to \infty$, the fluctuation corrections become suppressed, and one recovers the recursion relations $r' = b^2 r$ and $u' = b^{4-d}u$ of the Gaussian theory. Near the critical point, i.e. for $r \ll 1$, one may expand

$$\begin{aligned} A(r) &= S_d\Lambda^{d-2}\,\frac{1-b^{2-d}}{d-2} \\ &\quad - r\,S_d\Lambda^{d-4}\,\frac{1-b^{4-d}}{d-4} + O(r^2)\,, \end{aligned} \quad (46)$$

$$B(r) = S_d\Lambda^{d-4}\,\frac{1-b^{4-d}}{d-4} + O(r)\,. \quad (47)$$

It is useful to consider instead differential RG flow equations that result from infinitesimal RG transformations. Setting $b = e^{\delta\ell}$ with $\delta\ell \to 0$, (44)-(47) turn into

$$\begin{aligned} \frac{d\tilde r(\ell)}{d\ell} &= 2\tilde r(\ell) + \frac{\tilde u(\ell)}{2}\,S_d\Lambda^{d-2} \\ &\quad - \frac{\tilde r(\ell)\tilde u(\ell)}{2}\,S_d\Lambda^{d-4} + O(\tilde u\tilde r^2, \tilde u^2)\,, \end{aligned} \quad (48)$$

$$\begin{aligned} \frac{d\tilde u(\ell)}{d\ell} &= (4-d)\,\tilde u(\ell) - \frac{3}{2}\,\tilde u(\ell)^2 S_d\Lambda^{d-4} \\ &\quad + O(\tilde u\tilde r, \tilde u^2)\,. \end{aligned} \quad (49)$$

We specifically seek renormalization group *fixed points* (r^*, u^*) that describe scale-invariant behavior, to be determined by the conditions $d\tilde r(\ell)/d\ell = 0 = d\tilde u(\ell)/d\ell$. There is obviously always the *Gaussian* fixed point $u_0^* = 0$; linearizing (49) in terms of the deviation $\delta\tilde u_0(\ell) = \tilde u(\ell) - u_0^*$, one finds $d\delta\tilde u_0(\ell)/d\ell \approx (d-4)\delta\tilde u_0(\ell)$; u_0^* is hence stable for $d > d_c = 4$, but unstable for $d < 4$. Below the upper critical dimension, there exists also a positive *Ising* fixed point $u_I^* S_d = \frac{2}{3}(4-d)\Lambda^{4-d}$, which is then also stable since $d\delta\tilde u_I(\ell)/d\ell \approx (4-d)\delta\tilde u_I(\ell)$ for $\delta\tilde u_I(\ell) = \tilde u(\ell) - u_I^*$. Correspondingly, the critical behavior is governed by the Gaussian fixed point and associated scaling exponents in dimensions $d > 4$, but by the non-trivial Ising fixed point in low dimensions $d < d_c = 4$. Notice also that the numerical value of the Ising fixed point becomes small near the upper critical dimension, and indeed u_I^* emerges continuously from $u_0^* = 0$ as $\epsilon = 4 - d$ is increased from zero. At the non-trivial RG fixed point, ϵ may serve to provide a small effective expansion parameter for the perturbation expansion.

To lowest order in the non-linear coupling, (48) yields at the Ising fixed point $r_I^* = -\frac{1}{4}\, u_I^* S_d \Lambda^{d-2} = -\frac{1}{6}\,\epsilon\Lambda^2$, which describes a non-universal, fluctuation-induced downward shift of the critical temperature. Introducing the deviation $\tau = r - r_I^* = a(T - T_c)$ from the true critical temperature T_c, one may rewrite the flow equation (48) to obtain the recursion relation for this modified relevant *running coupling*:

$$\frac{d\tilde{\tau}(\ell)}{d\ell} = \tilde{\tau}(\ell)\left[2 - \frac{\tilde{u}(\ell)}{2} S_d \Lambda^{d-4}\right] . \qquad (50)$$

Its solution in the vicinity of the Ising fixed point reads $\tilde{\tau}(\ell) = \tilde{\tau}(0)\exp\left[\left(2 - \frac{\epsilon}{3}\right)\ell\right]$. Combining this result with $\widetilde{\xi}(\ell) = \xi(0)\, e^{-\ell}$, one identifies the correlation length exponent $\nu^{-1} = 2 - \frac{\epsilon}{3}$, or, in a consistent expansion to first order in $\epsilon = 4 - d$:

$$\nu = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2) , \quad \eta = 0 + O(\epsilon^2) . \qquad (51)$$

As anticipated, the critical exponents depend only on the spatial dimension, not on the strength of the non-linear coupling or other non-universal parameters. Note that at $d_c = 4$, (49) is solved by $\tilde{u}(\ell) = \tilde{u}(0)/[1 + 3\tilde{u}(0)\ell/16\pi^2]$, a very slow approach to the Gaussian fixed point that induces *logarithmic corrections* to the mean-field critical exponents, see (81) below. We finally remark that the RG procedure generates novel coupling terms $\sim S^6$, $\nabla^2 S^4$, etc. To order ϵ^3, their feedback into the recursion relations can however be safely neglected.

In summary, the renormalization group procedure as outlined above in Wilson's momentum shell formulation allows us to *derive* hitherto phenomenological scaling laws, and thereby gain deeper insights into scale-invariant features. We have also seen that the number of relevant couplings (two at standard critical points, namely τ and h) equals the number of *independent* critical exponents. Below the *upper critical dimension*, fluctuation corrections modify the critical scaling drastically as compared to the mean-field predictions. Finally, perturbative calculations that are safely carried out in the UV regime may be employed to systematically compute scaling exponents through a power series in the dimensional parameter $\epsilon = d_c - d$.

3. Field Theory Approach to Critical Phenomena

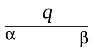
While Wilson's momentum shell scheme renders the basic philosophy of the renormalization group transparent, it becomes computationally quite cumbersome once nested momentum integrals appear beyond the first order in perturbation theory. Unnecessary technical complications in evaluating fluctuation loops can be avoided by extending the UV cutoff to $\Lambda \to \infty$, at the price of divergences in dimensions $d \geq d_c$. However, we already know that the Gaussian theory governs the infrared properties in that dimensional regime; thus in statistical physics theses UV singularities do not really pose a troublesome issue. We may however employ powerful various tools from quantum field theory, proceed to formally renormalize the UV divergences, and thereby gain crucial information about the desired IR scaling limit, provided an IR-stable RG fixed point can be identified that allows us to connect the UV and IR regimes. This chapter provides a succinct overview of how to construct the perturbation expansion in terms of *Feynman diagrams* for one-particle irreducible vertex functions, proceeds to analyze the resulting UV singularities, and finally utilizes the renormalization group equation to identify fixed points and determine the accompanying critical exponents. For more extensive treatments of the field-theoretic RG approach to critical phenomena, see Refs. [10]–[15] and other excellent texts.

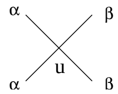
3.1. Perturbation expansion and Feynman diagrams

We now generalize our analysis to a LGW Hamiltonian with *continuous* $O(n)$ order parameter symmetry

$$\mathcal{H}[\vec{S}] = \int d^d x \sum_{\alpha=1}^{n} \left[\frac{r}{2}\, S^\alpha(x)^2 + \frac{1}{2}\, [\nabla S^\alpha(x)]^2 + \frac{u}{4!} \sum_{\beta=1}^{n} S^\alpha(x)^2\, S^\beta(x)^2 \right] , \qquad (52)$$

which encapsulates the critical behavior for the Heisenberg model for a three-component vector order parameter ($n = 3$), the planar XY model (and equivalently, superfluids with complex scalar order parameter) for $n = 2$, reduces to Ising Z_2 symmetry for $n = 1$, and in fact describes the scaling properties of self-avoiding polymers in the limit $n \to 0$. As in (40), one constructs the perturbation expansion for arbitrary N-point functions $\langle \prod_i S^{\alpha_i} \rangle$ in terms of averages within the Gaussian ensemble with $u = 0$ (keeping $k_B T = 1$). Diagrammatically, the Gaussian two-point functions or *propagators* $C_0(q)\,\delta^{\alpha\beta} = \delta^{\alpha\beta}/(r + q^2)$, diagonal in the field component indices, are represented through lines, to be connected through the non-linear *vertices* $-\frac{u}{6}$:

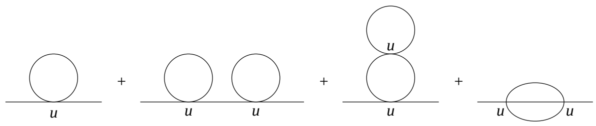




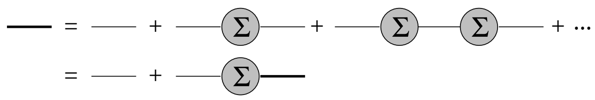
In the presence of external fields h^α, the partition function serves as *generating functional* for correlation functions (cumulants):

$$\mathcal{Z}[h] = \left\langle \exp \int d^d x \sum_\alpha h^\alpha S^\alpha \right\rangle ,$$
$$\left\langle \prod_i S^{\alpha_i} \right\rangle_{(c)} = \prod_i \frac{\delta (\ln) \mathcal{Z}[h]}{\delta h^{\alpha_i}} \Big|_{h=0} . \qquad (53)$$

The *cumulants* are graphically represented through *connected Feynman diagrams*; e.g., for the propagator:



Note that the second graph is a mere repetition of the first; indeed, upon defining the *self-energy* Σ as the sum of all *one-particle irreducible* Feynman graphs that cannot be split into lower-order contributions simply by cutting a propagator line, one infers the following general structure for the full propagator $C(q)$:



The second line is a graphical depiction of *Dyson's equation* that reads in Fourier space $C(q) = C_0(q) + C_0(q)\, \Sigma(q)\, C(q)$, solved by $C(q)^{-1} = C_0(q)^{-1} - \Sigma(q)$.

In order to similarly eliminate redundancies for arbitrary N-point functions, one proceeds with a Legendre transformation to construct the generating functional for *vertex functions*:

$$\Gamma[\Phi] = -\ln \mathcal{Z}[h] + \int d^d x \sum_\alpha h^\alpha \, \Phi^\alpha ,$$
$$\Gamma^{(N)}_{\{\alpha_i\}} = \prod_i^N \frac{\delta \Gamma[\Phi]}{\delta \Phi^{\alpha_i}} \Big|_{h=0} , \qquad (54)$$

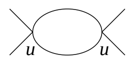
where $\Phi^\alpha = \delta \ln \mathcal{Z}[h] / \delta h^\alpha$. Through appropriate functional derivatives, these vertex functions can be related to the corresponding cumulants, for example for the two- and four-point functions:

$$\Gamma^{(2)}(q) = C(q)^{-1} ,$$
$$\left\langle \prod_{i=1}^4 S(q_i) \right\rangle_c = - \prod_{i=1}^4 C(q_i)\, \Gamma^{(4)}(\{q_i\}) . \qquad (55)$$

By means of these relations one easily confirms that the perturbation series for the vertex functions precisely consist of the *one-particle irreducible* Feynman graphs for the associated cumulants. Moreover, the perturbative expansion with respect to the non-linear coupling u diagrammatically translates to a *loop expansion*. Explicitly, one obtains for the *two-point vertex function* to two-loop order (resulting from the first, third, and fourth graph above):

$$\Gamma^{(2)}(q) = r + q^2 + \frac{n+2}{6}\, u \int_k \frac{1}{r + k^2} \qquad (56)$$
$$- \left(\frac{n+2}{6}\, u \right)^2 \int_k \frac{1}{r + k^2} \int_{k'} \frac{1}{(r + k'^2)^2}$$
$$- \frac{n+2}{18}\, u^2 \int_k \frac{1}{r + k^2} \int_{k'} \frac{1}{r + k'^2} \, \frac{1}{r + (q - k - k')^2} ;$$

for the *four-point vertex function* to one-loop order, we just have the single Feynman diagram



which yields at vanishing external wave vectors

$$\Gamma^{(4)}(\{q_i = 0\}) = u - \frac{n+8}{6}\, u^2 \int_k \frac{1}{(r + k^2)^2} . \qquad (57)$$

3.2. UV and IR divergences, renormalization

Let us now investigate the fluctuation correction (57) to the four-point vertex function. In dimensions $d < 4$, we can safely set the UV cutoff $\Lambda \to \infty$, and obtain after rendering the integral dimensionless:

$$u \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{(r + k^2)^2} =$$
$$\frac{u\, r^{-2+d/2}}{2^{d-1} \pi^{d/2} \Gamma(d/2)} \int_0^\infty \frac{x^{d-1}}{(1 + x^2)^2}\, dx . \qquad (58)$$

Observe that the true *effective* coupling in the perturbation expansion is not u, but $u\, r^{(d-4)/2} \to \infty$ as $r \to 0$, indicating the emerging *infrared* divergences that render a direct perturbative approach meaningless in the critical regime. We conclude once again that for $d < d_c = 4$, fluctuation corrections are IR-singular, and consequently expect the critical power laws to be modified as compared to the mean-field or Gaussian approximations. In contrast, the integral remains regular in the infrared for $d > 4$, but becomes ultraviolet-divergent. Indeed, keeping the cutoff finite, we have in dimensions larger or equal to the *upper critical dimension* $d_c = 4$:

$$\int_0^\Lambda \frac{k^{d-1}}{(r + k^2)^2}\, dk \sim \begin{cases} \ln(\Lambda^2 / r) & d = 4 \\ \Lambda^{d-4} & d > 4 \end{cases} , \qquad (59)$$

which both diverge as $\Lambda \to \infty$. Notice that at $d_c = 4$, the logarithmic IR and UV divergences are coupled, signaling the scale invariance of the LGW Hamiltonian in four dimensions.

One may take a convenient shortcut to determine the critical dimension through simple *power counting* in terms of an arbitrary momentum scale μ. Lengths then scale as $[x] = \mu^{-1}$, wave vectors as $[q] = \mu$, and fields consequently have the (naive) scaling dimension $[S^\alpha(x)] = \mu^\zeta = \mu^{-1+d/2}$. As noted before, the fields become dimensionless in two dimensions. For the LGW Hamiltonian (52) with continuous rotational symmetry in order parameter space, that is also just the *lower critical dimension*: For $d \leq d_{lc} = 2$, the system cannot maintain spatially homogeneous long-range order if $n \geq 3$; and at $d_{lc} = 2$ and for $n = 2$ only quasi-long-range order may exist with algebraically decaying correlations and temperature-dependent decay exponent (Berezinskii–Kosterlitz–Thouless scenario). For the couplings in (52) one infers $[r] = \mu^2$, which means that the temperature control parameter constitutes a *relevant* coupling, and $[u] = \mu^{4-d}$, which is relevant for $d < 4$, *marginal* at $d_c = 4$, and (dangerously) irrelevant for $d > 4$. Furthermore, dimensional analysis confirms that fluctuation loops become UV-divergent only for the vertex functions $\Gamma^{(2)}(q)$, but only up to order q^2 in a long-wavelength expansion, and $\Gamma^{(4)}(\{q_i = 0\})$. The following table summarizes the mathematical distinctions and their physical implications in the different dimensional regimes.

dim. range	perturb. series	$O(n)$ Φ^4 field theory	critical behavior
$d \leq 2$	IR-sing. UV-conv.	ill-defined u relevant	no long-range order ($n \geq 2$)
$d < 4$	IR-sing. UV-conv.	super-ren. u relevant	non-classical exponents
$d_c = 4$	log. IR-/ UV-div.	renorm. u marginal	logarithmic corrections
$d > 4$	IR-reg. UV-div.	non-renorm. u irrelevant	mean-field exponents

It is useful to perform the loop integrations in *dimensional regularization*; i.e., to assign the following values to wave vector integrals, even for non-integer d and σ:

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^{2\sigma}}{(\tau + k^2)^s} = \frac{\Gamma(\sigma + d/2)\,\Gamma(s - \sigma - d/2)}{2^d\, \pi^{d/2}\, \Gamma(d/2)\, \Gamma(s)}\, \tau^{\sigma - s + d/2} \,. \tag{60}$$

For $d < d_c$, where no UV divergences appear, this result follows directly by introducing spherical coordinates in momentum space. Beyond the upper critical dimension, essentially divergent surface integrals are discarded in (60). UV singularities become manifest as *dimensional poles* in Euler's Γ functions.

We may now proceed with the renormalization program. The goal is to absorb the UV divergences into *renormalized* couplings that through this procedure become scale-dependent. Beginning with the order parameter susceptibility, we naturally demand that $\chi^{-1} = C(q = 0)^{-1} = \Gamma^{(2)}(q = 0) = \tau = r - r_c$, i.e., χ diverges at the true critical temperature T_c. From (56) we thus obtain to first order in u,

$$r_c = -\frac{n+2}{6}\, u \int_k \frac{1}{r_c + k^2} + O(u^2) = -\frac{n+2}{6}\, u\, S_d\, \frac{\Lambda^{d-2}}{d-2} + O(u^2) \,, \tag{61}$$

which is to be interpreted as a non-universal fluctuation-induced downward shift of the critical temperature, the analog of r^* in Wilson's scheme. As $\Lambda \to \infty$, r_c becomes quadratically UV-divergent near four dimensions; this divergence becomes absorbed into the new proper temperature variable τ by means of an *additive renormalization*. Inserting (61) into (56) then yields to order u

$$\chi(q)^{-1} = q^2 + \tau \left[1 - \frac{n+2}{6}\, u \int_k \frac{1}{k^2(\tau + k^2)} \right] . \tag{62}$$

At the upper critical dimension, (62) and (57) are logarithmically divergent as $\Lambda \to \infty$, showing up as $1/\epsilon$ poles in the dimensionally regularized integral values. These UV poles are subsequently absorbed into *renormalized* fields S^α_R and parameters through the following *multiplicative renormalization* prescription:

$$S^\alpha_R = Z_S^{1/2}\, S^\alpha \;\Rightarrow\; \Gamma^{(N)}_R = Z_S^{-N/2}\, \Gamma^{(N)} \,; \tag{63}$$

$$\tau_R = Z_\tau\, \tau\, \mu^{-2} \,, \quad u_R = Z_u\, u\, A_d\, \mu^{d-4} \,, \tag{64}$$

which defines *dimensionless renormalized couplings* τ_R and u_R, and where $A_d = \Gamma(3 - d/2)/2^{d-1}\, \pi^{d/2}$ is a regular (near $d_c = 4$) geometric factor. To this end, one must carefully avoid the IR-singular regime, i.e., evaluate the fluctuation integrals at a safe *normalization point*, e.g., $\tau_R = 1$ (or $q = \mu$). In the *minimal subtraction* procedure, the renormalization constants Z_S in (63) and Z_τ, Z_u in (64) are chosen to contain *only* the $1/\epsilon$ poles and their residua. This leads to the following Z factors

$$O(u_R):\; Z_\tau = 1 - \frac{n+2}{6}\, \frac{u_R}{\epsilon} \,, \tag{65}$$

$$Z_u = 1 - \frac{n+8}{6}\, \frac{u_R}{\epsilon} \,, \tag{66}$$

$$O(u_R^2):\; Z_S = 1 + \frac{n+2}{144}\, \frac{u_R^2}{\epsilon} \,, \tag{67}$$

all calculated to first non-trivial order in u_R by means of dimensional regularization (60) and within the minimal subtraction prescription. Z_τ and Z_u follow directly from the one-loop results (62) and (57), whereas $Z_S = 1$ to $O(u_R)$ due to the absence of any wave vector dependence in the "Hartree" loop, whence field renormalization only ensues to two-loop order from the rightmost "sunset" Feynman diagram in the propagator self-energy, and the $1/\epsilon$ pole in the final expression in (56).

3.3. RG equation and critical exponents

Through the selection of a normalization point well outside the critical regime, the renormalized fields and parameters in (63), (64) explicitly carry the momentum scale μ. On the other hand, the *unrenormalized* quantities, including the N-point vertex functions, naturally do *not* depend on this arbitrary scale μ:

$$0 = \frac{d}{d\mu}\,\Gamma^{(N)}(\tau, u) = \frac{d}{d\mu}\left[Z_S^{N/2}\,\Gamma_R^{(N)}(\mu, \tau_R, u_R)\right]. \quad (68)$$

Carrying out the derivative with respect to μ by taking into account the scale dependence of Z_s, τ_R, and u_R, (68) can be rewritten as a partial differential equation,

$$\left[\mu\,\frac{\partial}{\partial\mu} + \frac{N}{2}\,\gamma_S + \gamma_\tau\,\tau_R\,\frac{\partial}{\partial\tau_R} + \beta_u\,\frac{\partial}{\partial u_R}\right] \Gamma_R^{(N)}(\mu, \tau_R, u_R) = 0\,. \quad (69)$$

This Gell-Mann–Low *renormalization group equation* carries crucial information on the fundamental scale dependence of the *renormalized* physical system, here rendered explicit for the vertex functions. In (69) we have introduced *Wilson's flow functions* defined as

$$\gamma_S = \mu\,\frac{\partial}{\partial\mu}\Big|_0\,\ln Z_S = -\frac{n+2}{72}\,u_R^2 + O(u_R^3)\,, \quad (70)$$

$$\gamma_\tau = \mu\,\frac{\partial}{\partial\mu}\Big|_0\,\ln\frac{\tau_R}{\tau} = -2 + \frac{n+2}{6}\,u_R + O(u_R^2)\,, \quad (71)$$

where the second lines follow from the lowest-order results (67) and (65), and the RG beta function for the non-linear coupling u,

$$\beta_u = \mu\,\frac{\partial}{\partial\mu}\Big|_0 u_R = u_R\left[d - 4 + \mu\,\frac{\partial}{\partial\mu}\Big|_0\,\ln Z_u\right] = u_R\left[-\epsilon + \frac{n+8}{6}\,u_R + O(u_R^2)\right], \quad (72)$$

where (66) has been inserted.

The first-order linear partial differential equation (69) can next be formally solved via the standard *method of characteristics*; to this end, one lets $\mu \to \mu(\ell) = \mu\,\ell$, with a dimensionless scale parameter ℓ; note that in contrast to the convention in Chap. 2.5, the IR regime is now reached in the limit $\ell \to 0$. Inserting this parametrization into the RG equation (69), one obtains an equivalent set of coupled first-order ordinary differential equations, namely the *RG flow equations* for the *running couplings*

$$\ell\,\frac{d\tilde\tau(\ell)}{d\ell} = \tilde\tau(\ell)\,\gamma_\tau(\ell)\,, \quad \ell\,\frac{d\tilde u(\ell)}{d\ell} = \beta_u(\ell)\,, \quad (73)$$

with initial values $\tilde\tau(1) = \tau_R$, $\tilde u(1) = u_R$, and similar differential equations for the N-point vertex functions, which involve their overall naive scaling dimensions and the anomalous contributions stemming from the field renormalization, as encoded in (70).

For example, for the susceptibility $\chi(q) = \Gamma^{(2)}(q)^{-1}$, one has $\chi_R(\mu, \tau_R, u_R, q)^{-1} = \mu^2\,\hat\chi_R(\tau_R, u_R, q/\mu)^{-1}$, and its RG flow correspondingly integrates to

$$\chi_R(\ell)^{-1} = \chi_R(1)^{-1}\,\ell^2\,\exp\left[\int_1^\ell \gamma_S(\ell')\,\frac{d\ell'}{\ell'}\right]. \quad (74)$$

Near an *infrared-stable* RG fixed point u^*, i.e., a zero of the RG beta function $\beta_u(u^*) = 0$ with $\beta_u'(u^*) > 0$, the flow equation for the running temperature variable is readily solved: $\tilde\tau(\ell) \approx \tau_R\,\ell^{\gamma_\tau^*}$, where $\gamma_\tau^* = \gamma_\tau(u^*)$. Inserting this and the fixed point value $\gamma_S^* = \gamma_S(u^*)$ into (74) yields the following general scaling form

$$\chi_R(\tau_R, q)^{-1} \approx \mu^2\,\ell^{2+\gamma_S^*}\,\hat\chi_R(\tau_R\,\ell^{\gamma_\tau^*}, u^*, q/\mu\,\ell)^{-1}\,. \quad (75)$$

With the *matching* condition $\ell = |q|/\mu$, one thus recovers (23), with the identifications $\eta = -\gamma_S^*$ and $\nu = -1/\gamma_\tau^*$ for the critical exponents.

Our perturbative RG analysis of the $O(n)$-symmetric LGW Hamiltonian (52) yielded the one-loop beta function (72), whose zeros are (i) the Gaussian fixed point $u_0^* = 0$, stable for $\epsilon < 0$ or $d > d_c = 4$, obviously resulting in the mean-field exponents $\eta = 0$ and $\nu = \frac{1}{2}$; and (ii) the non-trivial *Heisenberg fixed point*

$$u_H^* = \frac{6\,\epsilon}{n+8} + O(\epsilon^2)\,, \quad (76)$$

which exists and becomes IR-stable for $\epsilon > 0$, i.e., in dimensions $d < 4$. This allows us to compute the critical exponents in a systematic $\epsilon = 4 - d$ expansion,

$$\eta = \frac{n+2}{2\,(n+8)^2}\,\epsilon^2 + O(\epsilon^3)\,, \quad (77)$$

$$\nu^{-1} = 2 - \frac{n+2}{n+8}\,\epsilon + O(\epsilon^2)\,. \quad (78)$$

Aside from the dimensionality, these values only depend on the number of order parameter components n. Note that (78) reduces to the Ising exponent values (51) for $n = 1$; in the limit $n \to \infty$ one also finds the correct exponents for the exactly solvable *spherical model*, namely $\eta = 0$ and $\nu = 1/(2 - \epsilon) = 1/(d - 2)$, which diverges at the lower critical dimension $d_{lc} = 2$. At the upper critical dimension $d_c = 4$, the solution of the flow equation for the running non-linear coupling reads

$$\tilde{u}(\ell) = \frac{u_R}{1 - \frac{n+8}{6}\, u_R \, \ln \ell} \, , \tag{79}$$

whence approximately

$$\tilde{\tau}(\ell) \sim \frac{\tau_R}{\ell^2 (\ln |\ell|)^{(n+2)/(n+8)}} \, , \tag{80}$$

which in turn implies that the correlation length divergence picks up *logarithmic* corrections to the mean-field behavior,

$$\xi \propto \tau_R^{-1/2} \, (\ln \tau_R)^{(n+2)/2(n+8)} \, . \tag{81}$$

The field-theoretic formulation of the renormalization group provides us with an elegant and powerful tool to extract the proper infrared scaling properties in low dimensions $d \leq d_c$ from a continuum theory via a careful analysis of its ultraviolet singularities that appear in dimensions $d \geq d_c$. The renormalization group equation carries information on the scale dependence of physical parameters and correlation functions, and allows to make connections between the UV and IR limits provided a stable RG fixed point can be identified. One may then systematically derive scaling laws, and acquire a thorough understanding of the origin of universality and its realm of validity for a specific physical system. Moreover, we have seen that a perturbative analysis allows a controlled computation of critical exponents (and also of scaling functions) in the framework of a dimensional expansion near the upper critical dimension. This ϵ expansion certainly provides useful information on overall trends, and can in some instances even be rendered to a precision calculation if sufficiently high orders in the perturbation series can be evaluated and subsequently be refined through Borel resummations. In addition, modern non-perturbative “exact” numerical RG methods have succeeded in yielding very accurate results. It should also be stressed that it is of course the very concept of universality that also allows us to infer meaningful information from numerical simulations of simplified lattice or continuum models.

4. Critical Dynamics

We now venture to investigate *dynamic* critical phenomena near continuous phase transitions, first in the vicinity of critical point in thermal equilibrium, and later at genuine non-equilibrium phase transitions in externally driven, non-isolated systems. The natural time scale separation between the slow kinetics of the order parameter (along with any other conserved fields) and fast, non-critical degrees of freedom suggests a phenomenological description in terms of non-linear stochastic Langevin-type differential equations, and allows a generalization of universal scaling laws to time-dependent phenomena. Distinct dynamical universality classes ensue dependent on the order parameter itself representing a conserved quantity or not, and potentially its dynamical coupling to other conserved hydrodynamic modes [16]. We begin by writing down scaling laws for dynamical response and correlation functions, and subsequently introduce effective mesoscopic Langevin equations, with stochastic noise correlations that near thermal equilibrium are constrained by fluctuation-dissipation relations. It is then demonstrated how such non-linear stochastic partial differential equations can be mapped onto a field theory representation via the Janssen–De Dominicis response functional, which in turn may be analyzed by means of the field-theoretic renormalization group tools developed in the previous chapter. We will specifically construct a dynamic perturbation theory expansion and determine the universal scaling behavior along with the dynamic critical exponents for the relaxational models A and B with non-conserved and conserved order parameter, respectively; for more in-depth treatments, the reader is referred to Refs. [17]–[19] and [9, chaps. 4,5]. In addition to purely relaxational kinetics, we shall also address the critical dynamics of isotropic ferromagnets [20], as well as generic scale invariance and non-equilibrium phase transitions in driven diffusive systems [21].

4.1. Dynamical scaling hypothesis

Let us first recall the behavior of the static order parameter correlation function and susceptibility near a critical point ($h = 0$ and $\tau \to 0$), captured by the scaling laws (23) and (25), and induced by the divergence of the characteristic *correlation length*, $\xi(\tau) \sim |\tau|^{-\nu}$. As spatially correlated regions grow tremendously upon approaching the phase transition, one expects the typical relaxation time associated with the order parameter kinetics to increase as well, $t_c(\tau) \sim \xi(\tau)^z \sim |\tau|^{-z\nu}$. This phenomenon of *critical slowing-down* is governed by the *dynamic critical exponent* $z = \nu_t/\nu$, which can also be visualized as the ratio between the exponents that describe the divergence of correlations in the temporal and spatial "directions", respectively. We may thus write down a *dynamic scaling* ansatz for the corresponding wavevector-dependent *characteristic frequency* scale,

$$\omega_c(\tau, q) = |q|^z \, \hat{\omega}_\pm(q\,\xi) \, , \tag{82}$$

with $\hat{\omega}_\pm(y \to \infty) \to$ const., whence the *critical dispersion relation* becomes $\omega_c(0, q) \sim |q|^z$.

We are particularly interested in describing the time dependence for the order parameter response and correlation functions:

$$\chi(x - x', t - t') = \left. \frac{\partial \langle S(x,t) \rangle}{\partial h(x', t')} \right|_{h=0} , \tag{83}$$

$$C(x, t) = \langle S(x,t)\, S(0,0) \rangle - \langle S \rangle^2 \, , \tag{84}$$

where we are considering a stationary dynamical regime where spatial and temporal time translation invariance holds. In *thermal equilibrium* (only !), the spatio-temporal Fourier transforms of these functions are intimately connected through the *fluctuation-dissipation theorem* (FDT)

$$C(q, \omega) = 2 k_{\rm B} T \, {\rm Im} \, \frac{\chi(q, \omega)}{\omega} \, . \tag{85}$$

Generalizing the static scaling laws (23) and (25), we may then formulate the *dynamical scaling hypothesis* for the asymptotic critical properties of the time-dependent susceptibility and correlation function:

$$\chi(\tau, q, \omega) = |q|^{-2+\eta} \, \hat{\chi}_\pm(q\,\xi, \omega\,\xi^z) \, , \tag{86}$$

$$C(\tau, x, t) = |x|^{-(d-2+\eta)} \, \widetilde{C}_\pm(x/\xi, t/t_c) \, . \tag{87}$$

As a consequence of the stringent constraints imposed by the FDT (85), the same *three* independent critical exponents ν, η, and z characterize the universal scaling regimes in both (86) and (87). Away from thermal equilibrium, where the FDT restrictions do not apply, the dynamic response and correlation functions are however in general characterized by distinct scaling exponents. We remark that appropriate variants of the dynamical scaling hypothesis may also be postulated for transport coefficients.

4.2. Langevin dynamics and Gaussian theory

The critical slowing-down of the order parameter kinetics produces an effective time-scale separation between the critical degrees of freedom, additional conserved hydrodynamic modes that might be present, and all other comparatively "fast" variables. This observation naturally calls for a *mesoscopic Langevin description* of critical dynamics, where the fast degrees of freedom are treated as mere white *noise* that randomly affects the few slow modes in the system. In such a *coarse-grained* picture, one writes down coupled stochastic equations of motion for the order parameter and perhaps any other conserved fields that reflect their intrinsic microscopic reversible dynamics as well as irreversible relaxation kinetics, the latter connected in thermal equilibrium with the noise strengths through *Einstein relations* or FDTs. Generally the various possible *mode couplings* of the order parameter to additional conserved, and consequently diffusively slow modes leads to a splitting of the static into several *dynamic universality classes* [16, 18, 9].

Here we shall assume that the order parameter field is decoupled from any other slow modes, and first focus on its purely relaxational kinetics [9, Chaps. 4,5]. If the order parameter itself is not a conserved quantity, any deviation from thermal equilibrium will just tend to relax back to the minimizing configuration of the free energy, e.g. given by the $O(n)$-symmetric LGW Hamiltonian (52):

$$\frac{\partial S^\alpha(x,t)}{\partial t} = -D \, \frac{\delta \mathcal{H}[\vec{S}]}{\delta S^\alpha(x,t)} + \zeta^\alpha(x,t) \, , \tag{88}$$

with Gaussian white noise that is fully characterized by its first two moments,

$$\begin{aligned} &\langle \zeta^\alpha(x,t) \rangle = 0 \, , \\ &\langle \zeta^\alpha(x,t)\, \zeta^\beta(x',t') \rangle = \\ &\qquad 2 D k_{\rm B} T \, \delta(x - x') \delta(t - t') \delta^{\alpha\beta} \, . \end{aligned} \tag{89}$$

As can be inferred from the associated Fokker–Planck equation, the *Einstein relation* that connects the noise correlator strength with the relaxation constant D and temperature guarantees that the probability distribution for the field S^α asymptotically approaches the canonical stationary distribution $\mathcal{P}[\vec{S}, t] \to \mathcal{P}_s[\vec{S}] \propto \exp(-\mathcal{H}[\vec{S}]/k_{\rm B}T)$ as $t \to \infty$.

If the order parameter is *conserved* under the dynamics, satisfying a continuity equation, its spatial fluctuations can only relax *diffusively*; as a consequence, one needs to replace the constant relaxation rate D by the diffusion operator $-D\,\nabla^2$, both in the Langevin equation (88) and the noise correlation (90). In the following, we shall treat both these situations simultaneously, letting $D \to D(i\nabla)^a$, where either $a = 0$, corresponding to the purely relaxational *model A* of critical dynamics as appropriate for a non-conserved order parameter; or $a = 2$, which describes *model B* with a conserved order parameter field.

Let us again begin with the *Gaussian* or *mean-field approximation*, where we set the non-linear coupling

$u = 0$. A Fourier transform in space and time according to $S^\alpha(x,t) = \int \frac{d^d q}{(2\pi)^d} \int \frac{d\omega}{2\pi} S(q,\omega)\, e^{iq\cdot x - i\omega t}$ of the thus linearized Langevin equation (88) yields

$$\left[-i\omega + Dq^a(r+q^2)\right] S^\alpha(q,\omega) = Dq^a\, h^\alpha(q,\omega) + \zeta^\alpha(q,\omega) \, , \qquad (90)$$

with $\langle \zeta^\alpha(q,\omega)\rangle = 0$ and

$$\langle \zeta^\alpha(q,\omega)\zeta^\beta(q',\omega')\rangle = 2k_{\rm B}T\, Dq^a \, (2\pi)^{d+1}\delta(q+q')\delta(\omega+\omega')\delta^{\alpha\beta} \, , \qquad (91)$$

and where the external field term $-\sum_\alpha h^\alpha S^\alpha$ has been added to the LGW Hamiltonian (52). A derivative of (90) with respect to the external field then immediately gives the *dynamic response function*

$$\chi_0^{\alpha\beta}(q,\omega) = \frac{\partial\langle S^\alpha(q,\omega)\rangle}{\partial h^\beta(q,\omega)}\Big|_{h=0} = Dq^a G_0(q,\omega)\delta^{\alpha\beta} \, , \qquad (92)$$
$$G_0(q,\omega) = \frac{1}{-i\omega + Dq^a(r+q^2)} \, .$$

Its temporal Fourier backtransform of course satisfies *causality*: $G_0(q,t)$ vanishes for $t < 0$, and reads

$$G_0(q,t) = e^{-Dq^a(r+q^2)t}\, \Theta(t) \, , \qquad (93)$$

from which we infer the characteristic relaxation rate $t_c^{-1} = Dq^a(r+q^2)$: For model A ($a = 0$), the order parameter relaxes diffusively at the critical point (i.e., $z = 2$), while for model B ($a = 2$) critical slowing-down induces a crossover from Drq^2 to Dq^4 as $r \to 0$ ($z = 4$). For $h^\alpha = 0$, the dynamic correlation function is readily obtained from (90) and (91):

$$\langle S^\alpha(q,\omega)\, S^\beta(q',\omega')\rangle_0 = C_0(q,\omega) \, (2\pi)^{d+1}\delta(q+q')\delta(\omega+\omega')\delta^{\alpha\beta} \, ,$$
$$C_0(q,\omega) = \frac{2k_{\rm B}T\, Dq^a}{\omega^2 + [Dq^a(r+q^2)]^2} = 2k_{\rm B}T Dq^a\, |G_0(q,\omega)|^2 \, , \qquad (94)$$
$$C_0(q,t) = \frac{k_{\rm B}T}{r+q^2}\, e^{-Dq^a(r+q^2)|t|} \, . \qquad (95)$$

Comparing these results with (86) and (87), one again identifies the static *Gaussian critical exponents* $\nu = \frac{1}{2}$ and $\eta = 0$, and the *mean-field dynamic exponent* $z = 2 + a$ for the purely relaxational models A and B.

4.3. Field theory representations of Langevin dynamics

This subsection describes how stochastic Langevin equations of motion can be mapped onto continuous field theory representations. To this end, we consider the following general coupled Langevin equations for some mesoscopic stochastic variables S^α:

$$\frac{\partial S^\alpha(x,t)}{\partial t} = F^\alpha[S](x,t) + \zeta^\alpha(x,t) \, , \qquad (96)$$
$$\langle \zeta^\alpha(x,t)\zeta^\beta(x',t')\rangle = 2L^\alpha[S]\, \delta(x-x')\delta(t-t')\delta^{\alpha\beta} \, . \qquad (97)$$

Naturally we assume $\langle \zeta^\alpha(x,t)\rangle = 0$ here, since a non-vanishing mean of the *stochastic forces* or *noise* could just be included in the *systematic* forces $F^\alpha[S]$. Note that the *noise correlator* L^α may be an operator, as is the case for conserved variables, and could also be a functional of the slow fields S^α. The crucial input is that we assume the noise history to represent a *Gaussian* stochastic process, whose probability distribution if completely determined by the second moment (97):

$$\mathcal{W}[\zeta] \propto \exp\left[-\frac{1}{4}\int d^dx \int_0^{t_f} dt \sum_\alpha \zeta^\alpha (L^\alpha)^{-1} \zeta^\alpha\right] . \qquad (98)$$

Switching dynamical variables from the noise ζ^α to the slow stochastic fields S^α yields $\mathcal{W}[\zeta]\,\mathcal{D}[\zeta] = \mathcal{P}[S]\mathcal{D}[S] \propto e^{-\mathcal{G}[S]}\mathcal{D}[S]$, with the *Onsager-Machlup functional* providing the associated exponential weight that may be viewed as a field theory action:

$$\mathcal{G}[S] = \frac{1}{4}\int d^dx \int_0^{t_f} dt \sum_\alpha (\partial_t S^\alpha - F^\alpha[S]) \left[(L^\alpha)^{-1}(\partial_t S^\alpha - F^\alpha[S])\right] . \qquad (99)$$

The observant reader will have noticed that the functional determinant stemming from the variable transformation has been ignored here; however, upon utilizing a *forward* time discretization, i.e., the Itô interpretation for non-linear stochastic processes, this functional determinant turns out to be constant, and can simply be absorbed into the functional integral measure. Notice also that the overall normalization $\int \mathcal{D}[\zeta] W[\zeta] = 1$ implies the corresponding "partition function" to be unity, and hence to carry no information, in stark contrast with thermal equilibrium statistical mechanics. While the Onsager–Machlup functional (99) provides a desirable field theory representation of stochastic Langevin dynamics, it is also plagued by two technical problems: First, it contains $(L^\alpha)^{-1}$, which for conserved fields entails an inverse differential operator or Laplacian Green's function; second, it includes the square of the systematic force terms $F^\alpha[S]$ and consequently highly non-linear contributions. It is thus beneficial to partially linearize the above action by means of a Hubbard–Stratonovich transformation, at the expense of introducing an *additional dynamical field variable*.

In order to completely avoid any possible singularities incorporated in the inverse operator $(L^\alpha)^{-1}$, we follow another more direct route here. The goal is to compute averages of observables A that should be functionals of the slow modes S^α over noise "histories": $\langle A[S]\rangle_\zeta \propto \int \mathcal{D}[\zeta]\, A[S(\zeta)]\, W[\zeta]$. Inserting a rather involved representation of unity in terms of a product of Dirac delta distributions on each space-time point, and subsequently writing these as integrals over auxiliary fields $\widetilde{S}^\alpha$ along the imaginary axis, $1 = \int \mathcal{D}[S] \prod_\alpha \prod_{(x,t)} \delta\left(\partial_t S^\alpha(x,t) - F^\alpha[S](x,t) - \zeta^\alpha(x,t)\right) = \int \mathcal{D}[i\widetilde{S}] \int \mathcal{D}[S] \exp\left[-\int d^dx \int dt \sum_\alpha \widetilde{S}^\alpha\left(\partial_t S^\alpha - F^\alpha - \zeta^\alpha\right)\right]$, we arrive at

$$\langle A[S]\rangle_\zeta \propto \int \mathcal{D}[i\widetilde{S}] \int \mathcal{D}[S]\, A[S] \int \mathcal{D}[\zeta] \exp\left[-\int d^dx \int dt \sum_\alpha \widetilde{S}^\alpha\left(\partial_t S^\alpha - F^\alpha[S]\right)\right] \exp\left(-\int d^dx \int dt \sum_\alpha \left[\frac{\zeta^\alpha (L^\alpha)^{-1} \zeta^\alpha}{4} - \widetilde{S}^\alpha \zeta^\alpha\right]\right) .$$

Performing the Gaussian integral over the noise ζ^α finally yields

$$\langle A[S]\rangle_\zeta = \int \mathcal{D}[S]\, A[S]\, \mathcal{P}[S] \, , \quad \mathcal{P}[S] \propto \int \mathcal{D}[i\widetilde{S}]\, e^{-\mathcal{A}[\widetilde{S},S]} \, . \tag{100}$$

with the *Janssen–De Dominicis response functional*

$$\mathcal{A}[\widetilde{S},S] = \int d^dx \int_0^{t_f} dt \sum_\alpha \left[\widetilde{S}^\alpha\left(\partial_t S^\alpha - F^\alpha[S]\right) - \widetilde{S}^\alpha L^\alpha[S]\, \widetilde{S}^\alpha\right] . \tag{101}$$

The stochastic dynamics is now encoded in *two* distinct sets of mesoscopic fields, namely the original slow variables S^α and the associated auxiliary fields $\widetilde{S}^\alpha$. Once again, the Gaussian noise normalization implies $\int \mathcal{D}[i\widetilde{S}] \int \mathcal{D}[S]\, e^{-\mathcal{A}[\widetilde{S},S]} = 1$; furthermore, integrating out the auxiliary fields $\widetilde{S}^\alpha$ recovers the Onsager–Machlup functional (99).

Specifically for the purely relaxational models A and B, the response functional reads (we now set $k_\mathrm{B}T = 1$):

$$\mathcal{A}[\widetilde{S},S] = \int d^dx \int dt \sum_\alpha \Big(\widetilde{S}^\alpha \left[\partial_t + D(i\nabla)^a (r - \nabla^2)\right] S^\alpha - D\widetilde{S}^\alpha (i\nabla)^a \widetilde{S}^\alpha - D\widetilde{S}^\alpha (i\nabla)^a h^\alpha + D\,\frac{u}{6} \sum_\beta \widetilde{S}^\alpha (i\nabla)^a S^\alpha S^\beta S^\beta\Big) , \tag{102}$$

The first two lines here represent the Gaussian action $\mathcal{A}_0$, and the term $\sim u$ the non-linear contributions. By means of (100) and (102), the dynamical order parameter susceptibility becomes

$$\chi^{\alpha\beta}(x-x',t-t') = \frac{\delta\langle S^\alpha(x,t)\rangle}{\delta h^\beta(x',t')}\Big|_{h=0} = D\left\langle S^\alpha(x,t) (i\nabla)^a\, \widetilde{S}^\beta(x',t')\right\rangle ; \tag{103}$$

i.e., the response function can be expressed as an expectation value that involves both order parameter S^α and auxiliary fields $\widetilde{S}^\beta$, whence the latter are also referred to as *"response" fields*. Invoking causality and time inversion symmetry, it is a straightforward exercise to derive the *fluctuation-dissipation theorem*, equivalent to (85):

$$\chi^{\alpha\beta}(x-x',t-t') = \Theta(t-t')\, \frac{\partial}{\partial t'} \left\langle S^\alpha(x,t) S^\beta(x',t')\right\rangle . \tag{104}$$

In analogy with static, equilibrium statistical field theory (53), one defines the *generating functional* for correlation functions and cumulants,

$$\mathcal{Z}[\tilde{j},j] = \left\langle e^{\int d^dx \int dt \sum_\alpha \left(\tilde{j}^\alpha \widetilde{S}^\alpha + j^\alpha S^\alpha\right)}\right\rangle , \tag{105}$$

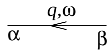
$$\left\langle \prod_{ij} S^{\alpha_i} \widetilde{S}^{\alpha_j}\right\rangle_{(c)} = \prod_{ij} \frac{\delta}{\delta j^{\alpha_i}} \frac{\delta(\ln)\mathcal{Z}[\tilde{j},j]}{\delta \tilde{j}^{\alpha_j}}\Big|_{\tilde{j}=j=0} .$$

4.4. Dynamic perturbation theory

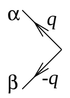
We may now proceed and treat the non-linear terms $\sim u$ by means of a *perturbation expansion*,

$$\left\langle \prod_{ij} S^{\alpha_i}\, \widetilde{S}^{\alpha_j}\right\rangle = \frac{\langle \prod_{ij} S^{\alpha_i}\, \widetilde{S}^{\alpha_j}\, e^{-\mathcal{A}_\mathrm{int}[\widetilde{S},S]}\rangle_0}{\langle e^{-\mathcal{A}_\mathrm{int}[\widetilde{S},S]}\rangle_0} = \left\langle \prod_{ij} S^{\alpha_i}\, \widetilde{S}^{\alpha_j} \sum_{l=0}^\infty \frac{1}{l!} \left(-\mathcal{A}_\mathrm{int}[\widetilde{S},S]\right)^l \right\rangle_0 . \tag{106}$$

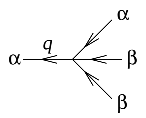
Since the denominator is one owing to noise normalization, there are *no "vacuum" contributions* in this dynamic field theory. Note furthermore that *causality* implies $\langle \widetilde{S}^\alpha(q,\omega)\, \widetilde{S}^\beta(q',\omega')\rangle_0 = 0$. From the Gaussian action $\mathcal{A}_0$ (with $u = 0$), one immediately recovers the expressions (92) and (94) for $G_0(q,\omega)$ and $C_0(q,\omega)$, respectively. Since the dynamic correlation function can be expressed in terms of the noise strength and the response function, the graphical representation in terms of Feynman diagrams can be based entirely on the causal *response propagators* $G_0(q,\omega)$, represented as *directed* lines (we use the convention that time propagates from right to left) that connect $\widetilde{S}^\beta$ to S^α fields, and join at either two-point noise or four-point non-linear relaxation *vertices*, all subject to wave vector and frequency conservation as a consequence of spatial and temporal time translation invariance:



$$= \frac{1}{-i\omega + D q^a (r + q^2)}\, \delta^{\alpha\beta}$$



$$= 2 D q^a\, \delta^{\alpha\beta}$$



$$= -D q^a \frac{u}{6}$$

Following standard field theory procedures, one next identifies the *cumulants* as represented by *connected* Feynman diagrams, and in complete analogy with the static theory establishes *Dyson's equation* for the full response propagator, $G(q,\omega)^{-1} = G_0(q,\omega)^{-1} - \Sigma(q,\omega)$. By means of the fields $\widetilde{\Phi}^\alpha = \delta \ln \mathcal{Z}/\delta \tilde{j}^\alpha$ and $\Phi^\alpha = \delta \ln \mathcal{Z}/\delta j^\alpha$ one constructs the *generating functional* for dynamical *vertex functions* via the Legendre transform

$$\Gamma[\widetilde{\Phi},\Phi] = -\ln \mathcal{Z}[\tilde{j}, j] + \int d^d x \int dt \sum_\alpha \left(\tilde{j}^\alpha\, \widetilde{\Phi}^\alpha + j^\alpha\, \Phi^\alpha \right) , \quad (107)$$

$$\Gamma^{(\widetilde{N},N)}_{\{\alpha_i\};\{\alpha_j\}} = \prod_i^{\widetilde{N}} \frac{\delta}{\delta \widetilde{\Phi}^{\alpha_i}} \prod_j^{N} \frac{\delta}{\delta \Phi^{\alpha_j}}\, \Gamma[\widetilde{\Phi},\Phi]\Big|_{\tilde{j}=0=j} .$$

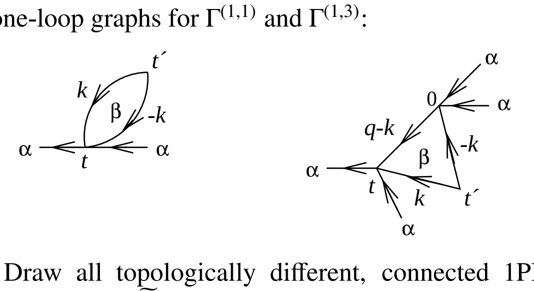
Functional derivatives then establish the following connections between two-point cumulants and vertex functions,

$$\Gamma^{(1,1)}(q,\omega) = G(-q,-\omega)^{-1} , \quad (108)$$

$$\Gamma^{(2,0)}(q,\omega) = -\frac{C(q,\omega)}{|G(q,\omega)|^2} = -\frac{2Dq^a}{\omega}\, \mathrm{Im}\, \Gamma^{(1,1)}(q,\omega) , \quad (109)$$

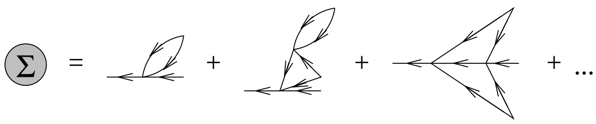
where the last relation follows from the equilibrium FDT (85). Moreover, $\Gamma^{(0,2)}(q,\omega) = 0$ as a consequence of causality. One thus easily sees that the vertex functions are graphically represented by the *one-particle (1PI) irreducible* Feynman diagrams.

We can now formulate the *Feynman rules* for the dynamical perturbation expansion for the l-th order contribution to the vertex function $\Gamma^{(\widetilde{N},N)}$, illustrated here for the one-loop graphs for $\Gamma^{(1,1)}$ and $\Gamma^{(1,3)}$:



1. Draw all topologically different, connected 1PI graphs with $\widetilde{N}$ out- / N incoming lines connecting l relaxation vertices. Do *not* allow closed response loops (since $\Theta(0) = 0$ in the Itô calculus).
2. Attach wave vectors q_i, frequencies ω_i / times t_i, and internal indices α_i to all directed lines, obeying *"momentum- energy" conservation* at each vertex.
3. Each *directed line* corresponds to a *response propagator* $G_0(-q,-\omega)$ or $G_0(q, t_i - t_j)$, the two-point vertex to the *noise strength* $2Dq^a$, and the four-point *relaxation vertex* to $-Dq^a u/6$. Closed loops imply integrals over the internal wave vectors and frequencies or times, subject to causality constraints, as well as sums over the internal vector indices. The residue theorem may be applied to evaluate frequency integrals.
4. Multiply with -1 and the *combinatorial factor* counting all possible ways of connecting the propagators, l relaxation vertices, and k two-point vertices leading to topologically identical graphs, including a factor $1/l!\,k!$ originating in the expansion of $\exp(-\mathcal{A}_{\rm int}[\widetilde{S}, S])$.

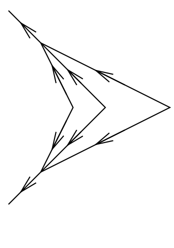
The perturbation series then graphically becomes a *loop expansion*. For example, the propagator self-energy diagrams up to two-loop order are



With the abbreviation $\Delta(q) = Dq^a(r + q^2)$, the corresponding explicit analytical expressions read:

$$\Gamma^{(1,1)}(q,\omega) = i\omega + Dq^a \Bigg[r + q^2 + \frac{n+2}{6}\, u \int_k \frac{1}{r+k^2} - \left(\frac{n+2}{6}\, u\right)^2 \int_k \frac{1}{r+k^2} \int_{k'} \frac{1}{(r+k'^2)^2} - \frac{n+2}{18}\, u^2 \int_k \frac{1}{r+k^2} \int_{k'} \frac{1}{r+k'^2}\, \frac{1}{r+(q-k-k')^2} \left(1 - \frac{i\omega}{i\omega + \Delta(k) + \Delta(k') + \Delta(q-k-k')} \right) \Bigg] . \quad (110)$$

The renormalized noise vertex is represented by the vertex function $\Gamma^{(2,0)}(q,\omega)$; the first non-vanishing fluctuation correction appears at two-loop order:



which translates to

$$\Gamma^{(2,0)}(q,\omega) = -2Dq^a \Bigg[1 + Dq^a \, \frac{n+2}{18} \, u^2 \int_k \frac{1}{r+k^2} \int_{k'} \frac{1}{r+k'^2} \, \frac{1}{r+(q-k-k')^2} \, \mathrm{Re} \, \frac{1}{i\omega + \Delta(k) + \Delta(k') + \Delta(q-k-k')} \Bigg] . \quad (111)$$

Finally, we shall require the renormalized relaxation vertex to one-loop order,

$$\Gamma^{(1,3)}(-3\underline{k}/2;\{\underline{k}/2\}) = D\Big(\frac{3q}{2}\Big)^a u \Bigg[1 - \frac{n+8}{6} \, u \int_k \frac{1}{r+k^2} \, \frac{1}{r+(q-k)^2} \Big(1 - \frac{i\omega}{i\omega + \Delta(k) + \Delta(q-k)} \Big) \Bigg] , \quad (112)$$

evaluated at equal incoming external wave vectors and frequencies $\underline{k} = (q,\omega)$.

4.5. Critical dynamics of the relaxational models

We may now proceed with the perturbative renormalization of the purely relaxational models A and B, generalizing the methods outlined in Chap. 3 to the dynamical action (102). The quadratic UV divergence (near the upper critical dimension $d_c = 4$) in (110) is taken care of by an appropriate *additive renormalization*; since we are concerned with near-equilibrium kinetics here, and $\chi(q,\omega = 0) = \chi(q)$, the result is precisely the T_c shift (61) evaluated in the static theory. In addition to (63) and (64), we need two new *multiplicative renormalization* factors associated with the response fields and the relaxation rate,

$$\widetilde{S}^\alpha_R = Z_{\widetilde{S}}^{1/2} \widetilde{S}^\alpha \ , \ D_R = Z_D \, D \, ; \quad (113)$$
$$\Rightarrow \ \Gamma_R^{(\widetilde{N},N)} = Z_{\widetilde{S}}^{-\widetilde{N}/2} Z_S^{-N/2} \, \Gamma^{(\widetilde{N},N)} \, .$$

As a consequence of the FDT (109), which must hold for both the unrenormalized and renormalized vertex functions, these Z factors are *not* independent in thermal equilibrium, but connected via $Z_D = \big(Z_S / Z_{\widetilde{S}}\big)^{1/2}$.

For model A with non-conserved order parameter ($a = 0$), extracting the UV poles in the minimal subtraction scheme from $\Gamma_R^{(2,0)}(0,0)$ or $\frac{\partial}{\partial i\omega}\Gamma_R^{(1,1)}(0,\omega)$ yields

$$Z_D = 1 - \frac{n+2}{144} \Big(6 \ln \frac{4}{3} - 1 \Big) \frac{u_R^2}{\epsilon} \, . \quad (114)$$

For model B with conserved order parameter ($a = 2$), on the other hand, the momentum dependence $\propto q^2$ of the relaxation vertex implies that to *all orders* in the perturbation expansion

$$\Gamma^{(1,1)}(q = 0,\omega) = i\omega \, ,$$
$$\partial_{q^2} \, \Gamma^{(2,0)}(q,\omega)\big|_{q=0} = -2D \, , \quad (115)$$

whence we arrive at the exact relations $Z_{\widetilde{S}} \, Z_S = 1$ and $Z_D = Z_S$. The conservation law thus allows us to reduce the dynamical multiplicative to static renormalizations.

With (113) taken into account, the *renormalization group equation* for the renormalized vertex functions $\Gamma_R^{(\widetilde{N},N)}(\mu, D, \tau_R, u_R)$ becomes in analogy with (69):

$$\Bigg[\mu \frac{\partial}{\partial\mu} + \frac{\widetilde{N}\gamma_{\widetilde{S}} + N\gamma_S}{2} + \gamma_D \, D_R \, \frac{\partial}{\partial D_R} + \gamma_\tau \, \tau_R \, \frac{\partial}{\partial \tau_R} + \beta_u \, \frac{\partial}{\partial u_R} \Bigg] \Gamma_R^{(\widetilde{N},N)} = 0 \, , \quad (116)$$

with the static *RG beta function* (72) and *Wilson's flow functions* (70), (71), supplemented with

$$\gamma_{\widetilde{S}} = \mu \, \frac{\partial}{\partial\mu}\Big|_0 \, \ln Z_{\widetilde{S}} \, , \quad (117)$$

$$\gamma_D = \mu \, \frac{\partial}{\partial\mu}\Big|_0 \, \ln \frac{D_R}{D} = \frac{\gamma_S - \gamma_{\widetilde{S}}}{2} \quad (118)$$

owing to the FDT. Employing characteristics $\mu \to \mu \, \ell$ to solve the Gell-Mann–Low RG equation (116), one has, in addition to (73),

$$\ell \, \frac{d\widetilde{D}(\ell)}{d\ell} = \widetilde{D}(\ell) \, \gamma_D(\ell) \, , \quad (119)$$

with $\widetilde{D}(1) = D_R$.

For the *dynamic susceptibility* near an infrared-stable RG fixed point, the static scaling law (75) generalizes to

$$\chi_R(\tau_R, q, \omega)^{-1} \approx \mu^2 \ell^{2+\gamma_S^*} \, \hat{\chi}_R\Big(\tau_R \, \ell^{\gamma_\tau^*}, u^*, \frac{q}{\mu \, \ell}, \frac{\omega}{D_R \mu^{2+a} \, \ell^{2+a+\gamma_D^*}} \Big)^{-1} , \quad (120)$$

which allows us to identify the static *critical exponents* as before, $\eta = -\gamma_S^*$, and $\nu = -1/\gamma_\tau^*$; and in addition $z = 2 + a + \gamma_D^*$ for the *dynamic critical exponent*. From the explicit two-loop result (114) one thus obtains for the $O(n)$-symmetric model A by inserting the Heisenberg fixed point (76) to order ϵ^2 in the $4 - \epsilon$ expansion

$$\text{model A} : z = 2 + c\,\eta \, , \ c = 6 \ln \frac{4}{3} - 1 + O(\epsilon) \, . \quad (121)$$

Yet if the order parameter is conserved, one has $\gamma_D^* = \gamma_S^*$, which implies the *exact scaling relation*

$$\text{model B} : z = 4 - \eta \, . \quad (122)$$

4.6. Critical dynamics of isotropic ferromagnets

So far we have only considered purely relaxational, dissipative kinetics. Often, however, the Langevin description of critical dynamics needs to take into account *reversible* systematic forces contributing to $F[S]$ in (96). The Langevin dynamics of *isotropic ferromagnets* provides a prominent example [20], [9, Chap. 6]. The order parameter here is a three-component vector field, namely the magnetization density $S^\alpha(x,t)$, which represent the coarse-grained mesoscopic counterpart of the microscopic local Heisenberg spins, also the generators of the rotation group $O(3)$. From the spin operator commutation relation $\left[S^\alpha, S^\beta\right] = i\hbar \sum_{\gamma=1}^3 \epsilon^{\alpha\beta\gamma} S^\gamma$ and Heisenberg's equation of motion, or their corresponding classical counterparts with commutators replaced by Poisson brackets, one readily obtains a spin precession term in the dynamics, in addition to the diffusive relaxation of the conserved magnetization density, and conserved stochastic noise:

$$\frac{\partial \vec{S}(x,t)}{\partial t} = -g\, \vec{S}(x,t) \times \frac{\delta \mathcal{H}[\vec{S}]}{\delta \vec{S}(x,t)} + D\nabla^2 \frac{\delta \mathcal{H}[\vec{S}]}{\delta \vec{S}(x,t)} + \vec{\zeta}(x,t) \, , \qquad (123)$$

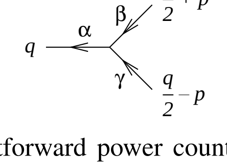
$$\left\langle \zeta^\alpha(x,t)\, \zeta^\beta(x',t') \right\rangle = -2D\, k_{\rm B} T\, \nabla^2 \delta(x-x') \delta(t-t') \delta^{\alpha\beta} \, . \qquad (124)$$

The coupled Langevin equations (123) with the three-component LGW Hamiltonian (52) and noise correlator (124) define the *model J* dynamic universality class.

The associated Janssen–De Dominicis response functional comprises the model B terms, (102) with $a = 2$, and the additional reversible *mode-coupling vertex*,

$$\mathcal{A}_J[\widetilde{S}, S] = -g \int d^d x \int dt \sum_{\alpha,\beta,\gamma} \epsilon^{\alpha\beta\gamma}\, \widetilde{S}^\alpha\, S^\beta \left(\nabla^2 S^\gamma + h^\gamma \right) \, . \qquad (125)$$

It is diagrammatically represented by a wave vector-dependent three-point vertex:



$= \varepsilon^{\alpha\beta\gamma}(q \cdot p) g$

Straightforward power counting gives $[g] = \mu^{3-d/2}$ for the mode coupling strength, which therefore becomes marginal at the *dynamical critical dimension* $d_c' = 6$, and irrelevant for $d > d_c'$. In principle, theories with competing upper critical dimension pose interesting non-trivial technical problems. Recall, however, that we are here considering near-equilibrium dynamics; thus the static critical properties completely decouple from the system's dynamics. Indeed, the dynamical critical exponent can be determined exactly from the underlying rotational symmetry as follows: We first note that according to (123) an external field h^γ induces a rotation of the magnetization density vector: $\left\langle S^\alpha(x,t) \right\rangle_h = g \int_0^t dt' \sum_\beta \epsilon^{\alpha\beta\gamma} \left\langle S^\beta(x,t') \right\rangle_h h^\gamma(t)$. Thus, we obtain an exact identity linking the *nonlinear susceptibility* $R^{\alpha;\beta\gamma} = \delta^2 \langle S^\alpha \rangle / \delta h^\beta\, \delta h^\gamma |_{h=0}$ with the linear order parameter response function:

$$\int d^d x'\, R^{\alpha;\beta\gamma}(x,t;x-x',t-t') = g\, \epsilon^{\alpha\beta\gamma}\, \chi^{\beta\beta}(x,t)\, \Theta(t)\, \Theta(t-t') \, . \qquad (126)$$

This identity provides crucial information for the renormalization of the UV singularities. In addition to (113), we define the renormalized dimensionless mode coupling strength

$$g_R^2 = Z_g\, g^2\, B_d\, \mu^{d-6} \, , \qquad (127)$$

where $B_d = \Gamma(4 - d/2)/2^d\, d\, \pi^{d/2}$. Just as for model B, one has to all orders in the perturbation expansion $\Gamma^{(1,1)}(q = 0, \omega) = i\omega$, whence again $Z_{\widetilde{S}}\, Z_S = 1$. Since (126) must hold for the renormalized susceptibilities as well, one then infers that $Z_g = Z_S$. Inspection of diagrams shows that the *effective coupling* in the dynamic perturbation expansion is $f = g^2/D^2$. The associated RG beta function becomes

$$\beta_f = \mu \frac{\partial}{\partial \mu}\Big|_0 f_R = f_R\, (d - 6 + \gamma_S - 2\,\gamma_D) \, . \qquad (128)$$

Consequently, at *any non-trivial*, stable RG fixed point $0 < f^* < \infty$ the terms in the bracket must cancel each other: $2\,\gamma_D^* = d - 6 + \gamma_S^*$ to *all orders* in the perturbation series with respect to f_R. Rotation invariance and the conservation law thus fix the dynamic critical exponent in dimensions $d < d_c' = 6$ to be

$$\text{model J}: z = 4 + \gamma_D^* = \frac{d + 2 - \eta}{2} \, . \qquad (129)$$

Indeed, an explicit one-loop calculation yields $\gamma_D = -f_R + O(u_R^2, f_R^2)$, which leads to the non-trivial model J RG fixed point $f_J^* = \frac{\varepsilon}{2} + O(\varepsilon^2)$, where $\varepsilon = 6 - d$. Note that $\eta = 0$ for $d > d_c = 4$, and $z = 4$ for $d > d_c' = 6$. Since the mode-coupling vertex does not contribute genuinely new IR singularities, dynamic scaling functions for isotropic ferromagnets and related models can be computed to exquisit precision by means of the *mode-coupling approximation*, which essentially amounts to a self-consistent one-loop theory for the propagators, ignoring vertex corrections [20].

4.7. Driven diffusive systems

This chapter on the application of field-theoretic RG methods to non-linear stochastic Langevin dynamics concludes with two paradigmatic non-equilibrium model systems that display *generic scale invariance* and a continuous phase transition, respectively. Both are driven lattice gases [21] consisting of particles that propagate via nearest-neighbor hopping which is biased along a specified 'drive' direction, and is subject to an exclusion constraint, i.e., only at most a single particle is allowed on each lattice site. If the system is set up with periodic boundary conditions, the biased diffusion generates a non-vanishing stationary mean particle current. At long times, the kinetics thus reaches a *non-equilibrium steady state* which turns out to be governed by *algebraic* rather than exponential temporal correlations. If in addition nearest-neighbor attractive interactions are included, the system displays a genuine non-equilibrium continuous phase transition in dimensions $d \geq 2$, from a disordered phase to an ordered state characterized by phase separation into low- and high-density regions, with the phase boundary oriented parallel to the drive and particle current. As the hopping bias vanishes, the phase transition is naturally described by the d-dimensional ferromagnetic equilibrium Ising model, since one may map the occupation numbers $n_i = 0, 1$ to binary spin variables $\sigma_i = 2n_i - 1 = \mp 1$.

We first consider the driven lattice gas with pure exclusion interactions, in one dimension also called *"asymmetric exclusion process"*. In order to construct a coarse-grained description for the non-equilibrium steady state of this system of particles with conserved density $\rho(x,t)$ and hard-core repulsion, driven along the '$\|$' spatial direction on a d-dimensional lattice, one first writes down the *continuity equation*

$$\frac{\partial}{\partial t} S(x,t) + \vec{\nabla} \cdot \vec{J}(x,t) = 0 \, , \qquad (130)$$

where the scalar field $S(x,t) = 2\rho(x,t) - 1$ represents a local magnetization in the spin language, whose mean remains fixed at $\langle S(x,t) \rangle = 0$, or $\langle \rho(x,t) \rangle = \frac{1}{2}$. Next the current density must be specified; in the $d_\perp$-dimensional transverse sector ($d_\perp = d - 1$), one may simply assert a noisy diffusion current, whereas along the drive, the bias and exclusion are crucial: $J_\| = -c\, D \, \nabla_\| S + 2Dg\rho(1-\rho) + \zeta$, where c measures the ratio of diffusivities parallel and transverse to the net current. In the comoving reference frame where $\langle J_\|(x,t) \rangle = 0$, therefore

$$\vec{J}_\perp(x,t) = -D\, \vec{\nabla}_\perp S(x,t) + \vec{\eta}(x,t) \, , \qquad (131)$$
$$J_\|(x,t) = -cD\, \nabla_\| S(x,t) - \frac{1}{2}\, Dg\, S(x,t)^2 + \zeta(x,t) \, ,$$

with $\langle \eta_i \rangle = 0 = \langle \zeta \rangle$, and the noise correlations

$$\langle \eta_i(x,t)\, \eta_j(x',t') \rangle = 2D\, \delta(x-x')\delta(t-t')\delta_{ij} \, , \qquad (132)$$
$$\langle \zeta(x,t)\, \zeta(x',t') \rangle = 2D\tilde{c}\, \delta(x-x')\delta(t-t') \, .$$

It is important to realize that Einstein's relations which connect the noise strengths and the relaxation rates need *not* be satisfied in the non-equilibrium steady state. Through straightforward rescaling of the field $S(x,t)$, one may however formally enforce this connection in the transverse sector, say; the deviation from the Einstein relation in the parallel direction is then encoded in (131) and (132) through the ratio $0 < w = \tilde{c}/c \neq 1$.

The Janssen–De Dominicis response functional for this driven diffusive system becomes

$$\mathcal{A}[\widetilde{S}, S] = \int d^d x \int dt\, \widetilde{S} \Big[\partial_t S - D \left(\nabla_\perp^2 + c \nabla_\|^2 \right) S + D \left(\nabla_\perp^2 + \tilde{c}\, \nabla_\|^2 \right) \widetilde{S} - \frac{D g}{2}\, \nabla_\| S^2 \Big] ; \qquad (133)$$

the action (133) represents a "massless" field theory, which hence displays *generic scale invariance* with no specific tuning of any control parameters required. One clearly has to allow for *anisotropic scaling* behavior owing to the very different dynamics parallel to the hopping bias; for example, (86) for the dynamic response function needs to be generalized to

$$\chi(q_\perp, q_\|, \omega) = |q_\perp|^{-2+\eta}\, \hat{\chi}\left(\frac{q_\|}{|q_\perp|^{1+\Delta}}, \frac{\omega}{|q_\perp|^z} \right), \qquad (134)$$

where Δ denotes the *anisotropy exponent* ($\Delta = 0$ in the mean-field approximation).

Following the renormalization procedures in the previous subsections, one first realizes that the drive generates a three-point vertex $\propto g\, iq_\|$; consequently no transverse fluctuations are affected by this non-linearity, and $Z_{\widetilde{S}} = Z_S = Z_D = 1$ *to all orders* in the perturbation expansion in g. The absence of any transverse propagator renormalization immediately implies that $\eta = 0$ and $z = 2$ in (134), which leaves merely the value of Δ to be determined. In a similar manner as for model J, one may in fact compute this exponent *exactly*; to this end, one observes that a generalized *Galilean transformation*

$$S(x_\perp, x_\|, t) \to S'(x'_\perp, x'_\|, t') = S(x_\perp, x_\| - Dgv\, t, t) - v \qquad (135)$$

leaves the Langevin equation (130), (131) or equivalent action (133) invariant. Thus the (arbitrary) speed v must scale as the order parameter field S, and since $Z_D = 1 = Z_S$, neither can the coupling g be altered by fluctuations, or (135) would be violated for the renormalized theory. Hence $Z_g = 1$ as well, and the only remaining non-trivial renormalizations are those for the dimensionless parameters $c_R = Z_c\, c$ and $\tilde{c}_R = Z_{\tilde{c}}\, \tilde{c}$.

An explicit one-loop calculation establishes the existence of an IR-stable RG fixed point for the coupling

$$v = g^2/c^{3/2} \ , \quad v_R = Z_c^{3/2}\, v\, C_d\, \mu^{d-2} \ , \tag{136}$$

with $C_d = \Gamma(2 - d/2)/2^{d-1}\pi^{d/2}$, and identifying $d_c = 2$ as the upper critical dimension for this problem. Evaluating the one-loop fluctuation corrections to the longitudinal propagator, one finds

$$\gamma_c = -\frac{v_R}{16}\,(3 + w_R) \ , \tag{137}$$
$$\gamma_{\tilde{c}} = -\frac{v_R}{32}\,\left(3w_R^{-1} + 2 + 3w_R\right)$$

for the anomalous scaling dimensions of c and $\tilde{c}$, or

$$\beta_w = w_R\,(\gamma_{\tilde{c}} - \gamma_c) = -\frac{v_R}{32}(w_R - 1)(w_R - 3) \ , \tag{138}$$

$$\beta_v = v_R\left(d - 2 - \frac{3}{2}\,\gamma_c\right) \tag{139}$$

for the associated RG beta functions of the ratio $w = \tilde{c}/c$ and the non-linear coupling v. At *any non-trivial* RG fixed point $0 < v^* < \infty$, (138) implies that either $w_N^* = 3$ or $w_E^* = 1$, but the latter is obviously *stable*; in the asymptotic scale-invariant regime, the Einstein relation is evidently restored, and the system effectively equilibrated. Moreover, in dimensions $d < d_c = 2$, (139) leads to the *exact* scaling exponents

$$\Delta = -\frac{\gamma_c^*}{2} = \frac{2-d}{3} \ , \quad z_\parallel = \frac{z}{1+\Delta} = \frac{6}{5-d} \ . \tag{140}$$

At $d = 1$, specifically, one has $z_\parallel = \frac{3}{2}$, which captures the dynamic scaling for the asymmetric exclusion process. In one dimension, the driven lattice gas with exclusion in fact maps onto the *noisy Burgers equation* for equilibrium fluid hydrodynamics, and also to the *Kardar–Parisi–Zhang equation* for curvature-driven surface or interface growth.

We finally briefly summarize the RG analysis for the driven lattice gas with conserved total density and attractive Ising interactions between the particles (and "holes") at its critical point. Since the system orders in stripes along the drive direction, only the *transverse* fluctuations become critical. Therefore one must amend the response functional (133) with a higher-order gradient term and non-linearity akin to the scalar model B, see (102) with $a = 2$; yet the noise terms too need only be retained in the transverse sector. For this driven model B, the effective critical action thus becomes

$$\mathcal{A}[\widetilde{S}, S] = \int d^d x \int dt\, \widetilde{S}\left[\partial_t S - D\nabla_\perp^2\left(r - \nabla_\perp^2\right) S - Dc\nabla_\parallel^2 S + D\left(\nabla_\perp^2 \widetilde{S} - \frac{g}{2}\,\nabla_\parallel S^2 - \frac{u}{6}\,\nabla_\perp^2 S^3\right)\right] , \tag{141}$$

and (134) is further generalized by adding a relevant temperature variable (now $\Delta = 1$ in mean-field theory):

$$\chi(\tau_\perp, q_\perp, q_\parallel, \omega) = |q_\perp|^{-2+\eta}\, \hat{\chi}\left(\frac{\tau}{|q_\perp|^{1/\nu}}, \frac{q_\parallel}{|q_\perp|^{1+\Delta}}, \frac{\omega}{|q_\perp|^z}\right) . \tag{142}$$

Straightforward power counting for the non-linear couplings yields $[g^2] = \mu^{5-d}$ and $[u] = \mu^{3-d}$; therefore the upper critical dimension is raised to $d_c = 5$ (compared to both the non-critical driven lattice gas and the equilibrium model B), and fluctuation corrections are dominated by the drive, while the static coupling u is (dangerously) *irrelevant* near d_c. The three-point vertex $\propto g\, iq_\parallel$ again does not allow any renormalizations in the transverse sector, whence $Z_{\widetilde{S}} = Z_S = Z_D = 1$; consequently $\eta = 0$, $\nu = \frac{1}{2}$, and $z = 4$ in (142) to all orders in the perturbation series, leaving only the anisotropy exponent to be determined. As before $Z_g = 1$ follows from Galilean invariance, imposing a simple structure for the RG beta function for the effective coupling (136):

$$\beta_v = v_R\left(d - 5 - \frac{3}{2}\,\gamma_c\right) . \tag{143}$$

In dimensions $d < d_c = 5$, at any non-trivial and finite RG fixed point, the scaling exponents are thus forced to assume the values

$$\Delta = 1 - \frac{\gamma_c^*}{2} = \frac{8-d}{3} \ , \ z_\parallel = \frac{4}{1+\Delta} = \frac{12}{11-d} \ . \tag{144}$$

These last examples clearly demonstrate how the powerful field-theoretic RG approach can help to exploit the basic symmetries for a given problem, allowing to determine certain non-trivial scaling exponents *exactly*.

5. Scale Invariance in Interacting Particle Systems

This last chapter details how the stochastic kinetics of classical interacting (reacting) particle systems, defined through a *microscopic* master equation, can also be mapped onto a dynamical field theory in the continuum limit [22]–[24]. For at most binary reactions, one can thus *derive* a corresponding mescoscopic Langevin representation, typically with multiplicative noise terms.

Furthermore, RG tools may be applied to extract the infrared properties in scale-invariant systems, as will be exemplified for diffusion-limited annihilation processes [22, 24, 19], and for non-equilibrium phase transitions from active to inactive, absorbing states, where all stochastic fluctuations cease [24]–[27]. Stochastic models in population dynamics and ecology are naturally formulated in a chemical reaction language, and hence amenable to these field-theoretic tools [28, 29].

5.1. Chemical reactions and population dynamics

Let us thus consider (classical) particles of various species $A, B, \ldots$ on a d-dimensional lattice that propagate by hops to nearest-neighbor sites, and either spontaneously decay or produce offspring, and/or upon encounter with other particles, undergo certain "chemical" reactions with prescribed rates. Our goal is to systematically construct a continuum description of such stochastic particle systems that however faithfully encodes the associated *intrinsic reaction noise*, and consequently allows us to properly address the effects of statistical fluctuations and spatio-temporal correlations [22, 24, 19].

To set the stage, we introduce three characteristic examples that we shall explore in more detail below. First, we address the general single-species *irreversible annihilation* reaction $k\,A \to m\,A$ with integers $m < k$, and rate λ_k. Assuming the system to be well-mixed, one may neglect spatial variations and focus on the mean particle density $a(t) = \langle a(x,t) \rangle$. Ignoring in addition any non-trivial correlations, one can write down the *rate equation* for this stochastic process, which in essence thus constitutes the simplest mean-field approximation:

$$\frac{\partial a(t)}{\partial t} = -(k-m)\,\lambda_k\, a(t)^k \, . \tag{145}$$

For $k = 1$ (and $m = 0$), this just describes spontaneous exponential decay, $a(t) = a(0)\, e^{-\lambda_1 t}$; for $k \geq 2$, (145) is easily integrated with the result

$$a(t) = \left[a(0)^{1-k} + (k-m)(k-1)\,\lambda_k\, t \right]^{-1/(k-1)} . \tag{146}$$

For the kth order annihilation reaction, the particle density decays algebraically $a(t) \sim (\lambda_k\, t)^{-1/(k-1)}$ at long times $t \gg \lambda_k^{-1}$, with an amplitude that does not even depend on the initial density $a(0)$ anymore. The replacement of an exponential decay by a power law signals scale invariance and indicates the potential importance of fluctuations and correlations. Indeed, the annihilation kinetics generates particle *anti-correlations*, whence the long-time kinetics is dominated by the ensuing *depletion zones* in low dimensions $d \leq d_c(k)$ that need to be traversed by any potentially reacting particles. As a consequence, one obtains *slower* decay power laws than predicted by the mean-field rate equation (146).

Next, we allow *competing reactions*, namely decay $A \to \emptyset$ (the empty state) with rate κ, and the reversible process $A \rightleftharpoons A + A$ with forward / backward rates σ and λ, respectively. Again, we begin with an analysis of the rate equation for this set of reactions,

$$\frac{\partial a(t)}{\partial t} = (\sigma - \kappa)\, a(t) - \lambda\, a(t)^2 \, , \tag{147}$$

which obviously predicts a *continuous non-equilibrium phase transition* at $\sigma_c = \kappa$: For $\sigma > \kappa$, the mean particle density approaches a finite value, $a(t \to \infty) \to a_\infty = (\sigma - \kappa)/\lambda$. One refers to this state as an *active* phase; ongoing reactions cause the particle number to fluctuate about its average. On the other hand, for $\sigma < \kappa$, the density can only decrease, whence ultimately $a(t) \to 0$; in this *inactive* phase, all reaction processes terminate since they all require the presence of a particle. Such a state is therefore called *absorbing*: once reached, the stochastic dynamics cannot escape from it anymore. Right at the critical point $\sigma = \kappa$, one recovers the long-time algebraic decay of the pair annihilation process, $a(t) \sim (\lambda\, t)^{-1}$; this suggests the interpretation of (146) as a *critical* power law induced by the precise cancellation of the contributions from first-order reactions that enter linearly proportional to the particle concentration. The obvious issues to be addressed by a more refined theoretical treatment are: How can internal reaction noise and correlations be systematically incorporated? What is the upper critical dimension d_c below which fluctuations crucially alter the mean-field power laws? Can certain *universality classes* be identified, and the associated critical exponents be computed, at least perturbatively in a dimensional expansion near d_c?

Finally, let us address a prominent textbook example from population dynamics, namely the classic *Lotka–Volterra predator-prey competition*. Invoking the stochastic chemical reaction framework, this model is defined via spontaneous death $A \to \emptyset$ (rate κ) and birth $B \to B + B$ (rate σ) processes for the "predators" A and "prey" B; absent any interactions between these two species, the predators must go extinct, while the prey population explodes exponentially. Interesting species competition and potentially coexistence is created by the binary *predation* reaction $A + B \to A + A$ (with rate λ). The associated coupled rate equations for the presumed homogeneous population densities read

$$\frac{\partial a(t)}{\partial t} = \lambda\, a(t) b(t) - \kappa\, a(t) \, ,$$
$$\frac{\partial b(t)}{\partial t} = \sigma\, b(t) - \lambda\, a(t) b(t) \, . \tag{148}$$

In this mean-field approximation, one easily confirms the existence of a conserved first integral for the ordinary differential equations (148): The quantity $K(t) = \lambda[a(t)+b(t)] - \ln[a(t)^\sigma b(t)^\kappa] = K(0)$ remains unchanged under the temporal evolution. Consequently, the mean-field trajectories are closed orbits in the phase space spanned by the population densities, and the dynamics is characterized by regular *population oscillations*, determined by the *initial* state. This is clearly not a biologically realistic feature, and indeed represents an artifact of the implicit mean-field factorization for the nonlinear predation process. Upon including the internal reaction noise and spatial degrees of freedom with diffusively spreading particles, as, e.g., in individual-based Monte Carlo simulations, one in fact observes striking *"pursuit and evasion" waves* in the species coexistence phase that generate complex dynamical patterns, locally discernible as *erratic* population oscillations which ultimately become overdamped in finite systems. Moreover, if the local "carrying capacity" is finite, i.e., only a certain maximum number of particles may occupy each lattice site, there emerges a predator *extinction threshold* which indicates a continuous phase transition to an absorbing state, namely the lattice filled with prey. Stochastic fluctuations as well as reaction-induced noise and correlations are thus crucial ingredients to properly describe the large-scale features of spatially extended Lotka–Volterra systems even and especially far away from the extinction threshold (for recent overviews, see Refs. [28, 29]).

5.2. Coherent-state path integral for master equations

In the following, reacting particle systems on a d-dimensional lattice shall be *defined* through the associated chemical *master equation* governing a Markovian stochastic process with prescribed, time-independent transition rates. Any possible configuration at time t of the stochastic dynamics is then uniquely characterized by a list of the integer occupation numbers $n_i = 0, 1, 2, \ldots$ for each particle species at sites i. The master equation governs the temporal evolution of the corresponding probability distribution $P(\{n_i\}; t)$ through a *balance* of gain and loss terms induced by the reaction processes; for example, for the binary reactions $A + A \to \emptyset$ and $A + A \to A$ with rates λ and λ':

$$\begin{aligned}\frac{\partial}{\partial t} P(n_i; t) &= \lambda\,(n_i + 2)\,(n_i + 1)\,P(\ldots, n_i + 2, \ldots; t)\\ &+\lambda'\,(n_i + 1)\,n_i\,P(\ldots, n_i + 1, \ldots; t) \qquad (149)\\ &-(\lambda + \lambda')\,n_i\,(n_i - 1)\,P(\ldots, n_i, \ldots; t)\,,\end{aligned}$$

with, say, an uncorrelated *initial Poisson distribution* $P(\{n_i\}, 0) = \prod_i \left(\bar{n}_0^{n_i}\, e^{-\bar{n}_0}/n_i!\right)$.

Since the dynamics merely consists of increasing or decreasing the particle occupation numbers on each site, it calls for a representation through *second-quantized bosonic ladder operators*, at least if arbitrary many particles are permitted per site, with standard commutation relations $[a_i, a_j] = 0$, $[a_i, a_j^\dagger] = \delta_{ij}$, and an empty vacuum state $|0\rangle$ that is annihilated by all operators a_i, $a_i|0\rangle = 0$. The Fock space of states $|\{n_i\}\rangle$ with n_i particles on sites i is then constructed through multiple creation operators acting on the vacuum, $|\{n_i\}\rangle = \prod_i \left(a_i^\dagger\right)^{n_i} |0\rangle$; note that a different normalization from standard many-particle quantum mechanics has been implemented here. Thus, $a_i\,|n_i\rangle = n_i\,|n_i - 1\rangle$ and $a_i^\dagger\,|n_i\rangle = |n_i + 1\rangle$, whence the states $|\{n_i\}\rangle$ are eigenstates of $\hat{n}_i = a_i^\dagger a_i$ with eigenvalues n_i. Next one defines the formal *state vector* $|\Phi(t)\rangle = \sum_{\{n_i\}} P(\{n_i\}; t)\,|\{n_i\}\rangle$, whose temporal evolution is determined by the master equation (149), and may be written in terms of a time-independent quasi-Hamiltonian or Liouvillian H that can be decomposed into a sum of local operators:

$$\frac{\partial}{\partial t}\,|\Phi(t)\rangle = -H\,|\Phi(t)\rangle\,, \; H = \sum_i H_i(a_i^\dagger, a_i)\,. \quad (150)$$

Note that (150) constitutes a *non-Hermitian imaginary-time Schrödinger equation*, with the formal solution $|\Phi(t)\rangle = \exp(-Ht)\,|\Phi(0)\rangle$.

For example, the quasi-Hamiltonian in this *Doi–Peliti bosonic operator formulation* reads for diffusion-limited annihilation and coagulation reactions

$$\begin{aligned} H = D &\sum_{<ij>} \left(a_i^\dagger - a_j^\dagger\right)\left(a_i - a_j\right) \qquad (151)\\ &- \sum_i \left[\lambda\left(1 - a_i^{\dagger 2}\right) a_i^2 + \lambda'\left(1 - a_i^\dagger\right) a_i^\dagger a_i^2\right],\end{aligned}$$

where the first line represents nearest-neighbor hopping, and the second encodes the processes in (149). For each stochastic reaction, H contains two contributions: the first one directly reflects the physical *process* under considerations, i.e., annihilation and production of particles, whereas the second term carries information on the *order* of the reaction (which powers of the particle concentrations enter the rate equations). In order to access the desired *statistical averages* with the time-dependent probability distribution $P(\{n_i\}; t)$, one needs the *projection* state $\langle\mathcal{P}| = \langle 0|\prod_i e^{a_i}$, with $\langle\mathcal{P}|0\rangle = 1$; the mean value for any observable F, necessarily a function of all occupation numbers n_i, at time t then follows as

$$\begin{aligned}\langle F(t)\rangle &= \sum_{\{n_i\}} F(\{n_i\})\,P(\{n_i\}; t)\\ &= \langle\mathcal{P}|\,F(\{a_i^\dagger a_i\})\,|\Phi(t)\rangle\,. \qquad (152)\end{aligned}$$

Probability conservation implies that $1 = \langle \mathcal{P}|\Phi(t)\rangle = \langle \mathcal{P}|e^{-Ht}|\Phi(0)\rangle$, and therefore $\langle \mathcal{P}|H = 0$. By means of $\left[e^a, a^\dagger\right] = e^a$, one may commute the product $e^{\sum_i a_i}$ through the quasi-Hamiltonian H, which effectively results in the operator shifts $a_i^\dagger \to 1 + a_i^\dagger$. H must consequently vanish if all creation operators are replaced with 1, $H_i(a_i^\dagger \to 1, a_i) = 0$. In averages, one may thus also replace $a_i^\dagger a_i \to a_i$; e.g., for the particle density one obtains simply $a(t) = \langle a_i \rangle$, while the two-point occupation number operator product becomes $a_i^\dagger a_i \, a_j^\dagger a_j \to a_i \delta_{ij} + a_i a_j$.

Starting with the Hamiltonian (150), and based on the expectation values (152), one may invoke standard procedures from quantum many-particle theory to construct a *path integral representation* based on *coherent states*, defined as eigenstates of the annihilation operators a_i with arbitrary complex eigenvalues ϕ_i: $a_i |\phi_i\rangle = \phi_i |\phi_i\rangle$. It is straightforward to confirm

$$|\phi_i\rangle = \exp\left(-\frac{1}{2}|\phi_i|^2 + \phi_i a_i^\dagger\right)|0\rangle \, , \tag{153}$$

$$1 = \int \prod_i \frac{d^2\phi_i}{\pi} \, |\{\phi_i\}\rangle \langle \{\phi_i\}| \, . \tag{154}$$

The *closure relation* (154) demonstrates that the coherent states for each site i form an *overcomplete* basis of Fock space. Upon splitting the time evolution into infinitesimal steps, and inserting (154) into (152) with the formal solution of (150), one eventually arrives at

$$\langle F(t)\rangle \propto \int \prod_i \mathcal{D}[\phi_i] \, \mathcal{D}[\phi_i^*] \, F(\{\phi_i\}) \, e^{-\mathcal{A}[\phi_i^*, \phi_i]},$$

$$\mathcal{A}[\phi_i^*, \phi_i] = \sum_i \Big[-\phi_i(t_f) \tag{155}$$

$$+ \int_0^{t_f} dt \Big[\phi_i^* \, \partial_t \phi_i + H(\phi_i^*, \phi_i)\Big] - \bar{n}_0 \phi_i^*(0)\Big].$$

In the end we take the *continuum limit* $\phi_i(t) \to a_0^d \, \psi(x, t)$ (with lattice constant a_0), and $\phi_i^*(t) \to \hat{\psi}(x, t)$; for diffusively propagating particles, the ensuing "bulk" action becomes

$$\mathcal{A}[\hat{\psi}, \psi] = \int d^d x \int_0^{t_f} dt \left[\hat{\psi} \left(\partial_t - D \nabla^2\right) \psi + \mathcal{H}_r\left(\hat{\psi}, \psi\right)\right]. \tag{156}$$

Here, $\mathcal{H}_r$ denotes the contributions stemming from the stochastic reaction kinetics; e.g., for *pair annihilation and coagulation*,

$$\mathcal{H}_r\left(\hat{\psi}, \psi\right) = -\lambda\left(1 - \hat{\psi}^2\right)\psi^2 - \lambda'\left(1 - \hat{\psi}\right)\hat{\psi}\,\psi^2 \, . \tag{157}$$

Appropriate factors of a_0 were absorbed into the continuum diffusion constant D and reaction rates λ,λ'. It is worthwhile emphasizing that the actions (155) based on a master equation should be viewed as *microscopic* stochastic field theories, which may well require additional coarse-graining steps. Yet the internal stochastic dynamics of the master equation is faithfully and consistently accounted for, since aside from the continuum limit no approximations have been invoked; specifically, no assumptions on the form or strength of any noise terms have been made. As exemplified next for the action (157), the Doi–Peliti coherent-state path integral representation of stochastic master equations may serve as a convenient starting point for systematic analytical approaches such as field-theoretic RG studies.

5.3. *Diffusion-limited annihilation processes*

Pair annihilation $A + A \to \emptyset$ or coagulation $A + A \to A$ represent the perhaps simplest but non-trivial diffusion-limited reactions. In order to reach beyond the mean-field rate equation approximation (146), we explore the corresponding Doi–Peliti field theory (156) with the specific reaction Hamiltonian (157). First we note that the associated *classical field equations* $\delta\mathcal{A}/\delta\psi = 0 = \delta\mathcal{A}/\delta\hat{\psi}$ are solved by $\hat{\psi} = 1$ (which just reflects probability conservation) and

$$\frac{\partial \psi(x,t)}{\partial t} = D \, \nabla^2 \, \psi(x,t) - (2\lambda + \lambda') \, \psi(x,t)^2 \, , \tag{158}$$

i.e., the rate equation for the local density field $\psi(x, t)$ augmented by diffusive spreading. It is convenient to shift the conjugate field about the mean-field solution, $\hat{\psi}(x, t) = 1 + \widetilde{\psi}(x, t)$, which turns the reactive action into

$$\mathcal{H}_r\left(\widetilde{\psi}, \psi\right) = (2\lambda + \lambda') \, \widetilde{\psi} \, \psi^2 + (\lambda + \lambda') \, \widetilde{\psi}^2 \, \psi^2 \, . \tag{159}$$

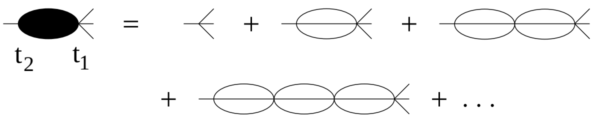
Since the annihilation and coagulation processes $\propto \lambda, \lambda'$ generate the very same vertices, we conclude that aside from non-universal amplitudes, both diffusion-limited reactions should follow *identical* scaling behavior. One may also formally interpret the ensuing field theory as a Janssen–De Dominicis response functional (101) originating from a "Langevin equation" (158) with added white noise $\zeta(x, t)$, whose second moment (97) is given by the functional $L[\psi] = -(\lambda + \lambda') \, \psi^2 < 0$. This negative variance, which reflects the emerging *anti*-correlations for surviving particles that are induced by the annihilation reactions, of course implies that a Langevin representation is not truly feasible for this stochastic process. One must also keep in mind that the fields ψ and $\hat{\psi}$ are complex-valued; indeed, the reaction noise can be recast as *"imaginary" multiplicative noise* $\propto i\psi(x, t) \, \zeta(x, t)$ in the associated stochastic differential equation.

As coagulation thus falls into the same universality class as annihilation, let us more generally study the Doi–Peliti action for *k-particle annihilation* $k\,A \to \emptyset$,

$$\mathcal{A}[\hat{\psi}, \psi] = \int d^d x \int dt \Big[\hat{\psi}\left(\partial_t - D\nabla^2\right)\psi - \lambda_k \left(1 - \hat{\psi}^k\right)\psi^k\Big] . \quad (160)$$

The corresponding mean-field rate equation (145) predicts algebraic decay $a(t) \sim (\lambda_k\, t)^{-1/(k-1)}$ at long times. Since the field $\hat{\psi}$ appears to higher than quadratic power for $k \geq 3$, *no* (obvious) equivalent Langevin description is possible for triplet and higher-order annihilation reactions. With $[\hat{\psi}(x,t)] = 1$ and $[\psi(x,t)] = \mu^d$, as $\langle\psi(x,t)\rangle = a(t)$ is just the particle density, power counting gives $[\lambda_k] = \mu^{2-(k-1)d}$; the upper critical dimension thus is $d_c(k) = 2/(k-1)$ for kth order annihilation, and one expects the mean-field power laws (146) to be accurate for $k > 3$ in all physical dimensions $d \geq 1$; for triplet reactions, one should encounter merely logarithmic corrections in one dimension, where genuine nontrivial exponents ensue only for pair annihilation.

Even for $k = 2$, one cannot construct any loop graphs from the vertices in (160) that would modify the massless diffusion propagator, implying that $\eta = 0$ and $z = 2$. This leaves the task to determine the reaction *vertex renormalization*, which can also be achieved to *all orders* by summing the diagrammatic geometric series (essentially a *Bethe–Salpeter equation*; here for $k = 3$):



With the factor $B_{kd} = k!\,\Gamma(2-d/d_c)\,d_c/k^{d/2}\,(4\pi)^{d/d_c}$, one finds the renormalized reaction rate

$$g_R = Z_g\,\frac{\lambda}{D}\,B_{kd}\,\mu^{-2(1-d/d_c)} ,$$
$$Z_g^{-1} = 1 + \frac{\lambda\,B_{kd}\,\mu^{-2(1-d/d_c)}}{D\,(d_c - d)} , \quad (161)$$

and the exact RG beta function and stable fixed point

$$\beta_g = -\frac{2g_R}{d_c}\,(d - d_c + g_R) , \quad g^* = d_c - d . \quad (162)$$

Next we write down the Gell-Mann–Low RG equation for the particle density $a(t)$, applying the matching condition $(\mu\ell)^2 = 1/Dt$:

$$\left[d + 2Dt\,\frac{\partial}{\partial(D\,t)} - d\,n_0\,\frac{\partial}{\partial n_0} + \beta_g\,\frac{\partial}{\partial g_R}\right] a(\mu, D, n_0, g_R, t) = 0 , \quad (163)$$

with the solution

$$a(\mu, D, n_0, g_R, t) = (D\mu^2\,t)^{-d/2}\,\hat{a}\Big(n_0\,(D\mu^2\,t)^{d/2}, \widetilde{g}(t)\Big) . \quad (164)$$

The particle density at time t naturally depends on its initial value n_0, clearly a *relevant* parameter in the RG sense. One therefore needs to establish through explicit calculation that the (tree level) scaling function $\hat{a}$ remains finite to *all orders* as $n_0 \to \infty$. In the end, (164) yields for pair annihilation,

$$k = 2 : \quad \begin{aligned} d < 2 &: \ a(t) \sim (D\,t)^{-d/2} , \\ d = 2 &: \ a(t) \sim (D\,t)^{-1}\ln(D\,t) , \\ d > 2 &: \ a(t) \sim (\lambda\,t)^{-1} ; \end{aligned} \quad (165)$$

while for the triplet reaction

$$k = 3 : \quad \begin{aligned} d = 1 &: \ a(t) \sim \left[(D\,t)^{-1}\ln(D\,t)\right]^{1/2} , \\ d > 1 &: \ a(t) \sim (\lambda\,t)^{-1/2} . \end{aligned} \quad (166)$$

At low dimensions $d \leq d_c(k) = 2/(k-1)$, the density decay is slowed down as compared to the mean-field power laws by the emergence of depletion zones around the surviving particles. Further annihilations require that the reactants traverse the diffusion length $L(t) \sim (D\,t)^{1/2}$, which sets the typical separation scale; the corresponding density must then scale as $L(t)^{-d}$. Beyond the upper critical dimension $d_c(k)$, the system becomes effectively well-mixed, diffusion plays no limiting role, and the time scale is set by the reaction rate.

5.4. Phase transitions from active to absorbing states

Turning to our second example in the introductory remarks, we now investigate diffusing particles subject to the *competing reactions* $A \to \emptyset$ and $A \rightleftharpoons A + A$; adding the diffusion term to (147), we arrive at the rate equation for the local particle density,

$$\frac{\partial a(x,t)}{\partial t} = -D\left(r - \nabla^2\right)a(x,t) - \lambda\,a(x,t)^2 , \quad (167)$$

where $r = (\kappa - \sigma)/D$; in mathematical biology and ecology, the partial differential equation (167) is known as the *Fisher–Kolmogorov equation*, and for example has been used to study population invasion fronts into empty regions. We shall instead focus on the critical region where the control parameter $r \to 0$, and a *continuous non-equilibrium phase transition* from an active to an inactive and absorbing state occurs.

The Doi–Peliti field theory action (156) capturing the above reactions reads

$$\mathcal{A}[\hat{\psi}, \psi] = \int d^d x \int dt \Big[\hat{\psi}\left(\partial_t - D\,\nabla^2\right)\psi - \kappa\Big(1 - \hat{\psi}\Big)\psi + \sigma\Big(1 - \hat{\psi}\Big)\hat{\psi}\,\psi - \lambda\Big(1 - \hat{\psi}\Big)\hat{\psi}\,\psi^2\Big] . \quad (168)$$

Upon shifting and rescaling the fields according to $\hat{\psi}(x,t) = 1 + \sqrt{\lambda/\sigma}\,\widetilde{S}(x,t)$ and $\psi(x,t) = \sqrt{\sigma/\lambda}\,S(x,t)$, one arrives at

$$\mathcal{A}[\widetilde{S},S] = \int d^dx \int dt \Big(\widetilde{S}\left[\partial_t + D\left(r - \nabla^2\right)\right] S - u\left(\widetilde{S} - S\right)\widetilde{S}\,S + \lambda \widetilde{S}^2 S^2 \Big), \quad (169)$$

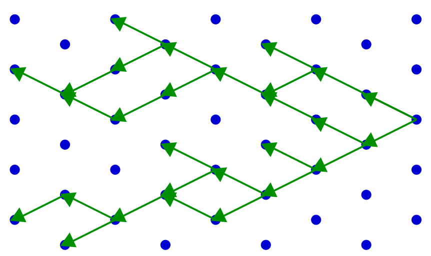
where the three-point vertices have been symmetrized and now are proportional to the coupling $u = \sqrt{\sigma\lambda}$ with scaling dimension $[u] = \mu^{2-d/2}$. The associated *upper critical dimension* is therefore $d_c = 4$, and the annihilation four-point vertex $[\lambda] = \mu^{2-d}$ consequently is *irrelevant* in the RG sense near d_c. In the effective critical action, one may set $\lambda \to 0$, whereupon (169) reduces to the familiar *Reggeon field theory* action, which is invariant under *rapidity inversion* $S(x,t) \leftrightarrow -\widetilde{S}(x,-t)$. Viewing (169) as a Janssen–De Dominicis functional (101), it is equivalent to the Langevin equation that amends the Fisher–Kolmogorov equation (167) with a noise term,

$$\frac{\partial S(x,t)}{\partial t} = D\left(\nabla^2 - r\right) S(x,t) - u S(x,t)^2 + \zeta(x,t), \quad (170)$$

with $\langle \zeta(x,t) \rangle = 0$ and the multiplicative "square-root" noise correlator

$$\langle \zeta(x,t)\zeta(x',t')\rangle = 2u\, S(x,t)\, \delta(x-x')\delta(t-t') . \quad (171)$$

Drawing a space-time plot (time running from right to left) for the branching $A \to A + A$, death $A \to \emptyset$, and coagulation $A + A \to A$ processes, starting from a single occupied site, as depicted below, one realizes that they generate a *directed percolation* (DP) cluster, with "time" playing the role of a specified "growth" direction. The field theory (169) (with $\lambda = 0$) thus also describes the universal scaling properties of critical DP.



Indeed, one expects that active to absorbing state phase transitions should *generically* be captured by this DP universality class, namely in the absence of coupling to other slow conserved modes, disorder, or special additional symmetries. The origin for this remarkable DP conjecture becomes evident in a complementary coarse-grained phenomenological approach that will be framed in the language of epidemic spreading [27]. Consider the following *simple epidemic process*:

1. A *"susceptible" medium* is locally *"infected"*, depending on the density of "sick" neighbors. Infected regions may later recover.
2. The "disease" *extinction* state is *absorbing*.
3. The disease *spreads diffusively* via infection, see 1.
4. Other fast microscopic degrees of freedom are incorporated as random *noise*. Yet according to condition 2, noise alone cannot regenerate the disease.

These decisive features can be encoded in a mesoscopic Langevin stochastic differential equation for the local density $n(x,t)$ of "active" (infected) individuals,

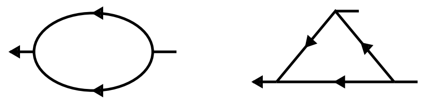
$$\frac{\partial n(x,t)}{\partial t} = D\left(\nabla^2 - R[n(x,t)]\right) n(x,t) + \zeta(x,t), \quad (172)$$

with the reactive functional $R[n]$, $\langle \zeta(x,t) \rangle = 0$, and noise correlator $L[n] = n\,N[n]$. In the spirit of Landau theory, near extinction one may expand these functionals in a Taylor series for small densities,

$$r \approx 0 : \; R[n] = r + u\,n + \ldots, \; N[n] = v + \ldots, \quad (173)$$

where higher-order terms are *irrelevant* in the RG sense. After rescaling, the corresponding Janssen–De Dominicis response functional (101) becomes identical to the Reggeon field theory action.

We now proceed to analyze the dynamic perturbation theory and renormalization for the DP action (169) to one-loop order. The only singular vertex functions are the propagator self-energy $\Gamma^{(1,1)}(q,\omega)$ and the three-point functions $\Gamma^{(1,2)} = -\Gamma^{(2,1)}$, owing to rapidity inversion symmetry, with the lowest-order Feynman graphs:



Explicit evaluation of the self-energy yields

$$\Gamma^{(1,1)}(q,\omega) = i\omega + D(r + q^2) + \frac{u^2}{D} \int_k \frac{1}{i\omega/2D + r + q^2/4 + k^2} . \quad (174)$$

As in critical statics, one needs to first ensure the *criticality condition*, namely that $\Gamma^{(1,1)}(0,0) = 0$ at the true percolation threshold $r = r_c$. To one-loop order, (174) results in the shift (additive renormalization)

$$r_c = -\frac{u^2}{D^2} \int_k \frac{1}{r_c + k^2} + O(u^4) , \quad (175)$$

and inserting $\tau = r - r_c$ in (174) subsequently leads to

$$\Gamma^{(1,1)}(q,\omega) = i\omega + D\left(\tau + q^2\right) - \frac{u^2}{D} \int_k \frac{i\omega/2D + \tau + q^2/4}{k^2\,(i\omega/2D + \tau + q^2/4 + k^2)} . \quad (176)$$

The three-point vertex function at vanishing external wave vectors and frequencies finally becomes

$$\Gamma^{(1,2)}(\{0\},(\{0\}) = -2u\left[1 - \frac{2u^2}{D^2}\int_k \frac{1}{(\tau+k^2)^2}\right]. \quad (177)$$

For the *multiplicative renormalizations*, we follow (63) and (64), but with the slight modification $u_R = Z_u\, u\, A_d^{1/2} \mu^{(d-4)/2}$, as well as (113) with $Z_{\tilde{S}} = Z_S$ owing to rapidity inversion invariance. Thus one obtains Wilson's RG flow functions to one-loop order,

$$\gamma_S = \frac{v_R}{2} + O(v_R^2)\,, \quad \gamma_D = -\frac{v_R}{4} + O(v_R^2)\,,$$
$$\gamma_\tau = -2 + \frac{3v_R}{4} + O(v_R^2)\,, \quad (178)$$

with the effective coupling and associated beta function

$$v_R = \frac{Z_u^2}{Z_D^2}\,\frac{u^2}{D^2}\,A_d\,\mu^{d-4}\,, \quad (179)$$

$$\beta_v = v_R\Big[-\epsilon + 3v_R + O(v_R^2)\Big]\,. \quad (180)$$

Below the upper critical dimension $d_c = 4$, the *IR-stable RG fixed point*

$$v_{\mathrm{DP}}^* = \frac{\epsilon}{3} + O(\epsilon^2) \quad (181)$$

appears, and solving the RG equation for the two-point *correlation function* in its vicinity yields

$$C_R(\tau_R, q, \omega)^{-1} \approx q^2\,\ell^{\gamma_S^*}$$
$$\hat{C}_R\left(\tau_R\,\ell^{\gamma_\tau^*}, v^*, \frac{q}{\mu\ell}, \frac{\omega}{D_R\,\mu^2\ell^{2+\gamma_D^*}}\right)^{-1}. \quad (182)$$

This allows us to identify the three independent *critical exponents* for directed percolation to order $\epsilon = 4 - d$:

$$\eta = -\gamma_S^* = -\frac{\epsilon}{6} + O(\epsilon^2)\,,$$
$$\nu^{-1} = -\gamma_\tau^* = 2 - \frac{\epsilon}{4} + O(\epsilon^2)\,, \quad (183)$$
$$z = 2 + \gamma_D^* = 2 - \frac{\epsilon}{12} + O(\epsilon^2)\,.$$

The DP universality class also applies to many active to absorbing state phase transitions with more than just one particle species. As an example, consider the predator extinction threshold in the spatially extended stochastic two-species Lotka–Volterra model with finite carrying capacity discussed in the chapter introduction. The associated Doi–Peliti field theory action reads

$$S[\hat{a}, a; \hat{b}, b] = \int d^d x \int dt \Big[\hat{a}\left(\partial_t - D_A\nabla^2\right) a$$
$$+\kappa\left(\hat{a} - 1\right) a + \hat{b}\left(\partial_t - D_B\nabla^2\right) b \quad (184)$$
$$+\sigma\left(1 - \hat{b}\right)\hat{b}\,b\,e^{-\rho^{-1}\hat{b}b} + \lambda\left(\hat{b} - \hat{a}\right)\hat{a}\,a\,b\Big]\,,$$

where diffusive spreading has been assumed, and the exponential term in the prey production term takes into account the local restriction to a maximum particle density ρ. As usual, one applies the field shifts $\hat{a} = 1 + \tilde{a}$, $\hat{b} = 1 + \tilde{b}$; realizing that ρ^{-1} constitutes an irrelevant perturbation (since the density scales as $[\rho] = \mu^d$), we furthermore expand to lowest order in ρ^{-1}, which yields

$$S[\tilde{a}, a; \tilde{b}, b] = \int d^d x \int dt \Big[\tilde{a}\left(\partial_t - D_A\nabla^2 + \kappa\right) a$$
$$+\tilde{b}\left(\partial_t - D_B\nabla^2 - \sigma\right) b - \sigma\,\tilde{b}^2\, b \quad (185)$$
$$+\sigma\,\rho^{-1}\left(1 + \tilde{b}\right)^2 \tilde{b}\,b^2 - \lambda\left(1 + \tilde{a}\right)\left(\tilde{a} - \tilde{b}\right) a\,b\Big].$$

Near the predator extinction threshold, the prey almost fill the entire system. We therefore define the properly *fluctuating fields* $c = b_s - b$ with $b_s \approx \rho$ and $\langle c\rangle = 0$, and $\tilde{c} = -\tilde{b}$. Rescaling to $\phi = \sqrt{\sigma}\,c$ and $\tilde{\phi} = \sqrt{\sigma}\,\tilde{c}$, and noting that asymptotically $\sigma \to \infty$ under the RG flow since $[\sigma] = \mu^2$, the ensuing action simplifies drastically. At last, we add the *growth-limiting reaction* $A + A \to A$ (with rate τ); the fields ϕ and $\tilde{\phi}$ can then be integrated out, leaving Reggeon field theory (169) as the resulting effective action, with the non-linear coupling $u = \sqrt{\tau\,\lambda\,b_s}$.

6. Concluding Remarks

These lecture notes can of course only provide a very sketchy and vastly incomplete introduction to the use of field theory tools and applications of the renormalization group in statistical physics. I have merely focused on continuous phase transitions in equilibrium, dynamic critical phenomena in simple relaxational models, and a few examples for universal scaling behavior in non-equilibrium dynamical systems. Among the many topics not covered or even mentioned here are critical phenomena in finite and disordered systems; universality classes of critical dynamics with reversible couplings to other conserved modes; universal short-time and non-equilibrium relaxation scaling properties in the aging regime; depinning transitions and driven interfaces in disordered media; spin glasses and structural glasses; and of course phase transitions and generic scale invariance in quantum systems. Nor have I addressed powerful representations through supersymmetric or conformally invariant quantum field theories, Monte Carlo algorithms, or numerical non-perturbative RG approaches, since the latter will be covered elsewhere in this volume; for their applications to non-equilibrium systems, see, e.g., Ref. [30].

Nevertheless, I hope to have conveyed the message that methods from field theory are ubiquitous in statistical physics, and the renormalization group has served as a remarkably powerful mathematical tool to address at least the universal scaling aspects of cooperative behavior governed by strong correlations and fluctuations. Thus, the RG has become a cornerstone of our understanding of complex interacting many-particle systems, and its language and basic philosophy now pervade the entire field, with applications that increasingly reach out beyond fundamental physics to material science, chemistry, biology, ecology, and even sociology.

Finally, I would like to express my gratitude to the organizers (and their funding agencies) for the opportunity to attend and lecture at the 49th Schladming Theoretical Physics Winter School. I thoroughly enjoyed the stimulating and informal atmosphere in the Styrian Alps, and profited from many discussions with colleagues and students. I can only hope that all attendants learned as much about new and exciting developments in theoretical physics from my fellow lecturers as I did.

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