Renormalization Group: Applications in Statistical Physics

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Physics at All Scales: The Renormalization Group

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Lecture plan

Critical Phenomena

- Continuous phase transitions
- Scaling theory
- Landau–Ginzburg–Wilson Hamiltonian
- Gaussian approximation
- Wilson's momentum shell renormalization group
- Dimensional expansion and critical exponents
- Literature

Field Theory Approach to Critical Phenomena

- Perturbation expansion and Feynman diagrams
- Ultraviolet and infrared divergences, renormalization
- Renormalization group equation and critical exponents
- Literature

Critical Phenomena

Landau expansion; mean-field theory

Expand free energy (density) in terms of order parameter (scalar field) ϕ near a *continuous* (second-order) phase transition at T_c:

$$f(\phi) = \frac{r}{2}\,\phi^2 + \frac{u}{4!}\,\phi^4 + \ldots - h\,\phi \ ,$$

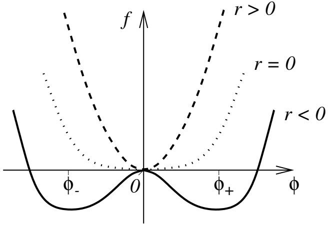
$r = a(T - T_c)$, $u > 0$; conjugate field h breaks $Z(2)$ symmetry $\phi \to -\phi$.

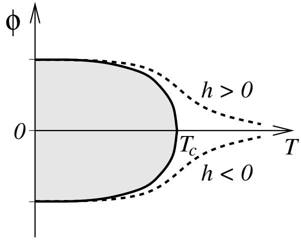
$f'(\phi) = 0 \Rightarrow$ *equation of state*:

$$h(T,\phi) = r(T)\,\phi + \frac{u}{6}\,\phi^3 \ ;$$

stability: $f''(\phi) = r + \frac{u}{2}\,\phi^2 > 0$.

- *Critical isotherm* at $T = T_c$: $h(T_c,\phi) = \frac{u}{6}\,\phi^3$.
- *Spontaneous order parameter* for $r < 0$: $\phi_\pm = \pm(6|r|/u)^{1/2}$.





Thermodynamic singularities at critical point

- Isothermal order parameter *susceptibility*:

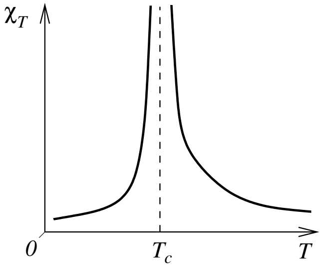
$$V\chi_T^{-1} = \left(\frac{\partial h}{\partial \phi}\right)_T = r + \frac{u}{2}\,\phi^2 \;\Rightarrow\; \frac{\chi_T}{V} = \begin{cases} 1/r^{1} & r > 0 \\ 1/2|r|^{1} & r < 0 \end{cases}$$

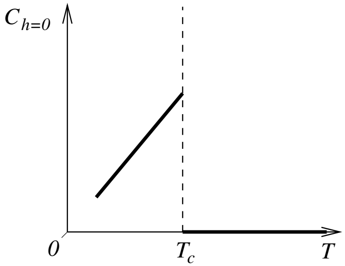
$\Rightarrow$ *divergence* at T_c.

- *Free energy* and *specific heat* for $T < T_c$:

$$f(\phi_\pm) = \frac{r}{4}\,\phi_\pm^2 = -\frac{3r^2}{2u}\,,\;\; C_{h=0} = -VT\left(\frac{\partial^2 f}{\partial T^2}\right)_{h=0} = VT\,\frac{3a^2}{u}$$

$\Rightarrow$ *discontinuity* at T_c.





Scaling hypothesis for free energy

Postulate: (sing.) free energy generalized *homogeneous* function:

$$f_{\rm sing}(\tau, h) = |\tau|^{2-\alpha}\, \hat{f}_\pm\left(h/|\tau|^\Delta\right) \ , \quad \tau = \frac{T - T_c}{T_c} \ ;$$

two-parameter scaling, with *scaling functions* $\hat{f}_\pm$, $\hat{f}_\pm(0) = \text{const.}$

Landau theory: *critical exponents* $\alpha = 0$, $\Delta = 3/2$.

- *Specific heat*:

$$C_{h=0} = -\frac{VT}{T_c^2}\left(\frac{\partial^2 f_{\rm sing}}{\partial \tau^2}\right)_{h=0} = C_\pm\, |\tau|^{-\alpha} \ .$$

- *Equation of state*:

$$\phi(\tau, h) = -\left(\frac{\partial f_{\rm sing}}{\partial h}\right)_\tau = -|\tau|^{2-\alpha-\Delta}\ \hat{f}'_\pm\left(h/|\tau|^\Delta\right) \ .$$

- *Coexistence line* $h = 0$, $\tau < 0$:

$$\phi(\tau, 0) = -|\tau|^{2-\alpha-\Delta}\ \hat{f}'_-(0) \propto |\tau|^\beta \ , \quad \beta = 2 - \alpha - \Delta \ .$$

Scaling relations

- *Critical isotherm*: τ-dependence in $\hat{f}'_\pm$ must cancel prefactor, $\hat{f}'_\pm(x) \propto x^{(2-\alpha-\Delta)/\Delta}$ as $x \to \infty$; hence

$$\phi(0, h) \propto h^{(2-\alpha-\Delta)/\Delta} = h^{1/\delta} \ , \quad \delta = \Delta/\beta \ .$$

- Isothermal *susceptibility*:

$$\frac{\chi_\tau}{V} = \left(\frac{\partial\phi}{\partial h}\right)_{\tau,\, h=0} = \chi_\pm \, |\tau|^{-\gamma} \ , \quad \gamma = \alpha + 2(\Delta - 1) \ .$$

Eliminate Δ $\Rightarrow$ *scaling relations*:

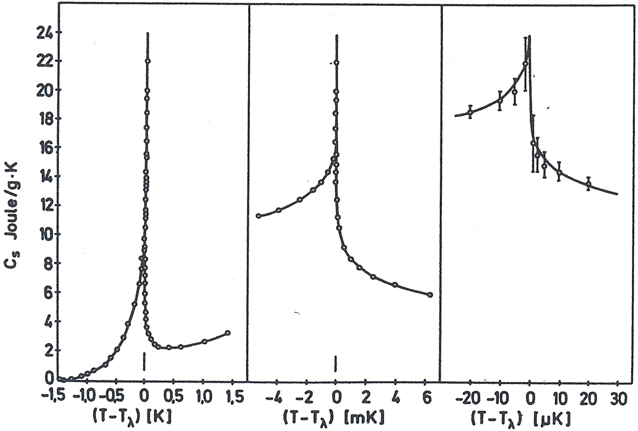
$$\Delta = \beta\delta \ , \quad \alpha + \beta(1+\delta) = 2 = \alpha + 2\beta + \gamma \ , \quad \gamma = \beta(\delta - 1) \ ;$$

$\Rightarrow$ only *two independent* (static) critical exponents.

Mean-field: $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = 1$, $\delta = 3$, $\Delta = \frac{3}{2}$ (dim. analysis).

Experimental exponent values different, but still *universal*: depend only on symmetry, dimension ..., *not* microscopic details.

Thermodynamic self-similarity in the vicinity of T_c



Temperature dependence of the specific heat near the normal- to superfluid transition of He 4, shown in successively reduced scales.

From: M.J. Buckingham and W.M. Fairbank, in: Progress in low temperature physics, *Vol. III, ed. C.J. Gorter, 80–112, North-Holland (Amsterdam, 1961).*

Landau–Ginzburg–Wilson Hamiltonian

Coarse-grained Hamiltonian, *order parameter field $S(x)$*:

$$\mathcal{H}[S] = \int d^d x \left[\frac{r}{2} S(x)^2 + \frac{1}{2} [\nabla S(x)]^2 + \frac{u}{4!} S(x)^4 - h(x) S(x) \right],$$

where $r = a(T - T_c^0)$, $u > 0$, $h(x)$ local external field;
gradient term $\sim [\nabla S(x)]^2$ suppresses spatial inhomogeneities.

Probability density for configuration $S(x)$: *Boltzmann factor*

$$\mathcal{P}_s[S] = \exp(-\mathcal{H}[S]/k_\mathrm{B} T)/\mathcal{Z}[h] \ ,$$

canonical *partition function* and moments ⇒ functional integrals:

$$\mathcal{Z}[h] = \int \mathcal{D}[S] \ e^{-\mathcal{H}[S]/k_\mathrm{B} T} , \ \phi = \langle S(x) \rangle = \int \mathcal{D}[S] \ S(x) \, \mathcal{P}_s[S] \ .$$

- Integral measure: discretize $x \to x_i \ \Rightarrow \ \mathcal{D}[S] = \prod_i dS(x_i)$;
- or employ Fourier transform: $S(x) = \int \frac{d^d q}{(2\pi)^d} \, S(q) \, e^{iq \cdot x}$,

$$\mathcal{D}[S] = \prod_q \frac{dS(q)}{V} = \prod_{q, q_1 > 0} \frac{d \, \mathrm{Re} \, S(q) \ d \, \mathrm{Im} \, S(q)}{V} \ .$$

Landau–Ginzburg approximation

Most likely configuration ⇒ *Ginzburg–Landau equation*:

$$0 = \frac{\delta \mathcal{H}[S]}{\delta S(x)} = \left[r - \nabla^2 + \frac{u}{6} \, S(x)^2 \right] S(x) - h(x) \; .$$

Linearize $S(x) = \phi + \delta S(x)$: $\delta h(x) \approx \left(r - \nabla^2 + \frac{u}{2} \, \phi^2 \right) \delta S(x)$.

Fourier transform ⇒ *Ornstein–Zernicke susceptibility*:

$$\chi_0(q) = \frac{1}{r + \frac{u}{2} \, \phi^2 + q^2} = \frac{1}{\xi^{-2} + q^2} \, , \; \xi = \begin{cases} 1/r^{1/2} & r > 0 \\ 1/|2r|^{1/2} & r < 0 \end{cases} \; .$$

Zero-field two-point *correlation function* (cumulant):

$$C(x - x') = \langle S(x) \, S(x') \rangle - \langle S(x) \rangle^2 = (k_{\rm B} T)^2 \frac{\delta^2 \ln \mathcal{Z}[h]}{\delta h(x) \, \delta h(x')} \bigg|_{h=0} \; .$$

Fourier transform $C(x) = \int \frac{d^d q}{(2\pi)^d} \, C(q) \, e^{iq \cdot x}$

⇒ *fluctuation–response theorem*: $C(q) = k_{\rm B} T \, \chi(q)$.

Scaling hypothesis for correlation function

Scaling ansatz, defines *Fisher exponent* η and *correlation length* ξ:

$$C(\tau, q) = |q|^{-2+\eta} \, \hat{C}_{\pm}(q\xi) \; , \quad \xi = \xi_{\pm} \, |\tau|^{-\nu} \; .$$

- Thermodynamic *susceptibility*:

$$\chi(\tau, q = 0) \propto \xi^{2-\eta} \propto |\tau|^{-\nu(2-\eta)} = |\tau|^{-\gamma} \; , \quad \gamma = \nu(2 - \eta) \; .$$

- Spatial *correlations* for $x \to \infty$:

$$C(\tau, x) = |x|^{-(d-2+\eta)} \, \widetilde{C}_{\pm}(x/\xi) \propto \xi^{-(d-2+\eta)} \propto |\tau|^{\nu(d-2+\eta)} \; .$$

$\langle S(x)S(0)\rangle \to \langle S\rangle^2 = \phi^2 \propto (-\tau)^{2\beta} \;\Rightarrow\;$ *hyperscaling relations*:

$$\beta = \frac{\nu}{2} \, (d - 2 + \eta) \; , \quad 2 - \alpha = d\nu \; .$$

Mean-field values: $\nu = \frac{1}{2}$, $\eta = 0$ (Ornstein–Zernicke).

Diverging spatial correlations induce thermodynamic singularities !

Gaussian approximation

High-temperature phase, $T > T_c$: neglect nonlinear contributions:

$$\mathcal{H}_0[S] = \int \frac{d^d q}{(2\pi)^d} \left[\frac{1}{2} \left(r + q^2 \right) |S(q)|^2 - h(q) S(-q) \right] .$$

Linear transformation $\widetilde{S}(q) = S(q) - \frac{h(q)}{r+q^2}$, $\int_q \ldots = \int \frac{d^d q}{(2\pi)^d} \ldots$:

$$\mathcal{Z}_0[h] = \int \mathcal{D}[S] \; \exp(-\mathcal{H}_0[S]/k_\mathrm{B} T) =$$

$$= \exp\left(\frac{1}{2 k_\mathrm{B} T} \int_q \frac{|h(q)|^2}{r + q^2} \right) \int \mathcal{D}[\widetilde{S}] \; \exp\left(- \int_q \frac{r + q^2}{2 k_\mathrm{B} T} \, |\widetilde{S}(q)|^2 \right)$$

$$\Rightarrow \; \left\langle S(q) S(q') \right\rangle_0 = \frac{(k_\mathrm{B} T)^2}{\mathcal{Z}_0[h]} \frac{(2\pi)^{2d} \, \delta^2 \mathcal{Z}_0[h]}{\delta h(-q) \, \delta h(-q')} \bigg|_{h=0}$$

$$= C_0(q) \, (2\pi)^d \delta(q + q') \; , \quad C_0(q) = \frac{k_\mathrm{B} T}{r + q^2} \; .$$

Gaussian model: free energy and specific heat

$$F_0[h] = -k_{\rm B} T \ln \mathcal{Z}_0[h] = -\frac{1}{2} \int_q \left(\frac{|h(q)|^2}{r + q^2} + k_{\rm B} T \, V \, \ln \frac{2\pi \, k_{\rm B} T}{r + q^2} \right) .$$

Leading singularity in *specific heat*:

$$C_{h=0} = -T \left(\frac{\partial^2 F_0}{\partial T^2} \right)_{h=0} \approx \frac{V k_{\rm B} \, (a T_c^0)^2}{2} \int_q \frac{1}{(r + q^2)^2} .$$

- $d > 4$: integral UV-divergent; regularized by cutoff Λ (Brillouin zone boundary) $\Rightarrow$ $\alpha = 0$ as in mean-field theory;
- $d = d_c = 4$: integral diverges logarithmically:

$$\int_0^{\Lambda\xi} \frac{k^3}{(1 + k^2)^2} \, dk \sim \ln(\Lambda\xi) \; ;$$

- $d < 4$: with $k = q/\sqrt{r} = q\xi$, surface area $K_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$:

$$C_{\rm sing} \approx \frac{V k_{\rm B} (a T_c^0)^2 \, \xi^{4-d}}{2^d \pi^{d/2} \, \Gamma(d/2)} \int_0^\infty \frac{k^{d-1}}{(1 + k^2)^2} \, dk \propto |T - T_c^0|^{-\frac{4-d}{2}}$$

$\Rightarrow$ diverges; stronger singularity than in mean-field theory.

Renormalization group program in statistical physics

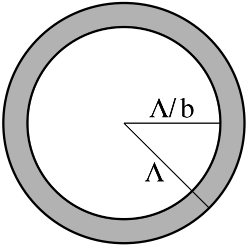
- Goal: *critical* (IR) singularities; perturbatively inaccessible.
- Exploit fundamental new symmetry:
  divergent correlation length induces *scale invariance*.
- Analyze theory in ultraviolet regime: integrate out
  short-wavelength modes / renormalize UV divergences.
- Rescale onto original Hamiltonian, obtain recursion relations
  for effective, now scale-dependent *running couplings*.
- Under such RG transformations:
  - $\rightarrow$ *Relevant* parameters grow: set to 0: *critical surface*.
  - $\rightarrow$ Certain couplings approach *IR-stable fixed point*:
    scale-invariant behavior.
  - $\rightarrow$ *Irrelevant* couplings vanish: origin of *universality*.
- Scale invariance at critical fixed point $\Rightarrow$ infer correct IR
  scaling behavior from (approximative) analysis of UV regime
  $\Rightarrow$ *derivation* of scaling laws.
- Dimensional expansion: $\epsilon = d_c - d$ small parameter, permits
  perturbational treatment $\Rightarrow$ *computation* of critical exponents.

Wilson's momentum shell renormalization group

RG transformation steps:

(1) Carry out the partition integral over all Fourier components $S(q)$ with wave vectors $\Lambda/b \leq |q| \leq \Lambda$, where $b > 1$: *eliminates short-wavelength modes*.

(2) *Scale transformation* with the same scale parameter $b > 1$:
$x \to x' = x/b \ , \quad q \to q' = b\, q \ .$



Accordingly, we also need to *rescale the fields*:

$$S(x) \to S'(x') = b^{\zeta} S(x) \ , \quad S(q) \to S'(q') = b^{\zeta - d} S(q) \ .$$

Proper choice of $\zeta \Rightarrow$ rescaled Hamiltonian assumes original form $\Rightarrow$ *scale-dependent effective couplings*, analyze dependence on b.

Notice *semi-group* character: RG transformation has no inverse.

Momentum shell RG: Gaussian model

$$\mathcal{H}_0[S_<] + \mathcal{H}_0[S_>] = \left(\int_q^< + \int_q^>\right)\left[\frac{r+q^2}{2}\,|S(q)|^2 - h(q)\,S(-q)\right],$$

where $\int_q^< \ldots = \int_{|q|<\Lambda/b} \frac{d^d q}{(2\pi)^d} \ldots$, $\int_q^> \ldots = \int_{\Lambda/b \le |q| \le \Lambda} \frac{d^d q}{(2\pi)^d} \ldots$.

Choose $\zeta = \frac{d-2}{2} \;\Rightarrow\; r \to r' = b^2 r$,

$$h(q) \to h'(q') = b^{-\zeta} h(q)\;, \quad h(x) \to h'(x') = b^{d-\zeta} h(x)\;.$$

$\Rightarrow$ r, h both *relevant* $\Rightarrow$ *critical surface* $r = 0 = h$.

- Correlation length: $\xi \to \xi' = \xi/b \;\Rightarrow\; \xi \propto r^{-1/2}$: $\nu = \frac{1}{2}$.
- Correlation function: $C'(x') = b^{2\zeta}\, C(x) \;\Rightarrow\; \eta = 0$.

Add other couplings:

- $c\int d^d x\,(\nabla^2 S)^2$: $c \to c' = b^{d-4-2\zeta} c = b^{-2} c$, *irrelevant*.
- $u\int d^d x\, S(x)^4$: $u \to u' = b^{d-4\zeta} u = b^{4-d} u$; *relevant* for $d < 4$, (dangerously) *irrelevant* for $d > 4$, *marginal* at $d = d_c = 4$.
- $v\int d^d x\, S(x)^6$: $v \to v' = b^{6-2d} v$, marginal for $d = 3$; *irrelevant* near $d_c = 4$: $v' = b^{-2} v$.

Momentum shell RG: general structure

General choice: $\zeta = \frac{d-2+\eta}{2} \Rightarrow \tau' = b^{1/\nu}\tau,\ h' = b^{(d+2-\eta)/2}h.$

- Only *two relevant* parameters τ and h.
- Few *marginal* couplings $u_i \to u_i' = u_i^* + b^{-x_i}u_i,\ x_i > 0.$
- Other couplings *irrelevant*: $v_i \to v_i' = b^{-y_i}v_i,\ y_i > 0.$

After single RG transformation:

$$f_{\text{sing}}(\tau, h, \{u_i\}, \{v_i\}) = b^{-d} f_{\text{sing}}\left(b^{1/\nu}\tau, b^{d-\zeta}h, \left\{u_i^* + \frac{u_i}{b^{x_i}}\right\}, \left\{\frac{v_i}{b^{y_i}}\right\}\right).$$

After sufficiently many $\ell \gg 1$ RG transformations:

$$f_{\text{sing}}(\tau, h, \{u_i\}, \{v_i\}) = b^{-\ell d} f_{\text{sing}}\left(b^{\ell/\nu}\tau, b^{\ell(d+2-\eta)/2}h, \{u_i^*\}, \{0\}\right).$$

Choose *matching condition* $b^\ell|\tau|^\nu = 1 \Rightarrow$ *scaling form*:

$$f_{\text{sing}}(\tau, h) = |\tau|^{d\nu}\, \hat{f}_\pm\left(h/|\tau|^{\nu(d+2-\eta)/2}\right)\ .$$

Correlation function scaling law: use $b^\ell = \xi/\xi_\pm \Rightarrow$

$$C(\tau, x, \{u_i\}, \{v_i\}) = b^{-2\ell\zeta}\, C\left(b^{\ell/\nu}\tau, \frac{x}{b^\ell}, \{u_i^*\}, \{0\}\right) \to \frac{\widetilde{C}_\pm(x/\xi)}{|x|^{d-2+\eta}}\ .$$

Perturbation expansion

Nonlinear interaction term:

$$\mathcal{H}_{\rm int}[S] = \frac{u}{4!} \int_{|q_i|<\Lambda} S(q_1) S(q_2) S(q_3) S(-q_1 - q_2 - q_3) \ .$$

Rewrite partition function and N-point correlation functions:

$$\mathcal{Z}[h] = \mathcal{Z}_0[h] \left\langle e^{-\mathcal{H}_{\rm int}[S]} \right\rangle_0 , \ \left\langle \prod_i S(q_i) \right\rangle = \frac{\left\langle \prod_i S(q_i)\, e^{-\mathcal{H}_{\rm int}[S]} \right\rangle_0}{\left\langle e^{-\mathcal{H}_{\rm int}[S]} \right\rangle_0} \ .$$

contraction: $\underbrace{S(q) S}(q') = \langle S(q) S(q') \rangle_0 = C_0(q)\, (2\pi)^d \delta(q + q')$

$\Rightarrow$ *Wick's theorem*:

$$\langle S(q_1) S(q_2) \ldots S(q_{N-1}) S(q_N) \rangle_0 =$$
$$= \sum_{\substack{\rm permutations \\ i_1(1) \ldots i_N(N)}} \underbrace{S(q_{i_1(1)}) S}(q_{i_2(2)}) \ \ldots \ \underbrace{S(q_{i_{N-1}(N-1)}) S}(q_{i_N(N)}) \ .$$

$\Rightarrow$ compute all expectation values in *Gaussian ensemble*.

First-order correction to two-point function

Consider $\langle S(q)S(q')\rangle = C(q)\,(2\pi)^d\delta(q+q')$ for $h=0$; to $O(u)$:

$$\left\langle S(q)S(q')\left[1-\frac{u}{4!}\int_{|q_i|<\Lambda} S(q_1)S(q_2)S(q_3)S(-q_1-q_2-q_3)\right]\right\rangle_0 .$$

- Contractions of external legs $\underbrace{S(q)S}(q')$:
  terms cancel with denominator, leaving $\langle S(q)S(q')\rangle_0$.
- The remaining twelve contributions are of the form
  $\int_{|q_i|<\Lambda} \underbrace{S(q)S}(q_1)\,\underbrace{S(q_2)S}(q_3)\,\underbrace{S(-q_1-q_2-q_3)S}(q') =$
  $= C_0(q)^2\,(2\pi)^d\delta(q+q')\int_{|p|<\Lambda} C_0(p).$

$$\Rightarrow\ C(q) = C_0(q)\left[1-\frac{u}{2}\,C_0(q)\int_{|p|<\Lambda} C_0(p)+O(u^2)\right];$$

re-interpret as first-order self-energy in Dyson's equation:

$$C(q)^{-1} = r+q^2+\frac{u}{2}\int_{|p|<\Lambda}\frac{1}{r+p^2}+O(u^2)\ .$$

Notice: to first order in u, there is only "mass" renormalization, no change in momentum dependence of $C(q)$.

Wilson RG procedure: first-order recursion relations

Split field variables in outer ($S_>$) / inner ($S_<$) momentum shell:

- simply re-exponentiate terms $\sim u \int S_<^4 \, e^{-\mathcal{H}_0[S]}$;
- contributions such as $u \int S_<^3 \, S_> e^{-\mathcal{H}_0[S]}$ vanish;
- terms $\sim u \int S_>^4 \, e^{-\mathcal{H}_0[S]} \to \mathrm{const.}$, contribute to free energy;
- contributions $\sim u \int S_<^2 \, S_>^2 \, e^{-\mathcal{H}_0}$: Gaussian integral over $S_>$.

With $S_d = K_d/(2\pi)^d = 1/2^{d-1}\pi^{d/2}\Gamma(d/2)$ and $\eta = 0$ to $O(u)$:

$$r' = b^2 \left[r + \frac{u}{2} \, A(r) \right] = b^2 \left[r + \frac{u}{2} \, S_d \int_{\Lambda/b}^{\Lambda} \frac{p^{d-1}}{r + p^2} \, dp \right] ,$$

$$u' = b^{4-d} u \left[1 - \frac{3u}{2} \, B(r) \right] = b^{4-d} u \left[1 - \frac{3u}{2} S_d \int_{\Lambda/b}^{\Lambda} \frac{p^{d-1} \, dp}{(r + p^2)^2} \right] .$$

- $r \gg 1$: fluctuation contributions disappear, Gaussian theory;
- $r \ll 1$: expand

$$A(r) = S_d \Lambda^{d-2} \frac{1 - b^{2-d}}{d-2} - r \, S_d \Lambda^{d-4} \, \frac{1 - b^{4-d}}{d-4} + O(r^2) \ ,$$

$$B(r) = S_d \Lambda^{d-4} \, \frac{1 - b^{4-d}}{d-4} + O(r) \ .$$

Differential RG flow, fixed point, dimensional expansion

Differential RG flow: set $b = e^{\delta\ell}$ with $\delta\ell \to 0$:

$$\frac{d\tilde{r}(\ell)}{d\ell} = 2\tilde{r}(\ell) + \frac{\tilde{u}(\ell)}{2} S_d \Lambda^{d-2} - \frac{\tilde{r}(\ell)\tilde{u}(\ell)}{2} S_d \Lambda^{d-4} + O(\tilde{u}\tilde{r}^2, \tilde{u}^2),$$

$$\frac{d\tilde{u}(\ell)}{d\ell} = (4-d)\tilde{u}(\ell) - \frac{3}{2}\,\tilde{u}(\ell)^2 S_d \Lambda^{d-4} + O(\tilde{u}\tilde{r}, \tilde{u}^2)\ .$$

Renormalization group *fixed points*: $d\tilde{r}(\ell)/d\ell = 0 = d\tilde{u}(\ell)/d\ell$.

- *Gauss*: $u_0^* = 0 \ \leftrightarrow\ $ *Ising*: $u_{\rm I}^* S_d = \frac{2}{3}\,(4-d)\Lambda^{4-d}$, $d < 4$.
- Linearize $\delta\tilde{u}(\ell) = \tilde{u}(\ell) - u_{\rm I}^*$: $\frac{d}{d\ell}\,\delta\tilde{u}(\ell) \approx (d-4)\delta\tilde{u}(\ell)$
  $\Rightarrow$ u_0^* stable for $d > 4$, $u_{\rm I}^*$ stable for $d < 4$.
- Small expansion parameter: $\epsilon = 4 - d = d_c - d$
  $u_{\rm I}^*$ emerges continuously from $u_0^* = 0$.
- Insert: $r_{\rm I}^* = -\frac{1}{4}\,u_{\rm I}^* S_d \Lambda^{d-2} = -\frac{1}{6}\,\epsilon\Lambda^2$: non-universal,
  describes fluctuation-induced downward T_c-shift.
- RG procedure generates new terms $\propto S^6$, $\nabla^2 S^4$, etc.
  To $O(\epsilon^3)$, feedback into recursion relations can be neglected.

Critical exponents

Deviation from true T_c: $\tau = r - r_{\rm I}^* \propto T - T_c$.
Recursion relation for this (relevant) *running coupling*:

$$\frac{d\tilde{\tau}(\ell)}{d\ell} = \tilde{\tau}(\ell) \left[2 - \frac{\tilde{u}(\ell)}{2} S_d \Lambda^{d-4} \right] .$$

Solve near Ising fixed point: $\tilde{\tau}(\ell) = \tilde{\tau}(0) \exp\left[\left(2 - \frac{\epsilon}{3}\right) \ell\right]$.
Compare with $\widetilde{\xi}(\ell) = \xi(0)\, e^{-\ell} \;\Rightarrow\; \nu^{-1} = 2 - \frac{\epsilon}{3}$.
Consistently to order $\epsilon = 4 - d$:

$$\nu = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2) \ , \quad \eta = 0 + O(\epsilon^2) \ .$$

Note at $d = d_c = 4$: $\tilde{u}(\ell) = \tilde{u}(0) / [1 + 3\, \tilde{u}(0)\, \ell / 16\pi^2]$
$\Rightarrow$ *logarithmic corrections* to mean-field exponents.

Renormalization group procedure:

- Derive scaling laws.
- Two relevant couplings $\Rightarrow$ independent critical exponents.
- Compute scaling exponents via power series in $\epsilon = d_c - d$.

Selected literature:

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Some exercises

1. *Landau theory for the ϕ^6 model.*
   Consider the *effective free energy*

$$f(\phi) = \frac{r}{2}\,\phi^2 + \frac{u}{4!}\,\phi^4 + \frac{v}{6!}\,\phi^6 - h\,\phi\ .$$

   Here, $r = a(T - T_0)$, $v > 0$, and h denotes an external field.
   (a) Show that for $u > 0$, there is a *second-order* phase transition at $h = 0$ and $T = T_0$ with the usual mean-field critical exponents β, γ, δ, and α. Why can v be neglected near the critical point ?
   (b) Compute $\beta_t, \gamma_t, \delta_t$, and α_t at the *tricritical point* $u = 0$.
   (c) Now assume $u = -|u| < 0$ and $h = 0$. Show that there is a *first-order* transition at $r_d = 5u^2/8v$, and calculate the jump in the order parameter and the associated free-energy barrier.
   (d) For non-zero external field $h \neq 0$ and $u < 0$, find parametric equations $r_c(|u|, v)$ and $h_c(|u|, v)$ for two additional *second-order* transition lines, with all three continuous phase boundaries merging at the tricritical point $u = 0$, $h = 0$.

2. *Gaussian approximation for the Heisenberg model.*
   Isotropic magnets with continuous rotational spin symmetry are described by the *Heisenberg model*. The corresponding effective Landau–Ginzburg–Wilson Hamiltonian reads

$$\mathcal{H}[S] = \int d^dx \sum_{\alpha=1}^{n} \left(\frac{r}{2}\,[S^\alpha(x)]^2 + \frac{1}{2}\,[\nabla S^\alpha(x)]^2 + \frac{u}{4!}\,\sum_{\beta=1}^{n} [S^\alpha(x)]^2 [S^\beta(x)]^2 - h^\alpha(x)\,S^\alpha(x) \right)\,,$$

   where $S^\alpha(x)$ is an n-component order parameter vector field.
   (a) Determine the two-point correlation functions in the high- and low-temperature phases in harmonic (Gaussian) approximation.
   Notice: For $T < T_c$, it is useful to expand about the spontaneous magnetization: e.g., $S^\alpha(x) = \pi^\alpha(x)$ for $\alpha = 1, \ldots, n-1$, and $S^n(x) = \phi + \sigma(x)$; then $\langle \pi^\alpha \rangle = 0 = \langle \sigma \rangle$. The components along and perpendicular to ϕ must be carefully distinguished.
   (b) For $d < d_c = 4$, compute the specific heat in Gaussian approximation on both sides of the phase transition, and show that $C_{h=0} = C_\pm |\tau|^{-(4-d)/2}$. Compute the *universal amplitude ratio* $C_+/C_- = 2^{-d/2} n$.

More exercises

3. *First-order recursion relations for the Heisenberg model.*
For the n-component *Heisenberg model* above, derive the renormalization group recursion relations

$$r' = b^2 \left[r + \frac{n+2}{6} \, u \, A(r) \right] \; , \quad u' = b^{4-d} u \left[1 - \frac{n+8}{6} \, u \, B(r) \right] \; .$$

Determine the associated RG fixed points and discuss their stability. Compute the critical exponent ν to first order in $\epsilon = 4 - d$.

4. *RG flow equations for the n-vector model with cubic anisotropy.*
The $O(n)$ rotational invariance of the Hamiltonian in the previous problems is broken by additional quartic terms with cubic symmetry,

$$\Delta \mathcal{H}[S] = \int d^d x \sum_{\alpha=1}^{n} \frac{v}{4!} \, [S^\alpha(x)]^4 \; .$$

(a) Derive the differential RG flow equations for the running couplings $\tilde{r}(\ell)$, $\tilde{u}(\ell)$, and $\tilde{v}(\ell)$.
(b) Discuss the ensuing RG fixed points and their stability as function of the number n of order parameter components, and compute the associated correlation length critical exponents ν.

Field Theory Approach to Critical Phenomena

Perturbation expansion

$O(n)$-symmetric Hamiltonian (set $k_{\mathrm{B}}T = 1$):

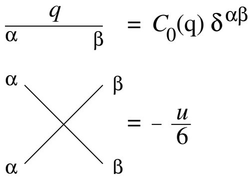
$$\mathcal{H}[S] = \int d^d x \sum_{\alpha=1}^{n} \left[\frac{r}{2} S^\alpha(x)^2 + \frac{1}{2} [\nabla S^\alpha(x)]^2 + \frac{u}{4!} \sum_{\beta=1}^{n} S^\alpha(x)^2 S^\beta(x)^2 \right] .$$

Construct *perturbation expansion* for $\left\langle \prod_{ij} S^{\alpha_i} S^{\alpha_j} \right\rangle$:

$$\frac{\left\langle \prod_{ij} S^{\alpha_i} S^{\alpha_j} e^{-\mathcal{H}_{\mathrm{int}}[S]} \right\rangle_0}{\left\langle e^{-\mathcal{H}_{\mathrm{int}}[S]} \right\rangle_0} = \frac{\left\langle \prod_{ij} S^{\alpha_i} S^{\alpha_j} \sum_{l=0}^{\infty} \frac{(-\mathcal{H}_{\mathrm{int}}[S])^l}{l!} \right\rangle_0}{\left\langle \sum_{l=0}^{\infty} \frac{(-\mathcal{H}_{\mathrm{int}}[S])^l}{l!} \right\rangle_0} .$$

Diagrammatic representation:

- *Propagator* $C_0(q) = (r + q^2)^{-1}$;
- *Vertex* $-\frac{u}{6}$.

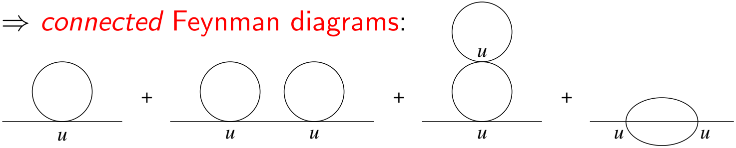


Generating functional for correlation functions (cumulants):

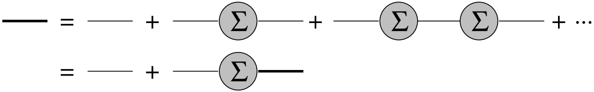
$$\mathcal{Z}[h] = \left\langle \exp \int d^d x \sum_\alpha h^\alpha S^\alpha \right\rangle , \quad \left\langle \prod_i S^{\alpha_i} \right\rangle_{(c)} = \prod_i \frac{\delta(\ln)\mathcal{Z}[h]}{\delta h^{\alpha_i}} \Big|_{h=0} .$$

Vertex functions

⇒ *connected* Feynman diagrams:



Dyson equation:



⇒ propagator self-energy: $C(q)^{-1} = C_0(q)^{-1} - \Sigma(q)$.

Generating functional for *vertex functions*, $\Phi^\alpha = \delta \ln \mathcal{Z}[h] / \delta h^\alpha$:

$$\Gamma[\Phi] = -\ln \mathcal{Z}[h] + \int d^d x \sum_\alpha h^\alpha \, \Phi^\alpha \, , \quad \Gamma^{(N)}_{\{\alpha_i\}} = \prod_i^N \frac{\delta \Gamma[\Phi]}{\delta \Phi^{\alpha_i}} \bigg|_{h=0} \, ;$$

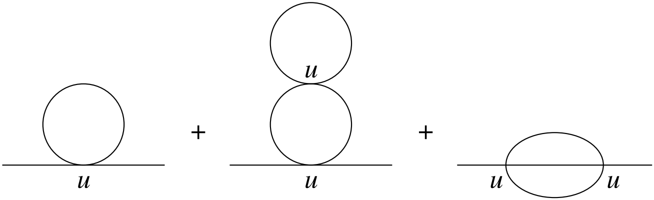
$$\Rightarrow \; \Gamma^{(2)}(q) = C(q)^{-1} \, , \; \left\langle \prod_{i=1}^4 S(q_i) \right\rangle_c = - \prod_{i=1}^4 C(q_i) \, \Gamma^{(4)}(\{q_i\})$$

⇒ *one-particle irreducible* Feynman graphs.

Perturbation series in nonlinear coupling u ⇔ *loop expansion*.

Explicit results

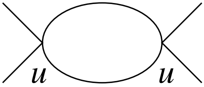
Two-point vertex function to two-loop order:



$$\Gamma^{(2)}(q) = r + q^2 + \frac{n+2}{6}\, u \int_k \frac{1}{r+k^2}$$
$$- \left(\frac{n+2}{6}\, u\right)^2 \int_k \frac{1}{r+k^2} \int_{k'} \frac{1}{(r+k'^2)^2}$$
$$- \frac{n+2}{18}\, u^2 \int_k \frac{1}{r+k^2} \int_{k'} \frac{1}{r+k'^2}\, \frac{1}{r+(q-k-k')^2}\ ;$$

four-point vertex function to one-loop order:

$$\Gamma^{(4)}(\{q_i = 0\}) = u - \frac{n+8}{6}\, u^2 \int_k \frac{1}{(r+k^2)^2}\ .$$



Ultraviolet and infrared divergences

Fluctuation correction to four-point vertex function:

$$d < 4 : \; u \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{(r + k^2)^2} = \frac{u \, r^{-2+d/2}}{2^{d-1} \pi^{d/2} \Gamma(d/2)} \int_0^\infty \frac{x^{d-1}}{(1 + x^2)^2} \, dx \, ,$$

effective coupling $u \, r^{(d-4)/2} \to \infty$ as $r \to 0$: *infrared* divergence,
$\Rightarrow$ fluctuation corrections singular, modify critical power laws.

$$\int_0^\Lambda \frac{k^{d-1}}{(r + k^2)^2} \, dk \sim \left\{ \begin{array}{cc} \ln(\Lambda^2/r) & d = 4 \\ \Lambda^{d-4} & d > 4 \end{array} \right\} \to \infty \quad \text{as } \Lambda \to \infty \, ,$$

ultraviolet divergences for $d > d_c = 4$: *upper critical dimension*.
Power counting in terms of arbitrary momentum scale μ:

- $[x] = \mu^{-1}$, $[q] = \mu$, $[S^\alpha(x)] = \mu^{-1+d/2}$;
- $\Rightarrow$ $[r] = \mu^2 \to$ *relevant*, $[u] = \mu^{4-d}$ *marginal* at $d_c = 4$;
- only divergent vertex functions: $\Gamma^{(2)}(q)$, $\Gamma^{(4)}(\{q_i = 0\})$;
- field dimensionless at *lower critical dimension* $d_{lc} = 2$.

Dimension regimes and dimensional regularization

dimension interval	perturbation series	$O(n)$-symmetric Φ^4 field theory	critical behavior
$d \leq d_{lc} = 2$	IR-singular UV-convergent	ill-defined u relevant	no long-range order ($n \geq 2$)
$2 < d < 4$	IR-singular UV-convergent	super-renormalizable u relevant	non-classical exponents
$d = d_c = 4$	logarithmic IR-/ UV-divergence	renormalizable u marginal	logarithmic corrections
$d > 4$	IR-regular UV-divergent	non-renormalizable u irrelevant	mean-field exponents

Integrals in *dimensional regularization*: even for non-integer d, σ:

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^{2\sigma}}{(\tau + k^2)^s} = \frac{\Gamma(\sigma + d/2)\,\Gamma(s - \sigma - d/2)}{2^d\,\pi^{d/2}\,\Gamma(d/2)\,\Gamma(s)}\;\tau^{\sigma - s + d/2}\;;$$

- $\Rightarrow$ discard divergent surface integrals;
- UV singularities $\rightarrow$ *dimensional poles* in Euler Γ functions.

Renormalization

Susceptibility $\chi^{-1} = C(q=0)^{-1} = \Gamma^{(2)}(q=0) = \tau = r - r_c$

$$\Rightarrow r_c = -\frac{n+2}{6}\, u \int_k \frac{1}{r_c + k^2} + O(u^2) = -\frac{n+2}{6}\, \frac{u\, K_d}{(2\pi)^d}\, \frac{\Lambda^{d-2}}{d-2} \ ,$$

(non-universal) T_c-shift: *additive renormalization*.

$$\Rightarrow \ \chi(q)^{-1} = q^2 + \tau \left[1 - \frac{n+2}{6}\, u \int_k \frac{1}{k^2(\tau + k^2)} \right].$$

Multiplicative renormalization:

absorb UV poles at $\epsilon = 0$ into *renormalized* fields and parameters:

$$S_R^\alpha = Z_S^{1/2}\, S^\alpha \ \Rightarrow \ \Gamma_R^{(N)} = Z_S^{-N/2}\, \Gamma^{(N)} \ ;$$

$$\tau_R = Z_\tau\, \tau\, \mu^{-2} \ , \qquad u_R = Z_u\, u\, A_d\, \mu^{d-4} \ , \ A_d = \frac{\Gamma(3 - d/2)}{2^{d-1}\, \pi^{d/2}} \ .$$

Normalization point outside IR regime, $\tau_R = 1$ or $q = \mu$:

$$O(u_R): \ Z_\tau = 1 - \frac{n+2}{6}\, \frac{u_R}{\epsilon} \ , \quad Z_u = 1 - \frac{n+8}{6}\, \frac{u_R}{\epsilon} \ ;$$

$$O(u_R^2): \ Z_S = 1 + \frac{n+2}{144}\, \frac{u_R^2}{\epsilon} \ .$$

Renormalization group equation

Unrenormalized quantities can*not* depend on arbitrary scale μ:

$$0 = \mu \frac{d}{d\mu} \Gamma^{(N)}(\tau, u) = \mu \frac{d}{d\mu} \left[Z_S^{N/2} \Gamma_R^{(N)}(\mu, \tau_R, u_R) \right]$$

$\Rightarrow$ *renormalization group* equation:

$$\left[\mu \frac{\partial}{\partial \mu} + \frac{N}{2} \gamma_S + \gamma_\tau \tau_R \frac{\partial}{\partial \tau_R} + \beta_u \frac{\partial}{\partial u_R} \right] \Gamma_R^{(N)}(\mu, \tau_R, u_R) = 0 \ .$$

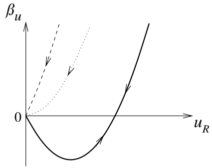
with *Wilson's flow* and RG beta functions:

$$\gamma_S = \mu \frac{\partial}{\partial \mu}\Big|_0 \ln Z_S = -\frac{n+2}{72} u_R^2 + O(u_R^3) \ ,$$

$$\gamma_\tau = \mu \frac{\partial}{\partial \mu}\Big|_0 \ln \frac{\tau_R}{\tau} = -2 + \frac{n+2}{6} u_R + O(u_R^2) \ ,$$

$$\beta_u = \mu \frac{\partial}{\partial \mu}\Big|_0 u_R = u_R \left[d - 4 + \mu \frac{\partial}{\partial \mu}\Big|_0 \ln Z_u \right]$$

$$= u_R \left[-\epsilon + \frac{n+8}{6} u_R + O(u_R^2) \right].$$



Method of characteristics

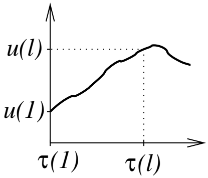
Susceptibility $\chi(q) = \Gamma^{(2)}(q)^{-1}$:

$$\chi_R(\mu, \tau_R, u_R, q)^{-1} = \mu^2 \, \hat{\chi}_R(\tau_R, u_R, q/\mu)^{-1} \ .$$

solve RG equation: *method of characteristics*

$$\mu \to \mu(\ell) = \mu \, \ell \ ,$$

$$\chi_R(\ell)^{-1} = \chi_R(1)^{-1} \, \ell^2 \, \exp\left[\int_1^\ell \gamma_S(\ell') \, \frac{d\ell'}{\ell'}\right],$$



with *running couplings*, initial values $\tilde{\tau}(1) = \tau_R$, $\tilde{u}(1) = u_R$:

$$\ell \, \frac{d\tilde{\tau}(\ell)}{d\ell} = \tilde{\tau}(\ell) \, \gamma_\tau(\ell) \ , \quad \ell \, \frac{d\tilde{u}(\ell)}{d\ell} = \beta_u(\ell) \ .$$

Near *infrared-stable RG fixed point*: $\beta_u(u^*) = 0$, $\beta'_u(u^*) > 0$

$$\tilde{\tau}(\ell) \approx \tau_R \, \ell^{\gamma_\tau^*} \ , \ \chi_R(\tau_R, q)^{-1} \approx \mu^2 \, \ell^{2+\gamma_S^*} \, \hat{\chi}_R(\tau_R \, \ell^{\gamma_\tau^*}, u^*, q/\mu \, \ell)^{-1},$$

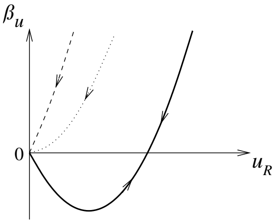
matching $\ell = |q|/\mu \Rightarrow$ scaling form with $\eta = -\gamma_S^*$, $\quad \nu = -1/\gamma_\tau^*$.

Critical exponents

Systematic $\epsilon = 4 - d$ expansion:

$$\beta_u = u_R \left[-\epsilon + \frac{n+8}{6} \, u_R + O(u_R^2) \right]$$

$$\Rightarrow \;\; u_0^* = 0 \, , \;\; u_H^* = \frac{6 \, \epsilon}{n+8} + O(\epsilon^2) \;\; ;$$



IR stability: $\beta_u'(u^*) > 0$

- $d > 4$: *Gaussian* fixed point u_0^* $\Rightarrow$ $\eta = 0$, $\nu = \frac{1}{2}$ (mean-field);
- $d < 4$: *Heisenberg* fixed point u_H^* stable

$$\Rightarrow \;\; \eta = \frac{n+2}{2\,(n+8)^2} \, \epsilon^2 + O(\epsilon^3) \; , \quad \frac{1}{\nu} = 2 - \frac{n+2}{n+8} \, \epsilon + O(\epsilon^2) \;\; ;$$

- $d = d_c = 4$: *logarithmic corrections*:

$$\tilde{u}(\ell) = \frac{u_R}{1 - \frac{n+8}{6} \, u_R \, \ln \ell} \; , \quad \tilde{\tau}(\ell) \sim \frac{\tau_R}{\ell^2 (\ln |\ell|)^{(n+2)/(n+8)}}$$

$$\Rightarrow \;\; \xi \propto \tau_R^{-1/2} \, (\ln \tau_R)^{(n+2)/2(n+8)} \quad .$$

- Accurate exponent values: Monte Carlo simulations; or: Borel resummation; non-perturbative "exact" (numerical) RG.

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- G. Parisi, *Statistical field theory*, Addison–Wesley (Redwood City, 1988).
- P. Ramond, *Field theory — A modern primer*, Benjamin–Cummings (Reading, 1981).
- J. Zinn-Justin, *Quantum field theory and critical phenomena*, Clarendon Press (Oxford, 1993).

Some exercises

1. *Relationship between cumulants and vertex functions.*
   By means of appropriate derivatives of the generating functional for the vertex functions, establish the relations

$$\Gamma^{(2)}(q) = C(q)^{-1} \; , \; \Big\langle \prod_{i=1}^{4} S(q_i) \Big\rangle_c = - \prod_{i=1}^{4} C(q_i) \, \Gamma^{(4)}(\{q_i\})$$

   between the two- and four-point vertex functions and cumulants.

2. *Explicit two-loop perturbation theory for the vertex functions.*
   Confirm the explicit two-loop result for $\Gamma^{(2)}(q)$ and the one-loop expression for $\Gamma^{(4)}(\{q_i = 0\})$.

3. *Singular contribution to the two-loop propagator self-energy.*
   Employ Feynman parametrization

$$\frac{1}{A^r \, B^s} = \frac{\Gamma(r+s)}{\Gamma(r)\,\Gamma(s)} \int_0^1 \frac{x^{r-1}\,(1-x)^{s-1}}{[x\,A + (1-x)\,B]^{r+s}} \, dx$$

   to extract the UV-singular part of the two-loop integral

$$D(q) = \int_k \frac{1}{\tau + k^2} \int_{k'} \frac{1}{\tau + k'^2} \, \frac{1}{\tau + (q - k - k')^2} \; ,$$

$$\Rightarrow \; \frac{\partial D(q)}{\partial q^2}\bigg|_{q=0}^{\text{sing.}} = - \, \frac{A_d^2 \, \tau^{-\epsilon}}{8\,\epsilon} \; .$$