Renormalization Group: Applications in Statistical Physics

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Lecture plan

Critical Dynamics
- Dynamical scaling hypothesis
- Langevin dynamics and response functional
- Dynamic perturbation theory
- Critical dynamics of the relaxational models A and B
- Critical dynamics of isotropic ferromagnets
- Driven diffusive systems
- Literature

Scale Invariance, Phase Transitions In Interacting Particle Systems
- Chemical reactions and population dynamics
- Master equation and coherent-state path integral representation
- Diffusion-limited annihilation processes
- Phase transitions from active to absorbing states
- Literature
Critical Dynamics
Critical slowing down

Recall behavior of static correlation function near critical point:

- **correlation length** diverges, \( \xi(\tau) \sim |\tau|^{-\nu} \), \( \tau = (T - T_c)/T_c \).
- \( \chi(\tau, q) = |q|^{-2+\eta} \hat{\chi}^\pm(q\xi) \), \( C(\tau, x) = |x|^{-(d-2+\eta)} \tilde{C}^\pm(x/\xi) \).

Expect **critical slowing-down** as correlated regions grow:

- **relaxation time** diverges \( t_c(\tau) \sim \xi(\tau)^z \sim |\tau|^{-z\nu} \);
- **characteristic frequency** scale:
  \( \omega_c(\tau, q) = |q|^z \hat{\omega}^\pm(q\xi); \)
  \( \Rightarrow \) **dynamic critical exponent** \( z \).

Dynamic response and correlations:

\[
\chi(x - x', t - t') = \frac{\partial \langle S(x, t) \rangle}{\partial h(x', t')} \bigg|_{h=0},
\]
\[
C(x, t) = \langle S(x, t) S(0, 0) \rangle - \langle S \rangle^2.
\]

**Fluctuation-dissipation theorem:**
\[
C(q, \omega) = 2k_B T \text{Im} \chi(q, \omega)/\omega.
\]

Magnetization relaxation time of a Fe bilayer (on a W substrate) near \( T_c = 453 \) K. The data for this two-dimensional Ising ferromagnet yield \( z\nu = 2.09 \pm 0.06 \).

Dynamical scaling hypothesis

Generalize static scaling hypothesis:

\[ \chi(\tau, q, \omega) = |q|^{-2+\eta} \tilde{\chi}_\pm(q \xi, \omega \xi^z), \]
\[ C(\tau, x, t) = |x|^{-(d-2+\eta)} \tilde{C}_\pm(x/\xi, t/t_c). \]

Dynamical scaling hypothesis applies also to transport coefficients:

(a) Phase and (b) scaled amplitude of the linear conductivity vs. rescaled frequency measured for the normal- to superconducting phase transition \( (T_g \approx 56 \text{ K}) \) in an external magnetic field \( (B = 12 \text{ T}) \) in a YBCO crystal. From: J. Kötzler, M. Kaufmann, G. Nakielski, R. Behr, and W. Assmus, Phys. Rev. Lett. 72, 2081 (1994).
Langevin description of critical dynamics

Coarse-grained description:

- fast modes $\rightarrow$ random noise;
- mesoscopic Langevin equation for slow variables $S^\alpha(x, t)$.

Example: purely relaxational critical dynamics:

$$ \frac{\partial S^\alpha(x, t)}{\partial t} = -D \frac{\delta H[S]}{\delta S^\alpha(x, t)} + \zeta^\alpha(x, t), \quad \langle \zeta^\alpha(x, t) \rangle = 0 ,$$

$$ \langle \zeta^\alpha(x, t) \zeta^\beta(x', t') \rangle = 2D k_B T \delta(x - x') \delta(t - t') \delta^{\alpha\beta} ;$$

Einstein relation guarantees that $\mathcal{P}[S, t] \rightarrow \mathcal{P}_s[S]$ as $t \rightarrow \infty$.

Non-conserved order parameter: $D = \text{const}$.
Conserved order parameter: relaxes diffusively, $D \rightarrow -D \nabla^2$;
model A / B: $D(i\nabla)^a, a = 0, 2$: non-conserved / conserved.

Generally: mode couplings to additional conserved, slow fields $\Rightarrow$ various dynamic universality classes.
Relaxational models A/B: Gaussian approximation

**Gaussian (mean-field) approximation**: \( u = 0 \); Fourier transform:

\[
-i \omega + Dq^a (r + q^2) \right] S^\alpha(q, \omega) = Dq^a h^\alpha(q, \omega) + \zeta^\alpha(q, \omega),
\]

\[
\langle \zeta^\alpha(q, \omega) \zeta^\beta(q', \omega') \rangle = 2k_B T Dq^a (2\pi)^{d+1} \delta(q + q') \delta(\omega + \omega') \delta^{\alpha\beta}.
\]

⇒ **Dynamic response and correlation functions**:

\[
\chi^\alpha_0(q, \omega) = \frac{\partial \langle S^\alpha(q, \omega) \rangle}{\partial h^\beta(q, \omega)} \bigg|_{h=0} = Dq^a G^0(q, \omega) \delta^{\alpha\beta},
\]

\[
G^0(q, \omega) = \frac{1}{-i \omega + Dq^a (r + q^2)}; \quad G^0(q, t) = e^{-Dq^a (r+q^2)t} \Theta(t).
\]

\[
\langle S^\alpha(q, \omega) S^\beta(q', \omega') \rangle_0 = C^0(q, \omega) (2\pi)^{d+1} \delta(q + q') \delta(\omega + \omega') \delta^{\alpha\beta},
\]

\[
C^0(q, \omega) = \frac{2k_B T Dq^a}{\omega^2 + [Dq^a (r + q^2)]^2} = 2k_B T Dq^a |G^0(q, \omega)|^2;
\]

\[
C^0(q, t) = \frac{k_B T}{r + q^2} e^{-Dq^a (r+q^2)|t|} .
\]

⇒ **Gaussian critical exponents**: \( \nu = \frac{1}{2}, \eta = 0, \) and \( z = 2 + a. \)
Onsager–Machlup functional

Coupled *Langevin equations* for mesoscopic stochastic variables:

\[
\frac{\partial S^\alpha(x, t)}{\partial t} = F^\alpha[S](x, t) + \zeta^\alpha(x, t), \quad \langle \zeta^\alpha(x, t) \rangle = 0 ,
\]

\[
\langle \zeta^\alpha(x, t)\zeta^\beta(x', t') \rangle = 2 L^\alpha \delta(x - x') \delta(t - t') \delta^{\alpha\beta} ;
\]

- systematic forces \(F^\alpha[S]\), stochastic forces (noise) \(\zeta^\alpha\);
- noise correlator \(L^\alpha\): can be operator, functional of \(S^\alpha\).

Assume *Gaussian* stochastic process \(\Rightarrow\) probability distribution:

\[
\mathcal{W}[\zeta] \propto \exp \left[ -\frac{1}{4} \int d^d x \int_0^{t_f} dt \sum_\alpha \zeta^\alpha(x, t) \left[ (L^\alpha)^{-1} \zeta^\alpha(x, t) \right] \right],
\]

switch variables \(\zeta^\alpha \rightarrow S^\alpha\): \(\mathcal{W}[\zeta] \mathcal{D}[\zeta] = \mathcal{P}[S] \mathcal{D}[S] \propto e^{-G[S]} \mathcal{D}[S]\),

with *Onsager-Machlup functional* providing field-theory action:

\[
G[S] = \frac{1}{4} \int d^d x \int dt \sum_\alpha \left( \partial_t S^\alpha - F^\alpha[S] \right) \left[ (L^\alpha)^{-1} \left( \partial_t S^\alpha - F^\alpha[S] \right) \right].
\]

- Functional determinant = 1 with *forward* (Itô) discretization;
- normalization: \(\int \mathcal{D}[\zeta] \mathcal{W}[\zeta] = 1 \Rightarrow \text{“partition function”} = 1;\)
- problems: \((L^\alpha)^{-1}\), high non-linearities \(F^\alpha[S] (L^\alpha)^{-1} F^\alpha[S]\).
Janssen–De Dominicis response functional

Average over noise ‘histories’:

\[ \langle A[S] \rangle_\zeta \propto \int \mathcal{D}[\zeta] A[S(\zeta)] W[\zeta] : \]

use

\[ 1 = \int \mathcal{D}[S] \prod_\alpha \prod_{(x,t)} \delta(\partial_t S^\alpha(x, t) - F^\alpha[S](x, t) - \zeta^\alpha(x, t)) \]

\[ = \int \mathcal{D}[i\tilde{S}] \mathcal{D}[S] \exp \left[ - \int d^d x \int dt \sum_\alpha \tilde{S}^\alpha (\partial_t S^\alpha - F^\alpha[S] - \zeta^\alpha) \right]. \]

\[ \langle A[S] \rangle_\zeta \propto \int \mathcal{D}[i\tilde{S}] \mathcal{D}[S] \exp \left[ - \int d^d x \int dt \sum_\alpha \tilde{S}^\alpha (\partial_t S^\alpha - F^\alpha[S]) \right] \]

\[ \times A[S] \int \mathcal{D}[\zeta] \exp \left( - \int d^d x \int dt \sum_\alpha \frac{1}{4} \zeta^\alpha (L^\alpha)^{-1} \zeta^\alpha - \tilde{S}^\alpha \zeta^\alpha \right) \]

Perform Gaussian integral over noise \( \zeta^\alpha \):

\[ \langle A[S] \rangle_\zeta = \int \mathcal{D}[S] A[S] \mathcal{P}[S] , \quad \mathcal{P}[S] \propto \int \mathcal{D}[i\tilde{S}] e^{-\mathcal{A}[\tilde{S}, S]} , \]

with Janssen–De Dominicis response functional:

\[ \mathcal{A}[\tilde{S}, S] = \int d^d x \int_0^{t_f} dt \sum_\alpha \left[ \tilde{S}^\alpha (\partial_t S^\alpha - F^\alpha[S]) - \tilde{S}^\alpha L^\alpha \tilde{S}^\alpha \right] , \]

\[ \int \mathcal{D}[i\tilde{S}] \mathcal{D}[S] e^{-\mathcal{A}[\tilde{S}, S]} = 1 ; \text{ integrate out } \tilde{S}^\alpha : \text{ Onsager–Machlup.} \]
Purely relaxational models A and B

\[ A = A_0 + A_{\text{int}}, \text{ with } (k_B T = 1): \]

\[ A[\tilde{S}, S] = \int d^d x \int dt \sum_\alpha \left( \tilde{S}^\alpha \left[ \partial_t + D (i \nabla)^a (r - \nabla^2) \right] S^\alpha - D \tilde{S}^\alpha (i \nabla)^a \tilde{S}^\alpha - D \tilde{S}^\alpha (i \nabla)^a h^\alpha + D \frac{u}{6} \sum_\beta \tilde{S}^\alpha (i \nabla)^a S^\alpha S^\beta S^\beta \right), \]

\[ \chi^{\alpha\beta}(x - x', t - t') = \frac{\delta \langle S^\alpha(x, t) \rangle}{\delta h^\beta(x', t')} \bigg|_{h=0} = D \langle S^\alpha(x, t) (i \nabla)^a \tilde{S}^\beta(x', t') \rangle; \]

\[ \Rightarrow \tilde{S}^\alpha \text{ "response" fields; fluctuation–dissipation theorem:} \]

\[ \chi^{\alpha\beta}(x - x', t - t') = \Theta(t - t') \frac{\partial}{\partial t'} \left\langle S^\alpha(x, t) S^\beta(x', t') \right\rangle. \]

**Generating functional** for correlation functions, cumulants:

\[ Z[\tilde{j}, j] = \left\langle \exp \int d^d x \int dt \sum_\alpha \left( \tilde{j}^\alpha \tilde{S}^\alpha + j^\alpha S^\alpha \right) \right\rangle, \]

\[ \left\langle \prod_{ij} S^{\alpha_i} \tilde{S}^{\alpha_j} \right\rangle_{(c)} = \prod_{ij} \frac{\delta}{\delta j^{\alpha_i}} \frac{\delta}{\delta \tilde{j}^{\alpha_j}} (\ln Z[\tilde{j}, j] \bigg|_{\tilde{j} = j = 0}. \]
Dynamic perturbation expansion

Nonlinear terms $\sim u$: treat by means of \textit{perturbation expansion}:

$$
\langle \prod_{ij} S^\alpha_i \tilde{S}^\alpha_j \rangle = \frac{\langle \prod_{ij} S^\alpha_i \tilde{S}^\alpha_j \ e^{-A_{\text{int}}[\tilde{S}, S]} \rangle_0}{\langle e^{-A_{\text{int}}[\tilde{S}, S]} \rangle_0} = 
\langle \prod_{ij} S^\alpha_i \tilde{S}^\alpha_j \sum_{l=0}^{\infty} \frac{1}{l!} \left(-A_{\text{int}}[\tilde{S}, S]\right)^l \rangle_0,
$$

since the denominator $= 1$: \textit{no “vacuum” contributions}.

\textit{Causality}: $\langle \tilde{S}^\alpha(q, \omega) \tilde{S}^\beta(q', \omega') \rangle_0 = 0$.

Gaussian approximation ($u = 0$) recovers $G_0(q, \omega)$, $C_0(q, \omega)$.

Graphical representation: \textit{directed} (causality !) \textit{propagator lines} connect $\tilde{S}^\beta$ to $S^\alpha$, join at \textit{vertices} (subject to $q, \omega$ conservation).

$$
\frac{q, \omega}{\alpha} \leftrightarrow \beta = \frac{1}{-i \omega + D q^a (r + q^2)} \delta^{\alpha\beta}
$$

$$
\begin{align*}
\alpha \quad q \\
\beta \quad -q
\end{align*}
= 2 D q^a \delta^{\alpha\beta}
\begin{align*}
\alpha \quad q \\
\beta \quad \beta
\end{align*}
= - D q^a \frac{u}{6} \quad \frac{q, \omega}{\alpha} \leftrightarrow \beta
$$
Vertex functions

Follow standard procedures: \textit{cumulants} $\leftrightarrow$ \textit{connected} graphs;
\textit{Dyson equation} for propagator: $G(q, \omega)^{-1} = G_0(q, \omega)^{-1} - \Sigma(q, \omega)$.
$
\tilde{\Phi}^\alpha = \delta \ln Z / \delta \tilde{j}^\alpha, \Phi^\alpha = \delta \ln Z / \delta j^\alpha \quad \Longrightarrow \quad \text{generating functional:}
$

\Gamma[\tilde{\Phi}, \Phi] = - \ln Z[\tilde{j}, j] + \int d^d x \int dt \sum_\alpha \left( \tilde{j}^\alpha \tilde{\Phi}^\alpha + j^\alpha \Phi^\alpha \right),
$

\Gamma^{(\tilde{N}, N)}_{\{\alpha_i\};\{\alpha_j\}} = \prod_i^{\tilde{N}} \delta \tilde{\Phi}^{\alpha_i} \prod_j^N \delta \Phi^{\alpha_j} \Gamma[\tilde{\Phi}, \Phi] \bigg|_{\tilde{j} = 0 = j} \Rightarrow
$

\Gamma^{(1,1)}(q, \omega) = G(-q, -\omega)^{-1}, \quad \Gamma^{(0,2)}(q, \omega) = 0 \quad \text{(causality)},
\Gamma^{(2,0)}(q, \omega) = - \frac{C(q, \omega)}{|G(q, \omega)|^2} = - \frac{2D q^a}{\omega} \text{Im} \Gamma^{(1,1)}(q, \omega) \quad \text{(FDT)}.
$

\Rightarrow \ \text{Vertex functions} \leftrightarrow \ \text{one-particle (1PI) irreducible graphs.}
Feynman rules

\( l \)-th order contribution to vertex function \( \Gamma(\tilde{N}, N) \):

1. Draw all topologically different, connected one-particle irreducible graphs with \( \tilde{N} \) outgoing and \( N \) incoming lines connecting \( l \) relaxation vertices \( \propto u \). Do not allow closed response loops [Itô calculus: \( \Theta(0) = 0 \)].

2. Attach wave vectors \( q_i \), frequencies \( \omega_i \) or times \( t_i \), and vector indices \( \alpha_i \) to all directed lines, obeying “momentum and energy” conservation at each vertex.

3. Each directed line corresponds to a response propagator \( G_0(-q, -\omega) / G_0(q, t_i - t_j) \), the two-point vertex to the noise strength \( 2Dq^a \), and the four-point relaxation vertex to \( -Dq^a u / 6 \). Closed loops imply integrals over the internal wave vectors and frequencies or times, subject to causality constraints, as well as sums over the internal vector indices. Apply residue theorem to evaluate frequency integrals.

4. Multiply with \( -1 \) and the combinatorial factor counting all possible ways of connecting the propagators, \( l \) relaxation vertices, and \( k \) two-point vertices leading to topologically identical graphs, including a factor \( 1/l! k! \) originating in the expansion of \( \exp(-A_{\text{int}}[\tilde{S}, S]) \).
Explicit results

Perturbation series → loop expansion, $\Delta(q) = Dq^a(r + q^2)$:

$$
\Gamma^{(1,1)}(q, \omega) = i\omega + Dq^a\left[r + q^2 + \frac{n+2}{6} u \int_k \frac{1}{r+k^2} \right.
 \left.- \left(\frac{n+2}{6} u\right)^2 \int_k \frac{1}{r+k^2} \int_{k'} \frac{1}{(r+k'^2)^2} \right.
 \left.- \frac{n+2}{18} u^2 \int_k \frac{1}{r+k^2} \int_{k'} \frac{1}{r+k'^2} \frac{1}{r+(q-k-k')^2} \right.
 \left.\times \left(1 - \frac{i\omega}{i\omega + \Delta(k) + \Delta(k') + \Delta(q-k-k')}\right)\right] ;
$$

$$
\Gamma^{(2,0)}(q, \omega) = -2Dq^a\left[1 + Dq^a \frac{n+2}{18} u^2 \int_k \frac{1}{r+k^2} \int_{k'} \frac{1}{r+k'^2} \right.
 \left.\times \frac{1}{r+(q-k-k')^2} \text{Re} \frac{1}{i\omega + \Delta(k) + \Delta(k') + \Delta(q-k-k')}\right] ;
$$

$$
k = (q, \omega): \Gamma^{(1,3)}(-3k/2; \{k/2\}) = D\left(\frac{3q}{2}\right)^a u\left[1 - \frac{n+8}{6} u \right.
 \left.\times \int_k \frac{1}{r+k^2} \frac{1}{r+(q-k)^2} \left(1 - \frac{i\omega}{i\omega + \Delta(k) + \Delta(q-k)}\right)\right] .
$$
Renormalization

- **Additive renormalization**, $T_c$ shift: as in static theory.
- **Multiplicative renormalization**: two new $Z$ factors

$$\begin{align*}
S_R^\alpha &= Z_S^{1/2} S^\alpha, \\
\tilde{S}_R^\alpha &= Z_{\tilde{S}}^{1/2} \tilde{S}^\alpha; \\
D_R &= Z_D D, \\
\tau_R &= Z_\tau \tau \mu^{-2}, \\
u_R &= Z_u u A_d \mu^{d-4}.
\end{align*}$$

- $\Gamma_R^{(\tilde{N},N)} = Z_{\tilde{S}}^{-\tilde{N}/2} Z_S^{-N/2} \Gamma(\tilde{N},N)$; FDT $\Rightarrow Z_D = (Z_S/Z_{\tilde{S}})^{1/2}$.

- **Model A** ($a = 0$): from $\Gamma_R^{(2,0)}(0,0)$ or $\Gamma_R^{(1,1)}(0,\omega)$:

$$\Rightarrow Z_D = 1 - \frac{n + 2}{144} \left(6 \ln \frac{4}{3} - 1\right) \frac{u_R^2}{\epsilon}.$$ 

- **Model B** ($a = 2$): to all orders:

$$\Gamma_R^{(1,1)}(q = 0,\omega) = i\omega, \quad \partial_q^2 \Gamma_R^{(2,0)}(q,\omega)\big|_{q=0} = -2D \\
\Rightarrow Z_{\tilde{S}} Z_S = 1, \quad Z_D = Z_S.$$
Renormalization group equation and critical exponents

The renormalization group equation for $\Gamma_R^{(\tilde{N}, N)}(\mu, D, \tau_R, u_R)$:

$$
\left[ \mu \frac{\partial}{\partial \mu} + \tilde{N} \gamma_{\tilde{S}} + N \gamma_{S} \right] + \gamma_D D_R \frac{\partial}{\partial D_R} + \gamma_{\tau} \tau_R \frac{\partial}{\partial \tau_R} + \beta_u \frac{\partial}{\partial u_R} \right] \Gamma_R^{(\tilde{N}, N)} = 0,
$$

with Wilson's flow and RG beta functions as in static theory and:

$$
\gamma_{\tilde{S}} = \mu \frac{\partial}{\partial \mu} \ln Z_{\tilde{S}}, \quad \gamma_D = \mu \frac{\partial}{\partial \mu} \ln \frac{D_R}{D} = \frac{1}{2} (\gamma_S - \gamma_{\tilde{S}})
$$

Characteristics $\mu \to \mu \ell$: $\ell \frac{d\tilde{D}(\ell)}{d\ell} = \tilde{D}(\ell) \gamma_D(\ell), \tilde{D}(1) = D_R$.

Solve RG equation for dynamic susceptibility near RG fixed point:

$$
\chi_R(\tau_R, q, \omega)^{-1} \approx \mu^2 \ell^{2+\gamma_{\tilde{S}}^*} \hat{\chi}_R \left( \tau_R \ell^{\gamma_{\tau}^*} u^*, \frac{q}{\mu \ell}, \frac{\omega}{D_R \mu^{2+a} \ell^{2+a+\gamma_D^*}} \right)^{-1}
$$

$\Rightarrow$ critical exponents: $\eta = -\gamma_{\tilde{S}}^*$, $\nu = -1/\gamma_{\tau}^*$, and $z = 2 + a + \gamma_{D}^*$;

model A: $z = 2 + c \eta$, $c = 6 \ln \frac{4}{3} - 1 + O(\epsilon)$; model B: $z = 4 - \eta$. 
Langevin dynamics of isotropic ferromagnets

**Heisenberg ferromagnet:** $S^\alpha$ generators of $O(3)$; Poisson brackets $\Rightarrow$ reversible spin precession term in Langevin dynamics, *model J*:

$$\frac{\partial \vec{S}(x, t)}{\partial t} = -g \vec{S}(x, t) \times \frac{\delta \mathcal{H}[\vec{S}]}{\delta \vec{S}(x, t)} + D \nabla^2 \frac{\delta \mathcal{H}[\vec{S}]}{\delta \vec{S}(x, t)} + \vec{\zeta}(x, t),$$

$$\langle \zeta^\alpha(x, t) \zeta^\beta(x', t') \rangle = -2D k_B T \nabla^2 \delta(x - x') \delta(t - t') \delta^{\alpha\beta}. $$

**Additional mode-coupling vertex:**

$$\mathcal{A}_J[\vec{S}, S] = -g \int d^d x \int dt \sum_{\alpha, \beta, \gamma} \epsilon^{\alpha\beta\gamma} \vec{S}^\alpha S^\beta \left( \nabla^2 S^\gamma + h^\gamma \right).$$

$$[g] = \mu^{3-d/2} \Rightarrow \text{dynamical critical dimension } d'_c = 6.$$

**External field induces spin rotation:**

$$\langle S^\alpha(x, t) \rangle_h = g \int_0^t dt' \sum_\beta \epsilon^{\alpha\beta\gamma} \langle S^\beta(x, t') \rangle_h h^\gamma(t)$$

$\Rightarrow$ **nonlinear susceptibility** $R^{\alpha;\beta\gamma} = \frac{\delta^2 \langle S^\alpha \rangle}{\delta h^\beta \delta h^\gamma} \bigg|_{h=0}$:

$$\int d^d x' R^{\alpha;\beta\gamma}(x, t; x - x', t - t') = g \epsilon^{\alpha\beta\gamma} \chi^{\beta\beta}(x, t) \Theta(t) \Theta(t - t').$$
Critical dynamics of isotropic ferromagnets

Renormalization:

- as for model B: \( \Gamma^{(1,1)}(q = 0, \omega) = i\omega \Rightarrow Z_S^\infty Z_S = 1 \).
- Define \( g_R^2 = Z_g g^2 B_d \mu^{d-6} \), \( B_d = \Gamma(4 - d/2)/2^d d \pi^{d/2} \);
- from nonlinear response identity: \( Z_g = Z_S \).
- effective coupling: \( f = g^2 / D^2 \); associated RG beta function:
  \( \beta_f = \mu \partial_{\mu} |0 f_R = f_R (d - 6 + \gamma_S - 2 \gamma_D) \).

At any nontrivial \( 0 < f^* < \infty \) to all orders in \( f_R, d < 6 \):

\[
z = 4 + \frac{\gamma_D^*}{2} = \frac{d + 2 - \eta}{2}.
\]

Explicit one-loop calculation:

\[
\gamma_D = -f_R + \mathcal{O}(u_R^2, f_R^2) \Rightarrow f_R^* = \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2), \quad \varepsilon = 6 - d.
\]

Scaling functions:
self-consistent one-loop / mode-coupling theory.

The critical slowing-down for the relaxation time \( t_c(q) = t_0 |q|^{-z} \) in the 3D Heisenberg ferromagnet \( \text{CdCr}_2\text{S}_4 \), is fit to a power-law with \( z = 2.47 \pm 0.02, t_0 = 0.028 \text{ ns nm}^{-z} \).

Driven lattice gas

Nonequilibrium steady state of particles (conserved density) with hard-core repulsion (exclusion); driven along “∥” direction.

Continuity equation: \( \partial_t S(x, t) + \nabla \cdot J(x, t) = 0, \langle S(x, t) \rangle = 0 \).

- Transverse sector, \( d_\perp = d - 1 \):
  \( J_\perp = -D \nabla_\perp S + \eta \);
- along the drive:
  \( J_{\parallel} = \langle J_{\parallel} \rangle - Dc \nabla_{\parallel} S - \frac{1}{2} Dg S^2 + \zeta \);
- \( \langle \eta_i \rangle = 0 = \langle \zeta \rangle \), noise correlations (no FDT!):
  \( \langle \eta_i(x, t) \eta_j(x', t') \rangle = 2D \delta_{ij} \delta(x - x') \delta(t - t') \),
  \( \langle \zeta(x, t) \zeta(x', t') \rangle = 2D \tilde{c} \delta(x - x') \delta(t - t') \).

Response functional for driven diffusive system (DDS):

\[
\mathcal{A}[\tilde{S}, S] = \int d^d x \int dt \tilde{S} \left[ \partial_t S - D \left( \nabla_\perp^2 + c \nabla_\parallel^2 \right) S 
+ D \left( \nabla_\perp^2 + \tilde{c} \nabla_\parallel^2 \right) \tilde{S} - \frac{Dg}{2} \nabla_{\parallel} S^2 \right];
\]

“massless” theory \( \Rightarrow \) generic scale invariance; no tuning required.

Dynamic response function: anisotropic scaling

\[
\chi(q_\perp, q_\parallel, \omega) = |q_\perp|^{-2+\eta} \hat{\chi}(q_{\parallel}/|q_\perp|^{1+\Delta}, \omega/|q_\perp|^z).
\]
Renormalization and scaling exponents for DDS

Vertex $\sim iq_\parallel \Rightarrow Z_\tilde{S} = Z_S = Z_D = 1 \Rightarrow \eta = 0, z = 2$.

**Galilean transformation:** $S'(x'_\perp, x'_\parallel, t') = S(x_\perp, x_\parallel - Dgt, t) - v$,
leaves Langevin equation / action invariant, $v \sim S \Rightarrow Z_g = 1$.

Explicit one-loop calculation, $C_d = \Gamma(2 - d/2)/2^{d-1} \pi^{d/2}$:

$w = \frac{\tilde{c}}{c}, \; v = \frac{g^2}{c^{3/2}}, \; v_R = Z_c^{3/2} v \; C_d \mu^{d-2} \Rightarrow d_c = 2$;

$\gamma_c = -\frac{v_R}{16} (3 + w_R), \; \gamma_{\tilde{c}} = -\frac{v_R}{32} (3w_R^{-1} + 2 + 3w_R)$.

$\Rightarrow \beta_w = w_R (\gamma_{\tilde{c}} - \gamma_c) = -\frac{v_R}{32} (w_R - 1) (w_R - 3),$

$\beta_v = v_R \left(d - 2 - \frac{3}{2} \gamma_c\right)$.

At any non-trivial RG fixed point $0 < v^* < \infty$: $w_R^* = 1$ stable;

$d < 2$:

$\Delta = -\frac{\gamma_c^*}{2} = \frac{2 - d}{3}, \; z_\parallel = \frac{z}{1 + \Delta} = \frac{6}{5 - d}$.

$d = 1$: $z_\parallel = \frac{3}{2}$, asymmetric exclusion process, Burgers equation, ...
Driven Ising lattice gas

**Driven model B, critical DDS:** conserved scalar field $S$ subject to nonequilibrium phase transition, only transverse sector critical:

$$\mathcal{A}[\tilde{S}, S] = \int d^d x \int dt \tilde{S} \left[ \partial_t S - D \nabla^2_\perp (r - \nabla^2_\perp) S - Dc \nabla^2_\parallel S \\
+ D \left( \nabla^2_\perp \tilde{S} - \frac{g}{2} \nabla_\parallel S^2 - \frac{u}{6} \nabla^2_\perp S^3 \right) \right].$$

Anisotropic scaling, dynamic susceptibility:

$$\chi(\tau_\perp, q_\perp, q_\parallel, \omega) = |q_\perp|^{-2+\eta} \hat{\chi} \left( \frac{\tau}{|q_\perp|^{1/\nu}}, \frac{q_\parallel}{|q_\perp|^{1+\Delta}}, \frac{\omega}{|q_\perp|^z} \right).$$

$[g^2] = \mu^{5-d} \Rightarrow d_c = 5$; $[u] = \mu^{3-d}$ (dangerously) irrelevant; vertex $\sim iq_\parallel \Rightarrow Z_{\tilde{S}} = Z_S = Z_D = 1 \Rightarrow \eta = 0, \nu = \frac{1}{2}, z = 4$.

**Galilean invariance** $\Rightarrow Z_g = 1, \nu = \frac{g^2}{c^{3/2}}, \beta_v = v_R \left( d - 5 - \frac{3}{2} \gamma_c \right)$.

$\Rightarrow$ scaling exponents to all orders in perturbation theory:

$$d < 5 : \quad \Delta = 1 - \frac{\gamma_c^*}{2} = \frac{8 - d}{3}, \quad Z_\parallel = \frac{4}{1 + \Delta} = \frac{12}{11 - d}.$$
Selected literature:

Some exercises

1. **Connected four-point functions to $O(u^2)$: one-loop diagrams.**
   Draw all possible Feynman diagrams to $O(u^2)$ for the connected four-point functions $\langle S^\alpha S^\beta \tilde{S}^\gamma \tilde{S}^\delta \rangle_c$, $\langle S^\alpha S^\beta S^\gamma \tilde{S}^\delta \rangle_c$, and $\langle S^\alpha S^\beta S^\gamma S^\delta \rangle_c$ for the relaxational models A and B. Convince yourself that the second-order terms contain precisely one closed loop, and that the only one-particle irreducible graphs to this order are given by the one-loop diagrams for the vertex functions $\Gamma^{(1,1)}$, $\Gamma^{(1,3)}$, and $\Gamma^{(2,2)}$.

2. **Two-point vertex functions for models A/B to two-loop order.**
   Evaluate the two-loop diagrams for the vertex functions $\Gamma^{(1,1)}(q, \omega)$ and $\Gamma^{(2,0)}(q, \omega)$ for the relaxational models A and B. Carry out the internal frequency integrals, confirm the explicit expressions listed above, and check the fluctuation-dissipation theorem to this order.

3. **Dynamic susceptibility for isotropic ferromagnets.**
   For the Langevin dynamics for isotropic ferromagnets (model J), show that the dynamic susceptibility is given by
   \[
   \chi^{\alpha \beta}(x - x', t - t') = -D \left\langle S^\alpha(x, t) \nabla^2 \tilde{S}^\beta(x', t') \right\rangle \\
   + g \sum_{\gamma, \delta} e^{\beta \gamma \delta} \left\langle S^\alpha(x, t) \left[ \tilde{S}^\gamma(x', t') S^\delta(x', t') \right] \right\rangle.
   \]
Scale Invariance, Phase Transitions In Interacting Particle Systems
Chemical reactions

‘Particles’ $A, B, \ldots$ hop to nearest neighbors, upon encounter: species change, annihilate, proliferate, $\ldots \Rightarrow$ diffusion-limited; assume mixing $\Rightarrow$ mean-field rate equations for density $a(t)$.

**Annihilation** $k A \rightarrow m A$ ($m < k$): $\partial_t a(t) = -(k - m) \lambda a(t)^k$;

$\quad \bullet$ $k = 1$: $a(t) = a(0) e^{-\lambda t}$;

$\quad \bullet$ $k \geq 2$: $a(t) = \left[a(0)^{1-k} + (k - m)(k - 1) \lambda t\right]^{-1/(k-1)}$.

But: chemical kinetics generates particle *anti-correlations*, expect depletion zones for $d \leq d_c(k) \Rightarrow$ slower decay power laws.

**Competing reactions**, e.g., $A \rightarrow \emptyset$, $A \rightleftharpoons A + A$:
rate equation $\partial_t a(t) = (\sigma - \kappa) a(t) - \lambda a(t)^2$;
$\Rightarrow$ continuous nonequilibrium phase transition at $\sigma_c = \kappa$:

$\quad \bullet$ $\sigma > \kappa$: $a(t) \rightarrow a_\infty = (\sigma - \kappa)/\lambda$ active phase;

$\quad \bullet$ $\sigma < \kappa$: $a(t) \rightarrow 0$ inactive, absorbing state (reactions cease);

$\quad \bullet$ $\sigma = \kappa$: $a(t) \sim (\lambda t)^{-1}$ critical power law.

Include fluctuations: critical exponents, *universality classes*?
Population dynamics

Same framework; e.g., *Lotka–Volterra predator-prey competition*:

- **death** $A \rightarrow \emptyset$, **birth** $B \rightarrow 2B$;
- **predation** $A + B \rightarrow A + A$:

$$
\partial_t a(t) = \lambda a(t) b(t) - \kappa a(t), \\
\partial_t b(t) = \sigma b(t) - \lambda a(t) b(t);
$$

Conserved first integral:

$$
K = \lambda [a(t) + b(t)] - \ln [a(t)^\sigma b(t)^\kappa] \\
\Rightarrow \text{regular population oscillations}, \text{ determined by initial state.}
$$

Include spatial degrees of freedom (diffusion), stochasticity:

- "pursuit and evasion" waves in coexistence phase;
- complex dynamical patterns, *erratic* population oscillations;
- predator *extinction threshold* ($\rightarrow$ absorbing state).
Master equation for chemical kinetics

*Master equation* for probability $P(\{n_i\}; t)$, $n_i = 0, 1, 2, \ldots$ provides *balance* of gain / loss terms; $A + A \rightarrow \emptyset$, $A + A \rightarrow A$:

$$
\partial_t P(n_i; t) = \lambda (n_i + 2) (n_i + 1) P(\ldots, n_i + 2, \ldots; t) \\
+ \lambda' (n_i + 1) n_i P(\ldots, n_i + 1, \ldots; t) \\
- (\lambda + \lambda') n_i (n_i - 1) P(\ldots, n_i, \ldots; t),
$$

with *initial Poisson distribution* $P(\{n_i\}, 0) = \prod_i \left( \bar{n}_0^{n_i} e^{-\bar{n}_0} / n_i! \right)$.

Introduce *second-quantized bosonic operator representation*:

$$
[a_i, a_j] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}, \quad a_i |0\rangle = 0, \quad a_i |n_i\rangle = n_i |n_i - 1\rangle,
$$

$$
a_i^\dagger |n_i\rangle = |n_i + 1\rangle \Rightarrow |\{n_i\}\rangle = \prod_i (a_i^\dagger)^{n_i} |0\rangle.
$$

Time evolution of *state vector* $|\Phi(t)\rangle = \sum_{\{n_i\}} P(\{n_i\}; t) |\{n_i\}\rangle$:

$$
\partial_t |\Phi(t)\rangle = -H |\Phi(t)\rangle, \quad H = \sum_i H_i(a_i^\dagger, a_i);
$$

$\Rightarrow$ *non-Hermitian imaginary-time Schrödinger equation*.
Doi–Peliti bosonic operator representation

Example: diffusion-limited annihilation and coagulation:

\[ H = D \sum_{<ij>} (a_i^\dagger - a_j^\dagger)(a_i - a_j) - \sum_i \left[ \lambda (1 - a_i^\dagger)^2 a_i^2 + \lambda' (1 - a_i^\dagger) a_i^\dagger a_i^2 \right]. \]

Note: first term \( \leftrightarrow \) physical process, second term: reaction order.

Need projection \( \langle P | = \langle 0 | \prod_i e^{a_i} \), \( \langle P | 0 \rangle = 1 \) for statistical averages:

\[ \langle F(t) \rangle = \sum_{\{n_i\}} F(\{n_i\}) P(\{n_i\}; t) = \langle P | F(\{a_i^\dagger a_i\}) | \Phi(t) \rangle. \]

Probability conservation:

\[ 1 = \langle P | \Phi(t) \rangle = \langle P | e^{-Ht} | \Phi(0) \rangle \quad \Longrightarrow \quad \langle P | H = 0. \]

\[ [e^a, a_i^\dagger] = e^a \quad \Rightarrow \quad \text{commuting } e^{\sum_i a_i} \text{ with } H \text{ shifts } a_i^\dagger \rightarrow 1 + a_i^\dagger \]

\[ \Rightarrow \quad H (a_i^\dagger \rightarrow 1, a_i) = 0, \text{ in averages replace } a_i^\dagger a_i \rightarrow a_i, \text{ i.e.,} \]

density \( a(t) = \langle a_i \rangle \), two-point operator \( a_i^\dagger a_i a_j^\dagger a_j \rightarrow a_i \delta_{ij} + a_i a_j. \)
Coherent-state path integral

Construct *path integral* representation via *coherent states*:

\[ a_i |\phi_i\rangle = \phi_i |\phi_i\rangle, \quad |\phi_i\rangle = \exp\left(-\frac{1}{2} |\phi_i|^2 + \phi_i a_i^\dagger\right)|0\rangle, \]

\[ \Rightarrow 1 = \int \prod_i \frac{d^2\phi_i}{\pi} |\{\phi_i\}\rangle \langle\{\phi_i\}| \quad \text{(overcomplete).} \]

Split time evolution into infinitesimal steps, standard procedures:

\[ \Rightarrow \langle F(t) \rangle \propto \int \prod_i \mathcal{D}[\phi_i] \mathcal{D}[\phi_i^*] F(\{\phi_i\}) e^{-\mathcal{A}[\phi_i^*, \phi_i]} , \]

\[ \mathcal{A}[\phi_i^*, \phi_i] = \sum_i \left[ -\phi_i(t_f) + \int_0^{t_f} dt \left[ \phi_i^* \partial_t \phi_i + H(\phi_i^*, \phi_i) \right] - \bar{n}_0 \phi_i^*(0) \right]. \]

**Continuum limit** \( \rightarrow \phi_i(t) \rightarrow \psi(x, t), \phi_i^*(t) \rightarrow \hat{\psi}(x, t) \); “bulk”:

\[ \mathcal{A}[\hat{\psi}, \psi] = \int d^d x \int_0^{t_f} dt \left( \hat{\psi} \left( \partial_t - D \nabla^2 \right) \psi + \mathcal{H}_r[\hat{\psi}, \psi] \right). \]

**Microscopic** stochastic field theory, *no assumptions on noise*!

\[ \Rightarrow \text{Basis for coarse-graining, renormalization group analysis.} \]
Field theory for diffusion-limited annihilation processes

Pair annihilation and coagulation $A + A \rightarrow \emptyset$, $A + A \rightarrow A$: 

$$\mathcal{H}_r(\hat{\psi}, \psi) = -\lambda (1 - \hat{\psi}^2) \psi^2 - \lambda' (1 - \hat{\psi}) \hat{\psi} \psi^2.$$ 

Classical field equations: 

$$\frac{\delta A}{\delta \psi} = 0 = \frac{\delta A}{\delta \hat{\psi}} \Rightarrow \hat{\psi} = 1$$ 

and 

$$\partial_t \psi(x, t) = D \nabla^2 \psi(x, t) - (2\lambda + \lambda') \psi(x, t)^2;$$ 

shift about mean-field solution $\hat{\psi}(x, t) = 1 + \tilde{\psi}(x, t)$: 

$$\mathcal{H}_r(\tilde{\psi}, \psi) = (2\lambda + \lambda') \tilde{\psi} \psi^2 + (\lambda + \lambda') \tilde{\psi}^2 \psi^2$$ 

$\Rightarrow$ aside from amplitudes, expect identical scaling behavior; formally equivalent to “Langevin equation” with noise correlator 

$$L[\psi] = -(\lambda + \lambda') \psi^2 < 0 \Rightarrow \text{“imaginary” multiplicative noise}.$$ 

Field theory action for $k$-particle annihilation $k A \rightarrow \emptyset$: 

$$\mathcal{A}[\hat{\psi}, \psi] = \int d^d x \int dt \left[ \hat{\psi}(\partial_t - D \nabla^2) \psi - \lambda (1 - \hat{\psi}^k) \psi^k \right];$$ 

for $k \geq 3$ no (obvious) equivalent Langevin description. 

$$[\lambda] = \mu^{2-(k-1)d} \Rightarrow d_c(k) = 2/(k - 1), \text{ mean-field for } k > 3.$$
Renormalization and asymptotic power laws

No propagator renormalization, massless \( \Rightarrow \eta = 0, z = 2; \)

geometric series for *vertex renormalization* (Bethe–Salpeter):

renormalized rate:

\[
g_R = Z_g \frac{\lambda}{D} B_{kd} \mu^{-2(1-d/d_c)},
\]

\[
B_{kd} = \frac{k! \Gamma(2-d/d_c) d_c}{k^{d/2} (4\pi)^{d/d_c}}.
\]

\[
Z_g^{-1} = 1 + \frac{\lambda B_{kd} \mu^{-2(1-d/d_c)}}{D(d_c-d)} \Rightarrow \beta_g = -\frac{2g_R}{d_c} (d - d_c + g_R), \quad g^* = d_c - d.
\]

RG equation for *particle density* \( a(t) \), \([a] = \mu^d, (\mu \ell)^2 = 1/Dt:\)

\[
\left[ d + 2Dt \frac{\partial}{\partial (D t)} - d n_0 \frac{\partial}{\partial n_0} + \beta_g \frac{\partial}{\partial g_R} \right] a(\mu, D, n_0, g_R, t) = 0;
\]

solution:

\[
a(\mu, D, n_0, g_R, t) = (D \mu^2 t)^{-d/2} \hat{a}(n_0 (D \mu^2 t)^{d/2}, \tilde{g}(t))
\]

one needs to establish that result is finite to *all orders* in \( n_0 \to \infty \) !

\[
\Rightarrow \quad k = 2 : \quad d < 2 : \quad a(t) \sim (D t)^{-d/2} ,
\]

\[
d = 2 : \quad a(t) \sim (D t)^{-1} \ln(D t) ,
\]

\[
d > 2 : \quad a(t) \sim (\lambda t)^{-1} ;
\]

\[
k = 3 : \quad d = 1 : \quad a(t) \sim \left[ (D t)^{-1} \ln(D t) \right]^{1/2} ,
\]

\[
d > 1 : \quad a(t) \sim (\lambda t)^{-1/2} .
\]
Phase transitions from active to absorbing states

- **Competing reactions** $A \to \emptyset$, $A \rightleftharpoons A + A$ with diffusion:

\[
\partial_t a(x, t) = -D \left( r - \nabla^2 \right) a(x, t) - \lambda a(x, t)^2, \quad r = (\kappa - \sigma)/D;
\]


- $r = 0$: **continuous transition** from active to absorbing state.

\[
\mathcal{A}[\hat{\psi}, \psi] = \int d^d x \int dt \left[ \hat{\psi} \left( \partial_t - D \nabla^2 \right) \psi - \kappa (1 - \hat{\psi}) \psi + \sigma (1 - \hat{\psi}) \hat{\psi} \psi - \lambda (1 - \hat{\psi}) \hat{\psi} \psi^2 \right].
\]

Shift and rescale $\hat{\psi}(x, t) = 1 + \sqrt{\frac{\sigma}{\lambda}} \tilde{S}(x, t)$, $\psi(x, t) = \sqrt{\frac{\lambda}{\sigma}} S(x, t)$:

\[
\mathcal{A} = \int d^d x \int dt \left[ \tilde{S} \left[ \partial_t + D \left( r - \nabla^2 \right) \right] S - u (\tilde{S} - S) \tilde{S} S + \lambda \tilde{S}^2 S^2 \right],
\]

$u = \sqrt{\sigma \lambda}$, $[u] = \mu^{2-d/2} \Rightarrow d_c = 4$, $[\lambda] = \mu^{2-d}$ irrelevant $\rightarrow \lambda = 0 \Rightarrow$ Reggeon field theory; rapidity inversion $S(x, t) \leftrightarrow -\tilde{S}(x, -t)$;

formally equivalent to Langevin equation:

\[
\partial_t S(x, t) = D \left( \nabla^2 - r \right) S(x, t) - u S(x, t)^2 + \zeta(x, t),
\]

$\langle \zeta(x, t) \rangle = 0$, $\langle \zeta(x, t) \zeta(x', t') \rangle = 2u S(x, t) \delta(x - x') \delta(t - t')$. 
Epidemic processes and directed percolation

Phenomenological approach to simple epidemic process (SEP):

1. A “susceptible” medium is locally “infected”, depending on the density of “sick” neighbors. Infected regions may recover.
2. The state with $n = 0$ (“disease” extinction) is absorbing.
3. The disease spreads diffusively via infection (1).
4. Microscopic fast degrees of freedom are incorporated as noise, respecting (2): noise alone cannot regenerate the disease.

Coarse-grained mesoscopic Langevin representation:

$$\partial_t n = D (\nabla^2 - R[n]) n + \zeta , \quad L[n] = n N[n] ;$$

$$r \approx 0 : \quad R[n] = r + u n + \ldots , \quad N[n] = v + \ldots ,$$

higher-order terms irrelevant; after rescaling:

response functional $\Rightarrow$ Reggeon field theory: DP conjecture.
Perturbation theory and renormalization

Explicit evaluation by means of dynamic perturbation theory:

\[ \Gamma^{(1,1)}(q, \omega) = i\omega + D(r + q^2) + \frac{u^2}{D} \int_k \frac{1}{i\omega/2D + r + q^2/4 + k^2} \cdot \]

**Criticality condition**, percolation threshold shift \( \tau = r - r_c \):

\[ \Gamma^{(1,1)}(0, 0) = 0 \text{ at } r = r_c \]

\[ \Rightarrow r_c = -\frac{u^2}{D^2} \int_k \frac{1}{r_c + k^2} + O(u^4) \]

\[ \Gamma^{(1,1)}(q, \omega) = i\omega + D(\tau + q^2) - \frac{u^2}{D} \int_k \frac{i\omega/2D + \tau + q^2/4}{k^2(i\omega/2D + \tau + q^2/4 + k^2)} \]

\[ \Gamma^{(1,2)}(\{0\}) = -\Gamma^{(2,1)}(\{0\}) = -2u \left(1 - \frac{2u^2}{D^2} \int_k \frac{1}{(\tau + k^2)^2}\right). \]

**Renormalization**: \( S_R = Z_S^{1/2} S \), \( \tilde{S}_R = Z_S^{1/2} \tilde{S} \), \( D_R = Z_D D \),

\[ \tau_R = Z_\tau \tau \mu^{-2} , \ u_R = Z_u u A_d^{1/2} \mu^{(d-4)/2}, \ A_d = \frac{\Gamma(3-d/2)}{2^{d-1} \pi^{d/2}}. \]
DP critical exponents

\[ \gamma_S = \nu_R / 2, \quad \gamma_D = -\nu_R / 4, \quad \gamma_\tau = -2 + 3\nu_R / 4; \]
\[ \nu_R = \frac{Z^2_u}{Z_D^2} \frac{u^2}{D^2} A_d \mu^{d-4} \Rightarrow \beta_v = \nu_R \left[ -\epsilon + 3\nu_R + O(\nu_R^2) \right]. \]

**Stable RG fixed point** for \( \epsilon = 4 - d > 0: \quad \nu^* = \epsilon / 3 + O(\epsilon^2). \)

Solve RG equation for correlation function:

\[ C_R(\tau_R, q, \omega)^{-1} \approx q^2 \ell^{\gamma_S^*} \hat{C}_R \left( \tau_R, \ell^{\gamma_\tau^*}, \nu^*, \frac{q}{\mu \ell}, \frac{\omega}{D_R \mu^2 \ell^{2+\gamma_D^*}} \right)^{-1} \]

⇒ identify **critical exponents** for directed percolation to order \( \epsilon: \)

\[ \eta = -\gamma_S^* = -\frac{\epsilon}{6}, \quad \nu^{-1} = -\gamma_\tau^* = 2 - \frac{\epsilon}{4}, \quad z = 2 + \gamma_D^* = 2 - \frac{\epsilon}{12}. \]

<table>
<thead>
<tr>
<th>Scaling exponent</th>
<th>( d = 1 )</th>
<th>( d = 2 )</th>
<th>( d = 4 - \epsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi \sim</td>
<td>\tau</td>
<td>^{-\nu} )</td>
<td>( \nu \approx 1.100 )</td>
</tr>
<tr>
<td>( t_c \sim \xi^z \sim</td>
<td>\tau</td>
<td>^{-z\nu} )</td>
<td>( z \approx 1.576 )</td>
</tr>
<tr>
<td>( a_\infty \sim</td>
<td>\tau</td>
<td>^{\beta} )</td>
<td>( \beta \approx 0.2765 )</td>
</tr>
<tr>
<td>( a_c(t) \sim t^{-\alpha} )</td>
<td>( \alpha \approx 0.160 )</td>
<td>( \alpha \approx 0.46 )</td>
<td>( \alpha = 1 - \epsilon / 4 + O(\epsilon^2) )</td>
</tr>
</tbody>
</table>
Lotka–Volterra predator-prey competition

Doi–Peliti field theory action, with *site occupation restrictions* for $A \to \emptyset$ (rate $\mu$), $B \to B + B$ (rate $\sigma$), and $A + B \to A + A$ (rate $\lambda$):

$$S[\hat{a}, a; \hat{b}, b] = \int d^d x \int dt \left[ \hat{a} (\partial_t - D_A \nabla^2) a + \mu (\hat{a} - 1) a ight. \\
+ \hat{b} (\partial_t - D_B \nabla^2) b + \sigma (1 - \hat{b}) \hat{b} b e^{-\rho^{-1} \hat{b} b} + \lambda (\hat{b} - \hat{a}) \hat{a} a b \right].$$

Shift fields $\hat{a} = 1 + \tilde{a}$, $\hat{b} = 1 + \tilde{b}$, expand in $\rho^{-1}$ ($[\rho] = \kappa^d$):

$$S = \int d^d x \int dt \left[ \tilde{a} (\partial_t - D_A \nabla^2 + \mu) a + \tilde{b} (\partial_t - D_B \nabla^2 - \sigma) b \\
- \sigma \tilde{b}^2 b + \sigma \rho^{-1} (1 + \tilde{b})^2 \tilde{b} b^2 - \lambda (1 + \tilde{a}) (\tilde{a} - \tilde{b}) a b \right].$$

- define *fluctuating fields* $c = b_s - b$, $b_s \approx \rho$, $\langle c \rangle = 0$, $\tilde{c} = -\tilde{b}$;
- rescale fields $\phi = \sqrt{\sigma} c$, $\tilde{\phi} = \sqrt{\sigma} \tilde{c}$, $\sigma \to \infty$ ($[\sigma] = \kappa^2$);
- add *growth-limiting reaction* $A + A \to A$ (rate $\tau$);
- integrate out fields $\phi$ and $\tilde{\phi}$, $u = \sqrt{\tau \lambda b_s}$;

$\Rightarrow$ Reggeon field theory for directed percolation.
Selected literature:


1. Doi–Peliti Hamiltonian for reversible binary reactions.
Write down the master equation for the reversible chemical reaction \( k A + m B \rightleftharpoons \ell C \). Employ a bosonic operator representation for each particle species, and establish that the associated quasi-Hamiltonian is given by
\[
H = \lambda + \sum_i \left[ (a_i^\dagger)^k (b_i^\dagger)^m - (c_i^\dagger)^\ell \right] \left( \lambda + a_i^k b_i^m - \lambda - c_i^\ell \right).
\]

2. Density decay for pair annihilation in two dimensions.
Solve the renormalization group flow equation for the running coupling \( \tilde{g}(t) \) for diffusion-limited pair annihilation at the critical dimension \( d_c(2) = 2 \), and thereby compute the ensuing logarithmic corrections to the algebraic particle density decay.

3. Directed percolation: one-loop vertex functions.
Evaluate the one-loop Feynman diagrams for the vertex functions \( \Gamma^{(1,1)}(q, \omega) \), \( \Gamma^{(1,2)}(\{q_i\}, \{\omega_i\}) \) for directed percolation (Reggeon field theory). Confirm the explicit expressions listed above, and compute the associated Wilson flow function and RG beta function to this order.