Renormalization Group: Applications in Statistical Physics

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Physics at All Scales: The Renormalization Group

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Lecture plan

Critical Dynamics

- Dynamical scaling hypothesis
- Langevin dynamics and response functional
- Dynamic perturbation theory
- Critical dynamics of the relaxational models A and B
- Critical dynamics of isotropic ferromagnets
- Driven diffusive systems
- Literature

Scale Invariance, Phase Transitions In Interacting Particle Systems

- Chemical reactions and population dynamics
- Master equation and coherent-state path integral representation
- Diffusion-limited annihilation processes
- Phase transitions from active to absorbing states
- Literature

Critical Dynamics

Critical slowing down

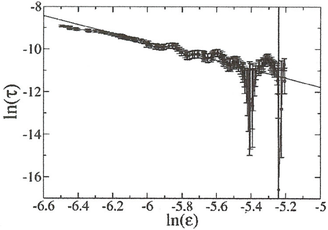
Recall behavior of static correlation function near critical point:

- *correlation length* diverges, $\xi(\tau) \sim |\tau|^{-\nu}$, $\tau = (T - T_c)/T_c$.
- $\chi(\tau, q) = |q|^{-2+\eta}\,\hat{\chi}_\pm(q\xi)$, $C(\tau, x) = |x|^{-(d-2+\eta)}\,\widetilde{C}_\pm(x/\xi)$.

Expect *critical slowing-down* as correlated regions grow:

- *relaxation time* diverges
  $t_c(\tau) \sim \xi(\tau)^z \sim |\tau|^{-z\nu}$;
- *characteristic frequency* scale:
  $\omega_c(\tau, q) = |q|^z\,\hat{\omega}_\pm(q\,\xi)$;

$\Rightarrow$ *dynamic critical exponent* z.



Magnetization relaxation time of a Fe bilayer (on a W substrate) near $T_c = 453\,\mathrm{K}$. The data for this two-dimensional Ising ferromagnet yield $z\nu = 2.09 \pm 0.06$.
From: M.J. Dunlavy & D. Venus, Phys. Rev. B **71**, *144406 (2005).*

Dynamic response and correlations:

$$\chi(x - x', t - t') = \left.\frac{\partial\langle S(x,t)\rangle}{\partial h(x', t')}\right|_{h=0},$$

$$C(x, t) = \langle S(x,t)\,S(0,0)\rangle - \langle S\rangle^2 .$$

Fluctuation-dissipation theorem:

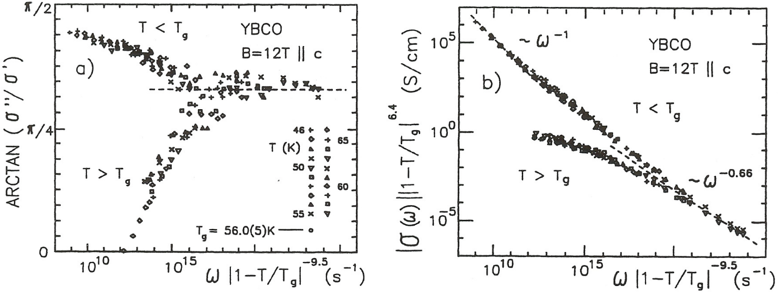
$$C(q, \omega) = 2k_{\mathrm{B}}T\,\mathrm{Im}\,\chi(q,\omega)/\omega.$$

Dynamical scaling hypothesis

Generalize static scaling hypothesis:

$$\chi(\tau, q, \omega) = |q|^{-2+\eta}\, \hat{\chi}_{\pm}(q\,\xi, \omega\,\xi^z) \ ,$$
$$C(\tau, x, t) = |x|^{-(d-2+\eta)}\, \widetilde{C}_{\pm}(x/\xi, t/t_c) \ .$$

Dynamical scaling hypothesis applies also to *transport coefficients*:



(a) Phase and (b) scaled amplitude of the linear conductivity vs. rescaled frequency measured for the normal- to superconducting phase transition ($T_g \approx 56\,\mathrm{K}$) in an external magnetic field ($B = 12\,\mathrm{T}$) in a YBCO crystal.
From: J. Kötzler, M. Kaufmann, G. Nakielski, R. Behr, and W. Assmus, Phys. Rev. Lett. **72**, *2081 (1994).*

Langevin description of critical dynamics

Coarse-grained description:

- *fast* modes $\rightarrow$ *random noise*;
- *mesoscopic Langevin equation* for *slow* variables $S^\alpha(x,t)$.

Example: purely *relaxational critical dynamics*:

$$\frac{\partial S^\alpha(x,t)}{\partial t} = -D\,\frac{\delta \mathcal{H}[S]}{\delta S^\alpha(x,t)} + \zeta^\alpha(x,t)\,,\ \ \langle \zeta^\alpha(x,t) \rangle = 0\ ,$$
$$\langle \zeta^\alpha(x,t)\,\zeta^\beta(x',t') \rangle = 2D\,k_{\rm B} T\,\delta(x-x')\delta(t-t')\delta^{\alpha\beta}\ ;$$

Einstein relation guarantees that $\mathcal{P}[S,t] \rightarrow \mathcal{P}_s[S]$ as $t \rightarrow \infty$.

Non-conserved order parameter: $D = \mathrm{const.}$
Conserved order parameter: relaxes *diffusively*, $D \rightarrow -D\,\nabla^2$;
model A / B: $D(i\nabla)^a$, $a = 0, 2$: non-conserved / conserved.

Generally: *mode couplings* to additional *conserved*, slow fields
$\Rightarrow$ various *dynamic universality classes*.

Relaxational models A/B: Gaussian approximation

Gaussian (mean-field) approximation: $u = 0$; Fourier transform:

$$\left[-i\omega + Dq^a(r + q^2)\right] S^\alpha(q,\omega) = Dq^a\, h^\alpha(q,\omega) + \zeta^\alpha(q,\omega) \ ,$$

$$\langle \zeta^\alpha(q,\omega)\zeta^\beta(q',\omega')\rangle = 2k_{\mathrm{B}} T\, Dq^a (2\pi)^{d+1}\delta(q+q')\delta(\omega+\omega')\delta^{\alpha\beta}.$$

$\Rightarrow$ *Dynamic response and correlation functions*:

$$\chi_0^{\alpha\beta}(q,\omega) = \left.\frac{\partial\langle S^\alpha(q,\omega)\rangle}{\partial h^\beta(q,\omega)}\right|_{h=0} = Dq^a\, G_0(q,\omega)\delta^{\alpha\beta} \ ,$$

$$G_0(q,\omega) = \frac{1}{-i\omega + Dq^a(r+q^2)} \ ; \ G_0(q,t) = e^{-Dq^a(r+q^2)t}\,\Theta(t).$$

$$\langle S^\alpha(q,\omega)S^\beta(q',\omega')\rangle_0 = C_0(q,\omega)(2\pi)^{d+1}\delta(q+q')\delta(\omega+\omega')\delta^{\alpha\beta},$$

$$C_0(q,\omega) = \frac{2k_{\mathrm{B}} T\, Dq^a}{\omega^2 + [Dq^a(r+q^2)]^2} = 2k_{\mathrm{B}} T Dq^a\, |G_0(q,\omega)|^2 \ ;$$

$$C_0(q,t) = \frac{k_{\mathrm{B}} T}{r+q^2}\, e^{-Dq^a\,(r+q^2)|t|} \ .$$

$\Rightarrow$ *Gaussian critical exponents*: $\nu = \frac{1}{2}$, $\eta = 0$, and $z = 2 + a$.

Onsager–Machlup functional

Coupled *Langevin equations* for mesoscopic stochastic variables:

$$\frac{\partial S^\alpha(x,t)}{\partial t} = F^\alpha[S](x,t) + \zeta^\alpha(x,t)\,, \ \langle \zeta^\alpha(x,t) \rangle = 0 \ ,$$
$$\langle \zeta^\alpha(x,t) \zeta^\beta(x',t') \rangle = 2L^\alpha \, \delta(x-x') \, \delta(t-t') \, \delta^{\alpha\beta} \ ;$$

- *systematic* forces $F^\alpha[S]$, *stochastic forces (noise)* ζ^α;
- noise *correlator* L^α: can be operator, functional of S^α.

Assume *Gaussian* stochastic process $\Rightarrow$ probability distribution:

$$\mathcal{W}[\zeta] \propto \exp\left[-\frac{1}{4} \int d^d x \int_0^{t_f} dt \sum_\alpha \zeta^\alpha(x,t) \left[(L^\alpha)^{-1} \zeta^\alpha(x,t) \right] \right],$$

switch variables $\zeta^\alpha \to S^\alpha$: $\mathcal{W}[\zeta]\,\mathcal{D}[\zeta] = \mathcal{P}[S]\mathcal{D}[S] \propto e^{-\mathcal{G}[S]}\mathcal{D}[S]$,
with *Onsager-Machlup functional* providing field-theory action:

$$\mathcal{G}[S] = \frac{1}{4} \int d^d x \int dt \sum_\alpha (\partial_t S^\alpha - F^\alpha[S]) \left[(L^\alpha)^{-1} (\partial_t S^\alpha - F^\alpha[S]) \right] .$$

- Functional determinant $= 1$ with *forward* (Itô) discrectization;
- normalization: $\int \mathcal{D}[\zeta] W[\zeta] = 1 \ \Rightarrow$ "partition function" $= 1$;
- problems: $(L^\alpha)^{-1}$, high non-linearities $F^\alpha[S] \, (L^\alpha)^{-1} \, F^\alpha[S]$.

Janssen–De Dominicis response functional

Average over noise 'histories': $\langle A[S]\rangle_\zeta \propto \int \mathcal{D}[\zeta]\, A[S(\zeta)]\, W[\zeta]$:

use $1 = \int \mathcal{D}[S] \prod_\alpha \prod_{(x,t)} \delta\big(\partial_t S^\alpha(x,t) - F^\alpha[S](x,t) - \zeta^\alpha(x,t)\big)$

$= \int \mathcal{D}[i\widetilde{S}] \int \mathcal{D}[S] \exp\big[-\int d^d x \int dt \sum_\alpha \widetilde{S}^\alpha(\partial_t S^\alpha - F^\alpha[S] - \zeta^\alpha)\big]$.

$$\langle A[S]\rangle_\zeta \propto \int \mathcal{D}[i\widetilde{S}] \int \mathcal{D}[S] \exp\left[-\int d^d x \int dt \sum_\alpha \widetilde{S}^\alpha(\partial_t S^\alpha - F^\alpha[S])\right]$$

$$\times A[S] \int \mathcal{D}[\zeta] \exp\left(-\int d^d x \int dt \sum_\alpha \left[\frac{1}{4}\zeta^\alpha (L^\alpha)^{-1} \zeta^\alpha - \widetilde{S}^\alpha \zeta^\alpha\right]\right).$$

Perform Gaussian integral over noise ζ^α:

$$\langle A[S]\rangle_\zeta = \int \mathcal{D}[S]\, A[S]\, \mathcal{P}[S]\,, \; \mathcal{P}[S] \propto \int \mathcal{D}[i\widetilde{S}]\, e^{-\mathcal{A}[\widetilde{S},S]}\,,$$

with *Janssen–De Dominicis response functional*:

$$\mathcal{A}[\widetilde{S},S] = \int d^d x \int_0^{t_f} dt \sum_\alpha \left[\widetilde{S}^\alpha\,(\partial_t S^\alpha - F^\alpha[S]) - \widetilde{S}^\alpha L^\alpha \widetilde{S}^\alpha\right],$$

$\int \mathcal{D}[i\widetilde{S}] \int \mathcal{D}[S]\, e^{-\mathcal{A}[\widetilde{S},S]} = 1$; integrate out $\widetilde{S}^\alpha$: Onsager–Machlup.

Purely relaxational models A and B

$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_{\rm int}$, with $(k_{\rm B} T = 1)$:

$$\mathcal{A}[\widetilde{S}, S] = \int d^d x \int dt \sum_\alpha \biggl(\widetilde{S}^\alpha \left[\partial_t + D(i\nabla)^a (r - \nabla^2) \right] S^\alpha$$

$$- D \widetilde{S}^\alpha (i\nabla)^a \widetilde{S}^\alpha - D \widetilde{S}^\alpha (i\nabla)^a h^\alpha + D \, \frac{u}{6} \sum_\beta \widetilde{S}^\alpha (i\nabla)^a S^\alpha S^\beta S^\beta \biggr),$$

$$\chi^{\alpha\beta}(x - x', t - t') = \frac{\delta \langle S^\alpha(x,t) \rangle}{\delta h^\beta(x',t')} \bigg|_{h=0} = D \bigl\langle S^\alpha(x,t) (i\nabla)^a \, \widetilde{S}^\beta(x',t') \bigr\rangle;$$

$\Rightarrow$ $\widetilde{S}^\alpha$ *"response" fields*; *fluctuation–dissipation theorem*:

$$\chi^{\alpha\beta}(x - x', t - t') = \Theta(t - t') \, \frac{\partial}{\partial t'} \Bigl\langle S^\alpha(x,t) S^\beta(x',t') \Bigr\rangle \ .$$

Generating functional for correlation functions, cumulants:

$$\mathcal{Z}[\tilde{j}, j] = \left\langle \exp \int d^d x \int dt \sum_\alpha \left(\tilde{j}^\alpha \, \widetilde{S}^\alpha + j^\alpha \, S^\alpha \right) \right\rangle \ ,$$

$$\Bigl\langle \prod_{ij} S^{\alpha_i} \, \widetilde{S}^{\alpha_j} \Bigr\rangle_{(c)} = \prod_{ij} \frac{\delta}{\delta j^{\alpha_i}} \, \frac{\delta}{\delta \tilde{j}^{\alpha_j}} \, (\ln) \mathcal{Z}[\tilde{j}, j] \Big|_{\tilde{j}=j=0} \ .$$

Dynamic perturbation expansion

Nonlinear terms $\sim u$: treat by means of *perturbation expansion*:

$$\left\langle \prod_{ij} S^{\alpha_i} \widetilde{S}^{\alpha_j} \right\rangle = \frac{\langle \prod_{ij} S^{\alpha_i} \widetilde{S}^{\alpha_j} \, e^{-\mathcal{A}_{\rm int}[\widetilde{S},S]} \rangle_0}{\langle e^{-\mathcal{A}_{\rm int}[\widetilde{S},S]} \rangle_0} =$$

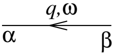
$$= \left\langle \prod_{ij} S^{\alpha_i} \widetilde{S}^{\alpha_j} \sum_{l=0}^{\infty} \frac{1}{l!} \left(-\mathcal{A}_{\rm int}[\widetilde{S}, S] \right)^l \right\rangle_0 ,$$

since the denominator $= 1$: *no "vacuum" contributions*.

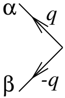
Causality: $\langle \widetilde{S}^\alpha(q,\omega) \widetilde{S}^\beta(q',\omega') \rangle_0 = 0$.

Gaussian approximation ($u = 0$) recovers $G_0(q,\omega)$, $C_0(q,\omega)$.

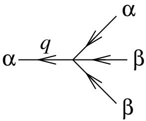
Graphical representation: *directed* (causality !) *propagator lines* connect $\widetilde{S}^\beta$ to S^α, join at *vertices* (subject to q, ω conservation).



$$= \frac{1}{-i\,\omega + D\,q^a (r + q^2)} \, \delta^{\alpha\beta}$$



$$= 2\,D\,q^a\,\delta^{\alpha\beta}$$



$$= -\,D\,q^a\,\frac{u}{6}$$

Vertex functions

Follow standard procedures: *cumulants* $\longleftrightarrow$ *connected* graphs;
Dyson equation for propagator: $G(q,\omega)^{-1} = G_0(q,\omega)^{-1} - \Sigma(q,\omega)$.

$\widetilde{\Phi}^\alpha = \delta \ln \mathcal{Z} / \delta \tilde{j}^\alpha$, $\Phi^\alpha = \delta \ln \mathcal{Z} / \delta j^\alpha \implies$ *generating functional*:

$$\Gamma[\widetilde{\Phi}, \Phi] = - \ln \mathcal{Z}[\tilde{j}, j] + \int d^d x \int dt \sum_\alpha \left(\tilde{j}^\alpha \, \widetilde{\Phi}^\alpha + j^\alpha \, \Phi^\alpha \right) \ ,$$

$$\Gamma^{(\widetilde{N},N)}_{\{\alpha_i\};\{\alpha_j\}} = \prod_i^{\widetilde{N}} \frac{\delta}{\delta \widetilde{\Phi}^{\alpha_i}} \prod_j^{N} \frac{\delta}{\delta \Phi^{\alpha_j}} \, \Gamma[\widetilde{\Phi}, \Phi] \Big|_{\tilde{j}=0=j} \ \Rightarrow$$

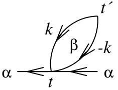
$$\Gamma^{(1,1)}(q,\omega) = G(-q,-\omega)^{-1} \ , \quad \Gamma^{(0,2)}(q,\omega) = 0 \quad \text{(causality)},$$

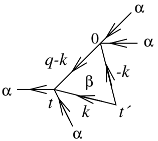
$$\Gamma^{(2,0)}(q,\omega) = - \frac{C(q,\omega)}{|G(q,\omega)|^2} = - \frac{2Dq^a}{\omega} \, \mathrm{Im} \, \Gamma^{(1,1)}(q,\omega) \ \ \text{(FDT)}.$$

$\Rightarrow$ *Vertex functions* $\longleftrightarrow$ *one-particle (1PI) irreducible graphs*.

Feynman rules

l-th order contribution to vertex function $\Gamma^{(\widetilde{N},N)}$:





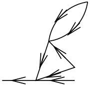
1. Draw all topologically different, connected *one-particle irreducible graphs* with $\widetilde{N}$ outgoing and N incoming lines connecting l relaxation vertices $\propto u$. Do *not* allow closed response loops [Itô calculus: $\Theta(0) = 0$].
2. Attach wave vectors q_i, frequencies ω_i or times t_i, and vector indices α_i to all directed lines, obeying *"momentum and energy" conservation* at each vertex.
3. Each *directed line* corresponds to a *response propagator* $G_0(-q, -\omega)$ / $G_0(q, t_i - t_j)$, the two-point vertex to the *noise strength* $2Dq^a$, and the four-point *relaxation vertex* to $-Dq^a u/6$. Closed loops imply integrals over the internal wave vectors and frequencies or times, subject to causality constraints, as well as sums over the internal vector indices. Apply residue theorem to evaluate frequency integrals.
4. Multiply with -1 and the *combinatorial factor* counting all possible ways of connecting the propagators, l relaxation vertices, and k two-point vertices leading to topologically identical graphs, including a factor $1/l!\,k!$ originating in the expansion of $\exp(-\mathcal{A}_{\rm int}[\widetilde{S}, S])$.

Explicit results

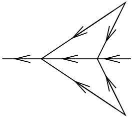
Perturbation series $\rightarrow$ *loop expansion*, $\Delta(q) = Dq^a(r+q^2)$:

$$\Gamma^{(1,1)}(q,\omega) = i\omega + Dq^a\Bigg[r + q^2 + \frac{n+2}{6}\,u\int_k \frac{1}{r+k^2}$$

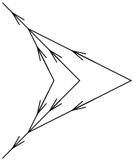
$$-\Big(\frac{n+2}{6}\,u\Big)^2 \int_k \frac{1}{r+k^2}\int_{k'}\frac{1}{(r+k'^2)^2}$$



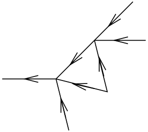
$$-\frac{n+2}{18}\,u^2 \int_k \frac{1}{r+k^2}\int_{k'}\frac{1}{r+k'^2}\,\frac{1}{r+(q-k-k')^2}$$
$$\times\Big(1-\frac{i\omega}{i\omega+\Delta(k)+\Delta(k')+\Delta(q-k-k')}\Big)\Bigg];$$



$$\Gamma^{(2,0)}(q,\omega) = -2Dq^a\Bigg[1 + Dq^a\,\frac{n+2}{18}\,u^2\int_k\frac{1}{r+k^2}\int_{k'}\frac{1}{r+k'^2}$$
$$\times\frac{1}{r+(q-k-k')^2}\,\mathrm{Re}\,\frac{1}{i\omega+\Delta(k)+\Delta(k')+\Delta(q-k-k')}\Bigg];$$



$$\underline{k} = (q,\omega):\ \ \Gamma^{(1,3)}(-3\underline{k}/2;\{\underline{k}/2\}) = D\Big(\frac{3q}{2}\Big)^a u\Bigg[1-\frac{n+8}{6}\,u$$
$$\times\int_k\frac{1}{r+k^2}\,\frac{1}{r+(q-k)^2}\Big(1-\frac{i\omega}{i\omega+\Delta(k)+\Delta(q-k)}\Big)\Bigg].$$



Renormalization

- *Additive renormalization*, T_c shift: as in static theory.
- *Multiplicative renormalization*: two new Z factors

$$S_R^\alpha = Z_S^{1/2} S^\alpha \ , \quad \widetilde{S}_R^\alpha = Z_{\widetilde{S}}^{1/2} \widetilde{S}^\alpha \ ;$$
$$D_R = Z_D \, D \ , \quad \tau_R = Z_\tau \, \tau \, \mu^{-2} \ , \quad u_R = Z_u \, u \, A_d \, \mu^{d-4} \ .$$

- $\Gamma_R^{(\widetilde{N},N)} = Z_{\widetilde{S}}^{-\widetilde{N}/2} Z_S^{-N/2} \, \Gamma^{(\widetilde{N},N)}$; FDT $\Rightarrow$ $Z_D = \left(Z_S / Z_{\widetilde{S}} \right)^{1/2}$.
- Model A ($a = 0$): from $\Gamma_R^{(2,0)}(0,0)$ or $\Gamma_R^{(1,1)}(0,\omega)$:

$$\Rightarrow \ Z_D = 1 - \frac{n+2}{144} \left(6 \ln \frac{4}{3} - 1 \right) \frac{u_R^2}{\epsilon} \ .$$

- Model B ($a = 2$): to *all orders*:

$$\Gamma^{(1,1)}(q=0,\omega) = i\omega \, , \ \partial_{q^2} \, \Gamma^{(2,0)}(q,\omega) \big|_{q=0} = -2D$$
$$\Rightarrow \ Z_{\widetilde{S}} \, Z_S = 1 \ , \quad Z_D = Z_S \ .$$

Renormalization group equation and critical exponents

Renormalization group equation for $\Gamma_R^{(\widetilde{N},N)}(\mu, D, \tau_R, u_R)$:

$$\left[\mu\frac{\partial}{\partial\mu}+\frac{\widetilde{N}\gamma_{\widetilde{S}}+N\gamma_S}{2}+\gamma_D D_R\,\frac{\partial}{\partial D_R}+\gamma_\tau\tau_R\,\frac{\partial}{\partial\tau_R}+\beta_u\frac{\partial}{\partial u_R}\right]\Gamma_R^{(\widetilde{N},N)}=0\,,$$

with *Wilson's flow* and RG beta functions as in static theory and:

$$\gamma_{\widetilde{S}}=\mu\,\frac{\partial}{\partial\mu}\bigg|_0\ln Z_{\widetilde{S}}\ ,\quad \gamma_D=\mu\,\frac{\partial}{\partial\mu}\bigg|_0\ln\frac{D_R}{D}=\frac{1}{2}\left(\gamma_S-\gamma_{\widetilde{S}}\right)$$

Characteristics $\mu \to \mu\,\ell$: $\ell\,\frac{d\widetilde{D}(\ell)}{d\ell}=\widetilde{D}(\ell)\,\gamma_D(\ell)$, $\widetilde{D}(1)=D_R$.

Solve RG equation for *dynamic susceptibility* near RG fixed point:

$$\chi_R(\tau_R,q,\omega)^{-1}\approx\mu^2\ell^{2+\gamma_S^*}\,\hat{\chi}_R\!\left(\tau_R\,\ell^{\gamma_\tau^*},u^*,\frac{q}{\mu\,\ell},\frac{\omega}{D_R\mu^{2+a}\,\ell^{2+a+\gamma_D^*}}\right)^{-1}$$

$\Rightarrow$ *critical exponents*: $\eta=-\gamma_S^*$, $\nu=-1/\gamma_\tau^*$, and $z=2+a+\gamma_D^*$;

model A: $z=2+c\,\eta$, $c=6\ln\frac{4}{3}-1+O(\epsilon)$; model B: $z=4-\eta$.

Langevin dynamics of isotropic ferromagnets

Heisenberg ferromagnet: S^α generators of $O(3)$; Poisson brackets
$\Rightarrow$ *reversible* spin precession term in Langevin dynamics, *model J*:

$$\frac{\partial \vec{S}(x,t)}{\partial t} = -g\, \vec{S}(x,t) \times \frac{\delta \mathcal{H}[\vec{S}]}{\delta \vec{S}(x,t)} + D\nabla^2 \frac{\delta \mathcal{H}[\vec{S}]}{\delta \vec{S}(x,t)} + \vec{\zeta}(x,t) \ ,$$

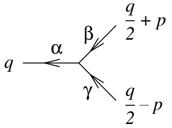
$$\left\langle \zeta^\alpha(x,t)\, \zeta^\beta(x',t') \right\rangle = -2D\, k_{\mathrm{B}} T\, \nabla^2 \delta(x-x') \delta(t-t') \delta^{\alpha\beta} \ .$$

Additional *mode-coupling vertex*:

$$\mathcal{A}_J[\widetilde{S}, S] = -g \int d^d x \int dt \sum_{\alpha,\beta,\gamma} \epsilon^{\alpha\beta\gamma} \widetilde{S}^\alpha S^\beta \left(\nabla^2 S^\gamma + h^\gamma \right).$$

$$[g] = \mu^{3-d/2} \ \Rightarrow$$

dynamical critical dimension $d_c' = 6$.



$= \epsilon^{\alpha\beta\gamma}(q \cdot p) g$

External field induces spin rotation:

$$\left\langle S^\alpha(x,t) \right\rangle_h = g \int_0^t dt' \sum_\beta \epsilon^{\alpha\beta\gamma} \left\langle S^\beta(x,t') \right\rangle_h h^\gamma(t)$$

$\Rightarrow$ *nonlinear susceptibility* $R^{\alpha;\beta\gamma} = \left. \frac{\delta^2 \langle S^\alpha \rangle}{\delta h^\beta\, \delta h^\gamma} \right|_{h=0}$:

$$\int d^d x'\, R^{\alpha;\beta\gamma}(x,t;x-x',t-t') = g\, \epsilon^{\alpha\beta\gamma}\, \chi^{\beta\beta}(x,t)\, \Theta(t)\, \Theta(t-t') \ .$$

Critical dynamics of isotropic ferromagnets

Renormalization:

- as for model B: $\Gamma^{(1,1)}(q=0,\omega) = i\omega \Rightarrow Z_{\tilde{S}}\, Z_S = 1$.
- Define $g_R^2 = Z_g\, g^2\, B_d\, \mu^{d-6}$, $B_d = \Gamma(4-d/2)/2^d\, d\, \pi^{d/2}$;
- from nonlinear response identity: $Z_g = Z_S$.
- *effective coupling*: $f = g^2/D^2$; associated RG beta function: $\beta_f = \mu\, \partial_\mu|_0 f_R = f_R\,(d - 6 + \gamma_S - 2\,\gamma_D)$.

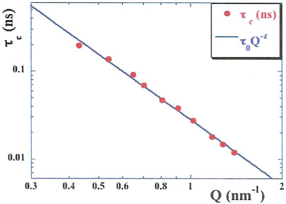
At *any nontrivial* $0 < f^* < \infty$ to *all orders* in f_R, $d < 6$:

$$z = 4 + \gamma_D^* = \frac{d+2-\eta}{2} \; .$$

Explicit one-loop calculation:

$\gamma_D = -f_R + \mathcal{O}(u_R^2, f_R^2) \Rightarrow$
$f_J^* = \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2)$, $\varepsilon = 6 - d$.

Scaling functions: self-consistent one-loop / *mode-coupling* theory.



The critical slowing-down for the relaxation time $t_c(q) = t_0\,|q|^{-z}$ in the 3D Heisenberg ferromagnet $CdCr_2S_4$, is fit to a power-law with $z = 2.47 \pm 0.02$, $t_0 = 0.028$ ns nm^{-z}.

M. Alba, S. Pouget, P. Fouquet, B. Farago, and C. Pappas, `arXiv:cond-mat/0703702`.

Driven lattice gas

Nonequilibrium steady state of particles (conserved density) with *hard-core repulsion (exclusion)*; *driven* along "$\|$" direction.
Continuity equation: $\partial_t S(x,t) + \nabla \cdot J(x,t) = 0$, $\langle S(x,t) \rangle = 0$.

- Transverse sector, $d_\perp = d - 1$: $J_\perp = -D \nabla_\perp S + \eta$;
- along the drive: $J_\| = \langle J_\| \rangle - Dc \, \nabla_\| S - \frac{1}{2} \, Dg \, S^2 + \zeta$;
- $\langle \eta_i \rangle = 0 = \langle \zeta \rangle$, noise correlations (*no* FDT !):

$$\left\langle \eta_i(x,t) \, \eta_j(x',t') \right\rangle = 2D \, \delta_{ij} \delta(x-x') \delta(t-t') \ ,$$

$$\left\langle \zeta(x,t) \, \zeta(x',t') \right\rangle = 2D \, \tilde{c} \, \delta(x-x') \delta(t-t') \ .$$

Response functional for *driven diffusive system (DDS)*:

$$\mathcal{A}[\widetilde{S}, S] = \int d^d x \int dt \ \widetilde{S} \Big[\partial_t S - D \left(\nabla_\perp^2 + c \, \nabla_\|^2 \right) S + D \left(\nabla_\perp^2 + \tilde{c} \, \nabla_\|^2 \right) \widetilde{S} - \frac{D \, g}{2} \, \nabla_\| S^2 \Big] ;$$

"massless" theory $\Rightarrow$ *generic scale invariance*; no tuning required.
Dynamic response function: *anisotropic scaling*

$$\chi(q_\perp, q_\|, \omega) = |q_\perp|^{-2+\eta} \, \hat{\chi}\big(q_\| / |q_\perp|^{1+\Delta}, \omega / |q_\perp|^z \big) \ .$$

Renormalization and scaling exponents for DDS

Vertex $\sim iq_\parallel \ \Rightarrow \ Z_{\widetilde{S}} = Z_S = Z_D = 1 \ \Rightarrow \ \eta = 0, \, z = 2.$
Galilean transformation: $S'(x'_\perp, x'_\parallel, t') = S(x_\perp, x_\parallel - Dgv\, t, t) - v,$
leaves Langevin equation / action *invariant*, $v \sim S \ \Rightarrow \ Z_g = 1.$

Explicit one-loop calculation, $C_d = \Gamma(2 - d/2)/2^{d-1}\pi^{d/2}$:

$$w = \frac{\tilde{c}}{c}\,, \ v = \frac{g^2}{c^{3/2}}\,, \ v_R = Z_c^{3/2}\, v\, C_d\, \mu^{d-2} \ \Rightarrow \ d_c = 2\ ;$$

$$\gamma_c = -\frac{v_R}{16}\,(3 + w_R)\ , \quad \gamma_{\tilde{c}} = -\frac{v_R}{32}\left(3w_R^{-1} + 2 + 3w_R\right).$$

$$\Rightarrow \ \beta_w = w_R\,(\gamma_{\tilde{c}} - \gamma_c) = -\frac{v_R}{32}\,(w_R - 1)\,(w_R - 3)\,,$$

$$\beta_v = v_R\Big(d - 2 - \frac{3}{2}\,\gamma_c\Big).$$

At *any non-trivial* RG fixed point $0 < v^* < \infty$: $w_R^* = 1$ *stable*;

$$d < 2: \quad \Delta = -\frac{\gamma_c^*}{2} = \frac{2-d}{3}\ , \quad z_\parallel = \frac{z}{1+\Delta} = \frac{6}{5-d}\ .$$

$d = 1$: $z_\parallel = \frac{3}{2}$, asymmetric exclusion process, Burgers equation, ...

Driven Ising lattice gas

Driven model B, critical DDS: conserved scalar field S subject to *nonequilibrium phase transition*, only *transverse* sector critical:

$$\mathcal{A}[\widetilde{S}, S] = \int d^d x \int dt \; \widetilde{S} \Big[\partial_t S - D \nabla_\perp^2 \left(r - \nabla_\perp^2 \right) S - Dc \, \nabla_\parallel^2 S$$
$$+ D \left(\nabla_\perp^2 \widetilde{S} - \frac{g}{2} \nabla_\parallel S^2 - \frac{u}{6} \nabla_\perp^2 S^3 \right) \Big] .$$

Anisotropic scaling, dynamic susceptibility:

$$\chi(\tau_\perp, q_\perp, q_\parallel, \omega) = |q_\perp|^{-2+\eta} \, \hat{\chi}\left(\frac{\tau}{|q_\perp|^{1/\nu}}, \frac{q_\parallel}{|q_\perp|^{1+\Delta}}, \frac{\omega}{|q_\perp|^z} \right) .$$

$[g^2] = \mu^{5-d} \;\Rightarrow\; d_c = 5$; $[u] = \mu^{3-d}$ (dangerously) *irrelevant*;
vertex $\sim i q_\parallel \Rightarrow Z_{\widetilde{S}} = Z_S = Z_D = 1 \Rightarrow \; \eta = 0, \; \nu = \frac{1}{2}, \; z = 4$.

Galilean invariance $\;\Rightarrow Z_g = 1, \; v = \frac{g^2}{c^{3/2}}, \; \beta_v = v_R \left(d - 5 - \frac{3}{2} \gamma_c \right)$.
$\Rightarrow$ scaling exponents to *all orders* in perturbation theory:

$$d < 5 : \quad \Delta = 1 - \frac{\gamma_c^*}{2} = \frac{8-d}{3} \, , \quad z_\parallel = \frac{4}{1+\Delta} = \frac{12}{11-d} \; .$$

Selected literature:

- R. Folk and G. Moser, *Critical dynamics: a field theoretical approach*, J. Phys. A: Math. Gen. **39**, R207–R313 (2006).
- E. Frey and F. Schwabl, *Critical dynamics of magnets*, Adv. Phys. **43**, 577–683 (1994).
- P.C. Hohenberg and B.I. Halperin, *Theory of dynamic critical phenomena*, Rev. Mod. Phys. **49**, 435–479 (1977).
- H.K. Janssen, *Field-theoretic methods applied to critical dynamics*, in: *Dynamical critical phenomena and related topics*, ed. C.P. Enz, Lecture Notes in Physics, Vol. **104**, Springer (Heidelberg), 26–47 (1979).
- B. Schmittmann and R.K.P. Zia, *Statistical mechanics of driven diffusive systems*, in: *Phase Transitions and Critical Phenomena*, ed. C. Domb and J.L. Lebowitz, Vol. **17**, Academic Press (London, 1995).
- U.C. Täuber, *Field theory approaches to nonequilibrium dynamics*, in: *Ageing and the Glass Transition*, eds. M. Henkel, M. Pleimling, and R. Sanctuary, Lecture Notes in Physics, Vol. **716**, Springer (Berlin), 295–348 (2007).
- U.C. Täuber, *Critical Dynamics — A Field Theory Approach to Equilibrium and Non-equilibrium Scaling Behavior*, Cambridge University Press (Cambridge, 201?); see `http://www.phys.vt.edu/~tauber/utaeuber.html`.

Some exercises

1. *Connected four-point functions to $O(u^2)$: one-loop diagrams.*
   Draw all possible Feynman diagrams to $O(u^2)$ for the *connected* four-point functions $\langle S^\alpha S^\beta \widetilde{S}^\gamma \widetilde{S}^\delta \rangle_c$, $\langle S^\alpha S^\beta S^\gamma \widetilde{S}^\delta \rangle_c$, and $\langle S^\alpha S^\beta S^\gamma S^\delta \rangle_c$ for the relaxational models A and B. Convince yourself that the second-order terms contain precisely one closed loop, and that the only one-particle irreducible graphs to this order are given by the one-loop diagrams for the vertex functions $\Gamma^{(1,1)}$, $\Gamma^{(1,3)}$, and $\Gamma^{(2,2)}$.

2. *Two-point vertex functions for models A/B to two-loop order.*
   Evaluate the *two-loop* diagrams for the vertex functions $\Gamma^{(1,1)}(q,\omega)$ and $\Gamma^{(2,0)}(q,\omega)$ for the *relaxational models* A and B. Carry out the internal frequency integrals, confirm the explicit expressions listed above, and check the *fluctuation-dissipation theorem* to this order.

3. *Dynamic susceptibility for isotropic ferromagnets.*
   For the Langevin dynamics for *isotropic ferromagnets* (model J), show that the *dynamic susceptibility* is given by

$$\chi^{\alpha\beta}(x-x',t-t') = -D \left\langle S^\alpha(x,t)\, \nabla^2\, \widetilde{S}^\beta(x',t') \right\rangle \\ + g \sum_{\gamma,\delta} \epsilon^{\beta\gamma\delta} \left\langle S^\alpha(x,t) \left[\widetilde{S}^\gamma(x',t')\, S^\delta(x',t') \right] \right\rangle \ .$$

Scale Invariance, Phase Transitions In Interacting Particle Systems

Chemical reactions

'Particles' $A, B, \ldots$ hop to nearest neighbors, upon encounter: species change, annihilate, proliferate, ... $\Rightarrow$ *diffusion-limited*; assume mixing $\Rightarrow$ *mean-field rate equations* for density $a(t)$.

Annihilation $k\,A \to m\,A$ $(m < k)$: $\partial_t\, a(t) = -(k-m)\,\lambda\, a(t)^k$;

- $k = 1$: $a(t) = a(0)\, e^{-\lambda\, t}$;
- $k \geq 2$: $a(t) = \left[a(0)^{1-k} + (k-m)(k-1)\,\lambda\, t\right]^{-1/(k-1)}$.

But: chemical kinetics generates particle *anti-correlations*, expect *depletion zones* for $d \leq d_c(k) \Rightarrow$ *slower* decay power laws.

Competing reactions, e.g., $A \to \emptyset$, $A \rightleftharpoons A + A$: rate equation $\partial_t\, a(t) = (\sigma - \kappa)\, a(t) - \lambda\, a(t)^2$;
$\Rightarrow$ *continuous nonequilibrium phase transition* at $\sigma_c = \kappa$:

- $\sigma > \kappa$: $a(t) \to a_\infty = (\sigma - \kappa)/\lambda$ *active* phase;
- $\sigma < \kappa$: $a(t) \to 0$ *inactive, absorbing* state (reactions cease);
- $\sigma = \kappa$: $a(t) \sim (\lambda\, t)^{-1}$ *critical* power law.

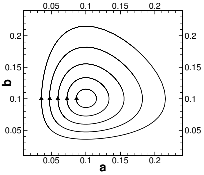
Include fluctuations: critical exponents, *universality classes* ?

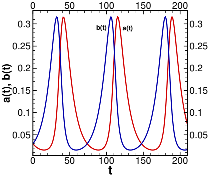
Population dynamics

Same framework; e.g., *Lotka–Volterra predator-prey competition*:

- *death* $A \to \emptyset$, *birth* $B \to 2B$;
- *predation* $A + B \to A + A$:

$$\partial_t\, a(t) = \lambda\, a(t)\, b(t) - \kappa\, a(t),$$
$$\partial_t\, b(t) = \sigma\, b(t) - \lambda\, a(t)\, b(t);$$

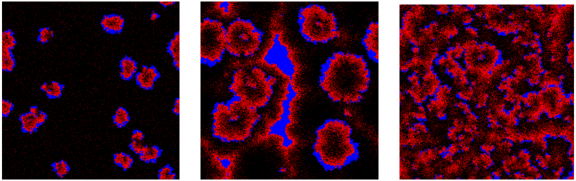




conserved first integral: $K = \lambda[a(t) + b(t)] - \ln[a(t)^\sigma\, b(t)^\kappa]$

$\Rightarrow$ regular *population oscillations*, determined by *initial* state.

Include spatial degrees of freedom (diffusion), stochasticity:



- *"pursuit and evasion" waves* in coexistence phase;
- complex dynamical patterns, *erratic* population oscillations;
- predator *extinction threshold* ($\to$ absorbing state).

Master equation for chemical kinetics

Master equation for probability $P(\{n_i\};t)$, $n_i = 0, 1, 2, \ldots$ provides *balance* of gain / loss terms; $A + A \to \emptyset$, $A + A \to A$:

$$\begin{aligned}\partial_t \, P(n_i;t) = \lambda \, (n_i + 2) \, (n_i + 1) \, P(\ldots, n_i + 2, \ldots; t) \\ + \lambda' \, (n_i + 1) \, n_i \, P(\ldots, n_i + 1, \ldots; t) \\ - (\lambda + \lambda') \, n_i \, (n_i - 1) \, P(\ldots, n_i, \ldots; t) \ ,\end{aligned}$$

with *initial Poisson distribution* $P(\{n_i\}, 0) = \prod_i \left(\bar{n}_0^{n_i} \, e^{-\bar{n}_0} / n_i! \right)$.

Introduce *second-quantized bosonic operator representation*:

$$[a_i, a_j] = 0 \, , \ [a_i, a_j^\dagger] = \delta_{ij} \, , \ a_i|0\rangle = 0 \ , \ a_i \, |n_i\rangle = n_i \, |n_i - 1\rangle \, ,$$

$$a_i^\dagger \, |n_i\rangle = |n_i + 1\rangle \ \Rightarrow \ |\{n_i\}\rangle = \prod_i \left(a_i^\dagger \right)^{n_i} |0\rangle \ .$$

Time evolution of *state vector* $|\Phi(t)\rangle = \sum_{\{n_i\}} P(\{n_i\};t) \, |\{n_i\}\rangle$:

$$\partial_t \, |\Phi(t)\rangle = -H \, |\Phi(t)\rangle \ , \quad H = \sum_i H_i(a_i^\dagger, a_i) \ ;$$

$\Rightarrow$ *non-Hermitian imaginary-time Schrödinger equation*.

Doi–Peliti bosonic operator representation

Example: diffusion-limited annihilation and coagulation:

$$H = D \sum_{<ij>} (a_i^\dagger - a_j^\dagger)(a_i - a_j) - \sum_i \left[\lambda (1 - a_i^{\dagger\,2})\, a_i^2 + \lambda' (1 - a_i^\dagger)\, a_i^\dagger a_i^2 \right].$$

Note: first term $\leftrightarrow$ physical *process*, second term: reaction *order*.

Need *projection* $\langle \mathcal{P} | = \langle 0 | \prod_i e^{a_i}$, $\langle \mathcal{P} | 0 \rangle = 1$ for *statistical averages*:

$$\langle F(t) \rangle = \sum_{\{n_i\}} F(\{n_i\})\, P(\{n_i\}; t) = \langle \mathcal{P} |\, F(\{a_i^\dagger\, a_i\})\, | \Phi(t) \rangle \ .$$

Probability conservation:

$$1 = \langle \mathcal{P} | \Phi(t) \rangle = \langle \mathcal{P} | e^{-H\,t} | \Phi(0) \rangle \implies \langle \mathcal{P} | H = 0 \ .$$

$\left[e^a, a^\dagger \right] = e^a \ \Rightarrow$ commuting $e^{\sum_i a_i}$ with H shifts $a_i^\dagger \to 1 + a_i^\dagger$

$\Rightarrow \ H_i(a_i^\dagger \to 1, a_i) = 0$, in averages replace $a_i^\dagger a_i \to a_i$, i.e.,

density $a(t) = \langle a_i \rangle$, two-point operator $a_i^\dagger a_i\, a_j^\dagger a_j \to a_i \delta_{ij} + a_i a_j$.

Coherent-state path integral

Construct *path integral* representation via *coherent states*:

$$a_i \, |\phi_i\rangle = \phi_i \, |\phi_i\rangle \ , \quad |\phi_i\rangle = \exp\Big(-\frac{1}{2} \, |\phi_i|^2 + \phi_i \, a_i^\dagger\Big) |0\rangle \ ,$$

$$\Rightarrow \ 1 = \int \prod_i \frac{d^2\phi_i}{\pi} \, |\{\phi_i\}\rangle \, \langle\{\phi_i\}| \quad \text{(overcomplete)}.$$

Split time evolution into infinitesimal steps, standard procedures:

$$\Rightarrow \ \langle F(t) \rangle \propto \int \prod_i \mathcal{D}[\phi_i] \, \mathcal{D}[\phi_i^*] \, F(\{\phi_i\}) \, e^{-\mathcal{A}[\phi_i^*, \phi_i]} \ ,$$

$$\mathcal{A}[\phi_i^*, \phi_i] = \sum_i \Big[-\phi_i(t_f) + \int_0^{t_f} dt \left[\phi_i^* \, \partial_t \phi_i + H(\phi_i^*, \phi_i) \right] - \bar{n}_0 \phi_i^*(0) \Big].$$

Continuum limit $\rightarrow \phi_i(t) \rightarrow \psi(x, t)$, $\phi_i^*(t) \rightarrow \hat{\psi}(x, t)$; "bulk":

$$\mathcal{A}[\hat{\psi}, \psi] = \int d^d x \int_0^{t_f} dt \left(\hat{\psi} \left(\partial_t - D \, \nabla^2 \right) \psi + \mathcal{H}_r \left[\hat{\psi}, \psi \right] \right) .$$

Microscopic stochastic field theory, *no assumptions on noise* !

$\Rightarrow$ Basis for coarse-graining, renormalization group analysis.

Field theory for diffusion-limited annihilation processes

Pair annihilation and coagulation $A + A \to \emptyset$, $A + A \to A$:

$$\mathcal{H}_r(\hat{\psi}, \psi) = -\lambda \left(1 - \hat{\psi}^2\right) \psi^2 - \lambda' \left(1 - \hat{\psi}\right) \hat{\psi}\, \psi^2 \ .$$

Classical field equations: $\delta \mathcal{A} / \delta \psi = 0 = \delta \mathcal{A} / \delta \hat{\psi} \ \Rightarrow \ \hat{\psi} = 1$ and

$$\partial_t \, \psi(x, t) = D \, \nabla^2 \, \psi(x, t) - (2\lambda + \lambda') \, \psi(x, t)^2 \ ;$$

shift about mean-field solution $\hat{\psi}(x, t) = 1 + \widetilde{\psi}(x, t)$:

$$\mathcal{H}_r\left(\widetilde{\psi}, \psi\right) = (2\lambda + \lambda') \, \widetilde{\psi} \, \psi^2 + (\lambda + \lambda') \, \widetilde{\psi}^2 \, \psi^2$$

$\Rightarrow$ aside from amplitudes, expect *identical* scaling behavior;
formally equivalent to "Langevin equation" with noise correlator
$L[\psi] = -(\lambda + \lambda') \, \psi^2 < 0 \ \Rightarrow$ *"imaginary" multiplicative noise*.

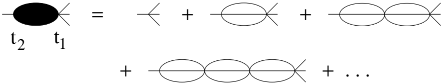
Field theory action for *k-particle annihilation* $k\,A \to \emptyset$:

$$\mathcal{A}[\hat{\psi}, \psi] = \int d^d x \int dt \left[\hat{\psi} \left(\partial_t - D \nabla^2 \right) \psi - \lambda \left(1 - \hat{\psi}^k \right) \psi^k \right];$$

for $k \geq 3$ *no* (obvious) equivalent Langevin description.
$[\lambda] = \mu^{2-(k-1)d} \ \Rightarrow \ d_c(k) = 2/(k-1)$, mean-field for $k > 3$.

Renormalization and asymptotic power laws

No propagator renormalization, massless $\Rightarrow$ $\eta = 0$, $z = 2$;
geometric series for *vertex renormalization* (Bethe–Salpeter):
renormalized rate:



$$g_R = Z_g \, \frac{\lambda}{D} \, B_{kd} \, \mu^{-2(1-d/d_c)} ,$$

$$B_{kd} = \frac{k! \, \Gamma(2-d/d_c) \, d_c}{k^{d/2} \, (4\pi)^{d/d_c}} :$$

$$Z_g^{-1} = 1 + \frac{\lambda B_{kd} \mu^{-2(1-d/d_c)}}{D(d_c-d)} \;\Rightarrow\; \beta_g = -\frac{2g_R}{d_c}(d-d_c+g_R) \, , \; g^* = d_c - d .$$

RG equation for *particle density* $a(t)$, $[a] = \mu^d$, $(\mu\ell)^2 = 1/Dt$:

$$\left[d + 2Dt \, \frac{\partial}{\partial(D\,t)} - d \, n_0 \, \frac{\partial}{\partial n_0} + \beta_g \, \frac{\partial}{\partial g_R} \right] a(\mu, D, n_0, g_R, t) = 0 ;$$

solution: $a(\mu, D, n_0, g_R, t) = (D\mu^2 \, t)^{-d/2} \, \hat{a}\big(n_0 \, (D\mu^2 \, t)^{d/2}, \widetilde{g}(t)\big)$,
one needs to establish that result is finite to *all orders* in $n_0 \to \infty$!

$$\Rightarrow \; k = 2 : \quad d < 2 : \; a(t) \sim (D\,t)^{-d/2} \; ,$$

$$d = 2 : \; a(t) \sim (D\,t)^{-1} \ln(D\,t) \; ,$$

$$d > 2 : \; a(t) \sim (\lambda\,t)^{-1} \; ;$$

$$k = 3 : \quad d = 1 \; : \; a(t) \sim \left[(D\,t)^{-1} \ln(D\,t) \right]^{1/2} \; ,$$

$$d > 1 : \; a(t) \sim (\lambda\,t)^{-1/2} \; .$$

Phase transitions from active to absorbing states

- *Competing reactions* $A \to \emptyset$, $A \rightleftharpoons A + A$ with diffusion:

$$\partial_t \, a(x,t) = -D \left(r - \nabla^2 \right) a(x,t) - \lambda \, a(x,t)^2 \; , \; r = (\kappa - \sigma)/D \; ;$$

- biology, ecology: *Fisher–Kolmogorov equation*, invasion fronts.
- $r = 0$: *continuous transition* from active to absorbing state.

$$\mathcal{A}[\hat{\psi}, \psi] = \int d^d x \int dt \left[\hat{\psi} \left(\partial_t - D \, \nabla^2 \right) \psi - \kappa \big(1 - \hat{\psi} \big) \psi \right.$$

$$\left. + \sigma \big(1 - \hat{\psi} \big) \hat{\psi} \, \psi - \lambda \big(1 - \hat{\psi} \big) \hat{\psi} \, \psi^2 \right] .$$

Shift and rescale $\hat{\psi}(x,t) = 1 + \sqrt{\frac{\sigma}{\lambda}} \, \widetilde{S}(x,t)$, $\psi(x,t) = \sqrt{\frac{\lambda}{\sigma}} \, S(x,t)$:

$$\mathcal{A} = \int d^d x \int dt \left[\widetilde{S} \left[\partial_t + D \left(r - \nabla^2 \right) \right] S - u \big(\widetilde{S} - S \big) \widetilde{S} \, S + \lambda \, \widetilde{S}^2 \, S^2 \right],$$

$u = \sqrt{\sigma \, \lambda}$, $[u] = \mu^{2-d/2} \; \Rightarrow \; d_c = 4$, $[\lambda] = \mu^{2-d}$ *irrelevant* $\to \lambda = 0$

$\Rightarrow$ *Reggeon field theory; rapidity inversion* $S(x,t) \leftrightarrow -\widetilde{S}(x,-t)$;

formally *equivalent* to Langevin equation:

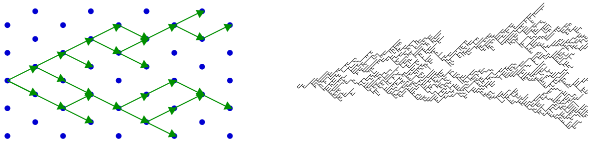
$$\partial_t \, S(x,t) = D \left(\nabla^2 - r \right) S(x,t) - u \, S(x,t)^2 + \zeta(x,t) \; ,$$

$$\langle \zeta(x,t) \rangle = 0 \, , \; \langle \zeta(x,t) \zeta(x',t') \rangle = 2u \, S(x,t) \, \delta(x - x') \delta(t - t') \, .$$

Epidemic processes and directed percolation

Phenomenological approach to *simple epidemic process (SEP)*:

1. A *"susceptible" medium* is locally *"infected"*, depending on the density of "sick" neighbors. Infected regions may recover.
2. The state with $n = 0$ ("disease" *extinction*) is *absorbing*.
3. The disease *spreads diffusively* via infection (1).
4. Microscopic fast degrees of freedom are incorporated as *noise*, respecting (2): noise alone cannot regenerate the disease.



Coarse-grained mesoscopic Langevin representation:

$$\partial_t n = D\left(\nabla^2 - R[n]\right) n + \zeta \ , \quad L[n] = n\, N[n] \ ;$$
$$r \approx 0 : \ R[n] = r + u\, n + \ldots \ , \ N[n] = v + \ldots \ ,$$

higher-order terms *irrelevant*; after rescaling:
response functional $\Rightarrow$ Reggeon field theory: DP conjecture.

Perturbation theory and renormalization

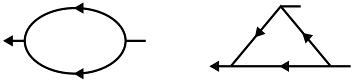
Explicit evaluation by means of dynamic perturbation theory:

$$\Gamma^{(1,1)}(q,\omega) = i\omega + D(r+q^2) + \frac{u^2}{D}\int_k \frac{1}{i\omega/2D + r + q^2/4 + k^2} \ .$$

Criticality condition, percolation threshold shift $\tau = r - r_c$:

$\Gamma^{(1,1)}(0,0) = 0$ at $r = r_c$

$$\Rightarrow r_c = -\frac{u^2}{D^2}\int_k \frac{1}{r_c + k^2} + O(u^4)$$



$$\Gamma^{(1,1)}(q,\omega) = i\omega + D(\tau+q^2) - \frac{u^2}{D}\int_k \frac{i\omega/2D + \tau + q^2/4}{k^2(i\omega/2D + \tau + q^2/4 + k^2)}$$

$$\Gamma^{(1,2)}(\{\underline{0}\}) = -\Gamma^{(2,1)}(\{\underline{0}\}) = -2u\Big(1 - \frac{2u^2}{D^2}\int_k \frac{1}{(\tau+k^2)^2}\Big).$$

Renormalization: $S_R = Z_S^{1/2}\, S$, $\widetilde{S}_R = Z_S^{1/2}\, \widetilde{S}$, $D_R = Z_D\, D$,
$\tau_R = Z_\tau\, \tau\, \mu^{-2}$, $u_R = Z_u\, u\, A_d^{1/2} \mu^{(d-4)/2}$, $A_d = \frac{\Gamma(3-d/2)}{2^{d-1}\pi^{d/2}}$.

DP critical exponents

$$\Rightarrow \quad \gamma_S = v_R/2 \; , \quad \gamma_D = -v_R/4 \; , \quad \gamma_\tau = -2 + 3v_R/4 \; ;$$

$$v_R = \frac{Z_u^2}{Z_D^2} \frac{u^2}{D^2} A_d \, \mu^{d-4} \Rightarrow \; \beta_v = v_R \left[-\epsilon + 3v_R + O(v_R^2) \right] .$$

Stable RG fixed point for $\epsilon = 4 - d > 0$: $\;v^* = \epsilon/3 + O(\epsilon^2)$.

Solve RG equation for *correlation function*:

$$C_R(\tau_R, q, \omega)^{-1} \approx q^2 \, \ell^{\gamma_S^*} \, \hat{C}_R \left(\tau_R \, \ell^{\gamma_\tau^*}, v^*, \frac{q}{\mu \ell}, \frac{\omega}{D_R \, \mu^2 \ell^{2+\gamma_D^*}} \right)^{-1}$$

$\Rightarrow$ identify *critical exponents* for directed percolation to order ϵ:

$$\eta = -\gamma_S^* = -\frac{\epsilon}{6} \, , \; \nu^{-1} = -\gamma_\tau^* = 2 - \frac{\epsilon}{4} \, , \; z = 2 + \gamma_D^* = 2 - \frac{\epsilon}{12} \; .$$

Scaling exponent	$d = 1$	$d = 2$	$d = 4 - \epsilon$
$\xi \sim \lvert\tau\rvert^{-\nu}$	$\nu \approx 1.100$	$\nu \approx 0.735$	$\nu = 1/2 + \epsilon/16 + O(\epsilon^2)$
$t_c \sim \xi^z \sim \lvert\tau\rvert^{-z\nu}$	$z \approx 1.576$	$z \approx 1.73$	$z = 2 - \epsilon/12 + O(\epsilon^2)$
$a_\infty \sim \lvert\tau\rvert^\beta$	$\beta \approx 0.2765$	$\beta \approx 0.584$	$\beta = 1 - \epsilon/6 + O(\epsilon^2)$
$a_c(t) \sim t^{-\alpha}$	$\alpha \approx 0.160$	$\alpha \approx 0.46$	$\alpha = 1 - \epsilon/4 + O(\epsilon^2)$

Lotka–Volterra predator-prey competition

Doi–Peliti field theory action, with *site occupation restrictions* for $A \to \emptyset$ (rate μ), $B \to B + B$ (rate σ), and $A + B \to A + A$ (rate λ):

$$S[\hat{a}, a; \hat{b}, b] = \int d^d x \int dt \, \Big[\hat{a} \big(\partial_t - D_A \nabla^2 \big) a + \mu \big(\hat{a} - 1 \big) a$$

$$+ \hat{b} \big(\partial_t - D_B \nabla^2 \big) b + \sigma \big(1 - \hat{b} \big) \hat{b} \, b \, e^{-\rho^{-1} \hat{b} \, b} + \lambda \big(\hat{b} - \hat{a} \big) \hat{a} \, a \, b \Big].$$

Shift fields $\hat{a} = 1 + \tilde{a}$, $\hat{b} = 1 + \tilde{b}$, expand in ρ^{-1} ($[\rho] = \kappa^d$):

$$S = \int d^d x \int dt \Big[\tilde{a} \big(\partial_t - D_A \nabla^2 + \mu \big) a + \tilde{b} \big(\partial_t - D_B \nabla^2 - \sigma \big) b$$

$$- \sigma \, \tilde{b}^2 \, b + \sigma \, \rho^{-1} \, \big(1 + \tilde{b} \big)^2 \, \tilde{b} \, b^2 - \lambda \, \big(1 + \tilde{a} \big) \, \big(\tilde{a} - \tilde{b} \big) \, a \, b \Big].$$

- define *fluctuating fields* $c = b_s - b$, $b_s \approx \rho$, $\langle c \rangle = 0$, $\tilde{c} = -\tilde{b}$;
- *rescale fields* $\phi = \sqrt{\sigma} \, c$, $\tilde{\phi} = \sqrt{\sigma} \, \tilde{c}$, $\sigma \to \infty$ ($[\sigma] = \kappa^2$);
- add *growth-limiting reaction* $A + A \to A$ (rate τ);
- *integrate out* fields ϕ and $\tilde{\phi}$, $u = \sqrt{\tau \, \lambda \, b_s}$;

$\Rightarrow$ Reggeon field theory for directed percolation.

Selected literature:

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- D.C. Mattis and M.L. Glasser, *The uses of quantum field theory in diffusion-limited reactions*, Rev. Mod. Phys. **70**, 979–1002 (1998).
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Some exercises

1. *Doi–Peliti Hamiltonian for reversible binary reactions.*
   Write down the master equation for the reversible chemical reaction $k\,A + m\,B \leftrightharpoons \ell\,C$. Employ a bosonic operator representation for each particle species, and establish that the associated quasi-Hamiltonian is given by
   $$H = \lambda_+ \sum_i \Big[\big(a_i^\dagger\big)^k \big(b_i^\dagger\big)^m - \big(c_i^\dagger\big)^\ell \Big] \Big(\lambda_+ a_i^k\, b_i^m - \lambda_- c_i^\ell \Big) \ .$$

2. *Density decay for pair annihilation in two dimensions.*
   Solve the renormalization group flow equation for the running coupling $\widetilde{g}(t)$ for *diffusion-limited pair annihilation* at the critical dimension $d_c(2) = 2$, and thereby compute the ensuing *logarithmic corrections* to the algebraic particle density decay.

3. *Directed percolation: one-loop vertex functions.*
   Evaluate the *one-loop* Feynman diagrams for the vertex functions $\Gamma^{(1,1)}(q, \omega)$, $\Gamma^{(1,2)}(\{q_i\}, \{\omega_i\})$ for *directed percolation (Reggeon field theory)*. Confirm the explicit expressions listed above, and compute the associated Wilson flow function and RG beta function to this order.