Current Distribution in Driven Diffusive Systems: Field Theory Approach

Uwe C. Täuber, Virginia Tech

Vivien Lecomte and Frédéric van Wijland,
Université Paris-Sud Orsay / Paris VII – Denis Diderot

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1 Introduction: Driven diffusive systems

H.K. Janssen, B. Schmittmann (1986);

K.-t. Leung, J.L. Cardy (1986)

Phenomenological description of driven diffusive systems:

- total particle number conserved
- site exclusion: $n_i = 0, 1$; $\phi_i = n_i - \frac{1}{2} = \mp\frac{1}{2}$
- driven in the '$\|$' spatial direction

$\Longrightarrow$ continuity equation: $\partial_t \rho(\mathbf{x}, t) = -\nabla \cdot \mathbf{j}(\mathbf{x}, t)$
particle current: $\mathbf{j} = -D\nabla\rho + \rho\,\mathbf{u}(\rho) + \eta$
with $\phi(\mathbf{x}, t) = \rho(\mathbf{x}, t) - \rho_0$: $u_\|(\phi) = \frac{Dg}{2}(1-\phi^2) + \ldots$

effective Langevin description:

$$\begin{aligned} \mathbf{j}_\perp - \mathbf{j}_{\perp 0} &= -D\,\nabla_\perp \phi + \eta_\perp \\ j_\| - j_{\|0} &= -D\,\lambda\,\nabla_\| \phi - \frac{Dg}{2}\,\phi^2 + \eta_\| \end{aligned}$$

$$\begin{aligned} \langle \eta_{\perp i}(\mathbf{x}, t)\,\eta_{\perp j}(\mathbf{x}', t')\rangle &= 2D\,\delta_{ij}\delta(\mathbf{x} - \mathbf{x}')\delta(t - t') \\ \langle \eta_\|(\mathbf{x}, t)\,\eta_\|(\mathbf{x}', t')\rangle &= 2D\sigma\,\delta(\mathbf{x} - \mathbf{x}')\,\delta(t - t') \end{aligned}$$

in terms of the density fluctuations, with $\zeta = -\nabla \cdot \eta$:

$$\frac{\partial \phi}{\partial t} = D\left(\nabla_\perp^2 + \lambda\,\nabla_\|^2\right)\phi + \frac{Dg}{2}\,\nabla_\|\,\phi^2 + \zeta$$

$$\langle \zeta(\mathbf{x}, t)\,\zeta(\mathbf{x}', t')\rangle = \\ = -2D\left(\nabla_\perp^2 + \sigma\,\nabla_\|^2\right)\delta(\mathbf{x} - \mathbf{x}')\,\delta(t - t')$$

Standard mapping to field theory, renormalization group
$\Longrightarrow$ universal scaling properties, $d \le d_c = 2$:

$$C(q_\|, \mathbf{q}_\perp, \omega) = |\mathbf{q}_\perp|^{-z_\perp - 2 + \eta} \, \hat{C} \left(\frac{q_\|}{|\mathbf{q}_\perp|^{1+\Delta}}, \frac{\omega}{|\mathbf{q}_\perp|^{z_\perp}} \right)$$

transverse exponents Gaussian: $\eta = 0 \; , \quad z_\perp = 2$
anisotropy and longitudinal dynamic exponent:

$$\Delta = \frac{2-d}{3} \; , \quad z_\| = \frac{z_\perp}{1+\Delta} = \frac{6}{5-d}$$

to *all* orders in $\epsilon = 2 - d$ ($d = 1$: $z_\| = 3/2$; see KPZ)
$d > 2$: mean-field exponents $\eta = \Delta = 0$, $z_\| = z_\perp = 2$

Add attractive Ising interactions
$\Longrightarrow$ driven lattice gas with continuous phase transition, orders transverse to drive; Langevin description:

$$\mathbf{j}_\perp - \mathbf{j}_{\perp 0} = -D \, \nabla_\perp \left(r - \nabla_\perp^2 + \frac{u}{6} \, \phi^2 \right) \phi + \eta_\perp$$

$$\frac{\partial \phi}{\partial t} = D \nabla_\perp^2 \left(r - \nabla_\perp^2 + \frac{u}{6} \, \phi^2 \right) \phi + \lambda D \, \nabla_\|^2 \, \phi + \frac{D g}{2} \, \nabla_\| \, \phi^2 + \zeta$$

$$\langle \zeta(\mathbf{x}, t) \, \zeta(\mathbf{x}', t') \rangle = -2D \left(\nabla_\perp^2 + \sigma \, \nabla_\|^2 \right) \delta(\mathbf{x} - \mathbf{x}') \, \delta(t - t')$$

scaling exponents (to all orders, $d_c = 5$): $\eta = 0$, $\nu = 1/2$,

$$z_\perp = 4 \; , \quad \Delta = \frac{8-d}{3} \; , \quad z_\| = \frac{12}{11-d}$$

2 Current distribution, large deviations

Total integrated longitudinal current up to time t:

$$Q(t) = \int_0^t dt' \int d^d x \, j_\parallel(\mathbf{x}, t')$$

large deviation function:

$$\pi(q) = \lim_{t \to \infty} \frac{1}{t} \ln P(Q = q\, t, t)$$

generating function for moments of $Q(t)$:

$$Z(s,t) = \left\langle e^{-s\, Q(t)} \right\rangle = \left\langle \exp\left[-s \int_0^t dt' \int d^d x \, j_\parallel(\mathbf{x}, t') \right] \right\rangle$$

$$\Longrightarrow \quad \langle Q(t)^n \rangle = (-1)^n \, \frac{d^n Z(s,t)}{ds^n} \Big|_{s=0}$$

Legendre transform ('dynamical free energy'):

$$\mu(s) = \lim_{t \to \infty} \frac{1}{t} \ln Z(s,t) \ , \quad \pi(q) = \max_s \{ \mu(s) + s\, q \}$$

$$\Longrightarrow \quad (-1)^n \, \frac{d^n \mu(s)}{ds^n} \Big|_{s=0} = \lim_{t \to \infty} \frac{\langle Q(t)^n \rangle_c}{t}$$

- non-interacting particles, $|s| \to 0$: extensive, analytic

$$\mu_0(s) = L^d \left(B_1\, s + B_2\, s^2 + \ldots \right)$$

- exactly solvable nonequilibrium systems: nonanalytic e.g., totally asymmetric exclusion process, $s < 0$:

$$\mu_{\mathrm{TASEP}}(s) = -\rho(1-\rho)\, s + \frac{(3\pi)^{\frac{2}{3}}}{5} \left[\rho(1-\rho) \right]^{\frac{4}{3}} |s|^{\frac{5}{3}} + \ldots$$

B. Derrida, J.L. Lebowitz (1998); B. Derrida, C. Appert (1999)

3 Field theory, RG analysis, results

Janssen–De Dominicis field theory representation for Langevin equations, periodic boundary conditions:

$$
\begin{aligned}
Z(s,t) \;&=\; \int \mathcal{D}\phi \mathcal{D}\bar{\phi}\ \exp\left(-S[\bar{\phi},\phi,s;t]\right) \\
S[\bar{\phi},\phi,s;t] \;&= \int d^dx \int_0^t dt' \left[\bar{\phi}\left(\partial_{t'} - D\nabla_\perp^2 - D\lambda\nabla_\parallel^2\right)\phi \right. \\
&\qquad -D\left(\nabla_\perp\bar{\phi}\right)^2 - D\sigma\left(\nabla_\parallel\bar{\phi}\right)^2 + \frac{Dg}{2}\left(\nabla_\parallel\bar{\phi}\right)\phi^2 \\
&\qquad \left. -s\,\frac{Dg}{2}\,\phi^2\right] + s\,j_0\,L^d t - s^2\,D\sigma\,L^d t
\end{aligned}
$$

evaluate in the Gaussian approximation:

$$
\begin{aligned}
&\ln \int \mathcal{D}\phi\,\mathcal{D}\bar{\phi}\ \exp\left(-S_0[\bar{\phi},\phi,s;t]\right) = \\
&\quad = -\frac{L^d t}{2}\int \frac{d^dq}{(2\pi)^d}\,\frac{d\omega}{2\pi}\,\ln\left[1 - \frac{2sgD^2(\mathbf{q}_\perp^2 + \sigma q_\parallel^2)}{\omega^2 + D^2(\mathbf{q}_\perp^2 + \lambda q_\parallel^2)^2}\right] \\
&\quad = -\frac{L^d\,Dt}{2}\int \frac{d^dq}{(2\pi)^d}\left[\sqrt{\left(\mathbf{q}_\perp^2 + \lambda q_\parallel^2\right)^2 - 2s\,g\left(\mathbf{q}_\perp^2 + \sigma q_\parallel^2\right)}\right. \\
&\qquad\qquad \left. - \left(\mathbf{q}_\perp^2 + \lambda q_\parallel^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow L^{-d}\,\mu(s) \;&= -j_0\,s + D\lambda\left[1 + \frac{C_d g^2}{\lambda^{3/2}}\,\frac{(-gs)^{-\varepsilon/2}}{4\varepsilon}\right]s^2 \\
&= -j_0\,s + D\lambda_R\,s^2
\end{aligned}
$$

with renormalized longitudinal diffusivity λ_R

asymptotic scaling behavior: $\lambda_R(\kappa) \sim \lambda\, \kappa^{\gamma_\lambda^\star}$,
with anomalous dimension $\gamma_\lambda^\star = \gamma_\sigma^\star = -\frac{2}{3}\,(2-d) = -2\Delta$
normalization point / matching condition: $\kappa = (-s\, g)^{1/2}$

$$\Longrightarrow d < d_c = 2 \;:\; L^{-d}\, \mu(s) = -j_0\, s + \mathcal{A}_d\, |s|^{(d+4)/3}$$

- power law for $\mu(s)$ as $|s| \to 0$ determined by anomalous dimension of noise strength

additional novel results:

- logarithmic corrections at $d_c = 2$:

$$L^{-2}\, \mu(s) = -j_0\, s + \mathcal{A}_2\, s^2\, (-\ln|s|)^{2/3}$$

- driven diffusive system with Ising interactions: requires anomalous dimension of irrelevant longitudinal noise strength

$$d < d_c = 5 \;:\; L^{-d}\, \mu(s) = -j_0\, s + \mathcal{B}_d\, s^{2-\epsilon/18}$$
$$d = 5 \;:\; L^{-5}\, \mu(s) = -j_0\, s + \mathcal{B}_5\, s^2\, (-\ln|s|)^{2/9}$$

- particles subject to anisotropic random advective field

see *V.L., U.C.T., and F.v.W.*,
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