Generalized symmetries and gauge theory multiverses

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Eric Sharpe, Virginia Tech

An overview of hep-th/0502027, 0502044, 0502053, 0606034,
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2204.09117, 2204.13708, 2206.14824, 2208.05982, & to appear

Recently there's been a lot of interest in "generalized symmetries" of QFT.

One of my goals today is to outline some basics aspects of those symmetries.

Furthermore, we'll see that sometimes, local theories with generalized symmetries
are equivalent to
disjoint unions of other local theories, known as "universes" in this context,
which gives rise to a notion of multiverses in gauge theories.

This is called **decomposition**, and explaining this will be the other goal of this talk.

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We'll see that decomposition intertwines with many physical phenomena, for example:

- Restrictions on instantons realized as `multiverse' interference effects
- Defects realized as `portals between universes'
- Wormholes between the universes arise in certain constructions

Decomposition arises when we study generalizations of ordinary symmetries, so let's begin with a quick review of symmetries in physics.

Let's begin with a quick overview of actions of ordinary groups.

Classical physics:

Recall a group defines a symmetry of a theory if the action S is invariant.

This leads to Noether's theorem, conserved currents.

Quantum mechanics:

Given a group G,

we can represent elements $g \in G$ by unitary operators $A(g) = \exp(iT(g))$,

such that $\quad A(g)A(h) = A(gh)$.

We say this is a symmetry if this commutes with the Hamiltonian, in the sense

$$A(g)\, H\, A(g)^{-1} \;=\; H \qquad\qquad \left[T(g), H\right] \;=\; 0$$

A simple common example: angular momentum in quantum mechanics

Symmetry: rotation invariance of the system

Generated by Lie algebra $\mathfrak{so}(3)$:

$$L_i = \epsilon_{ijk} x_j p_k \qquad [L_i, L_j] = i\hbar \epsilon_{ijk} L_k$$

The operators $\overrightarrow{L}^2, L_z$ commute,
and so can be simultaneously diagonalized:
states characterized by total angular momentum ℓ:

$$\overrightarrow{L}^2 |\psi\rangle = \hbar^2 \ell(\ell+1) |\psi\rangle$$

and by z-component m:

$$L_z |\psi\rangle = \hbar m |\psi\rangle$$

(So the states form a representation of $\mathfrak{so}(3)$.)

Another simple common example: gauge transformations in electromagnetism

Here, physically, if A is the gauge field of electromagnetism,

then $A \sim A + d\alpha$

for α any function,

because both define the same electric fields $\overrightarrow{E}$ and magnetic fields $\overrightarrow{B}$.

The α describes an infinitesimal action of the group $U(1)$,

and since we're identifying fields related by that action,

we say that we *gauged* the symmetry.

Now, how can this be generalized?

One way is to generalize the groups appearing to `higher' groups.
A higher group is much like a group, except that some axioms are weakened.

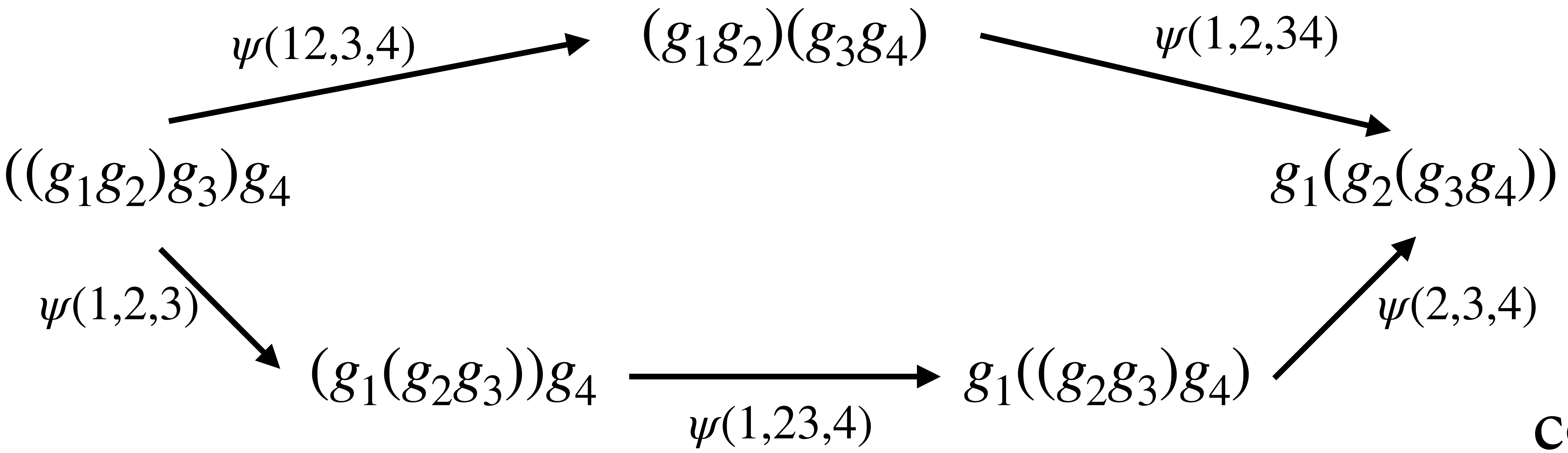
Example: associativity

In a group, we require $g_1(g_2g_3) = (g_1g_2)g_3$

In a higher group, we instead merely require the existence of isomorphisms

$$\psi(g_1, g_2, g_3) : (g_1g_2)g_3 \xrightarrow{\sim} g_1(g_2g_3)$$

such that



commutes.

Example: B fields

The B field is a two-form tensor potential $B = B_{\mu\nu} dx^\mu \wedge dx^\nu$ with a gauge invariance:

$$B \ \sim \ B + d\Lambda$$

where Λ is a connection on a line bundle.

Here, the gauge transformation itself admits gauge transformations:

a gauge transformation by Λ

is equivalent to

a gauge transformation by $\Lambda + d\alpha$

As a result, gauge transformations can merely hope to be isomorphic to one another, so associativity only holds up to isomorphism.

(Fields with this & related towers of gauge transformations for gauge transformations are common in string theory.)

Example: line bundles

In that same spirit, we're going to see examples of higher groups defined by line bundles.
Here's the idea:

To elements g of an ordinary group G, assign a line bundle L_g
together with isomorphisms $\psi_{g,h} : L_g \otimes L_h \xrightarrow{\sim} L_{gh}$ which essentially encode possible gauge transformations between the line bundles.

These structures may seem obscure,
but they've been known in various parts of physics for a long time.

A few examples:

- We've already discussed B fields and tensor field potentials that arise in sugrav

- The String 2-group has been known in elliptic genus circles for many decades

- Two-dimensional gauge theories with trivially-acting subgroups

(Pantev, ES '06)

I'll specialize to examples of this form next,
as they'll provide prototypical examples of decomposition.

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as they'll provide prototypical examples of decomposition.

In these examples, the generalized symmetry will act on the instantons,
so let me take a minute to remind everyone of instantons in gauge theories.

Briefly, a gauge theory is a theory of gauge fields = connections on principal bundles.

The physical theory sums over (equivalence classes of) bundles,
and then, for each such equivalence class,
one sees fluctuations of the gauge field / connection.

The instantons (= nonperturbative sectors) of the gauge theory
are precisely the (equiv classes of) bundles:

instanton = bundle

- Two-dimensional gauge theories with trivially-acting subgroups (Pantev, ES '06)

Example: Consider a theory of electromagnetism (a $U(1)$ gauge theory)
in which all of the matter fields (electrons, ...) have charges that are multiples of k
so that $\mathbb{Z}_k \subset U(1)$ acts trivially.

Technical point:

why is that different from a theory in which everything has charges that are multiples of 1 ?

Can't I just rescale all the charges ?

Answer: You can, but that's only one option. (Pantev, ES '06)

Another option: Add heavy charge ± 1 fields, with masses above cutoff scale.

This certainly distinguishes.
In 2d, their presence can be detected via θ angle periodicity.

Upshot: the difference is nonperturbative; are identical perturbatively.

- Two-dimensional gauge theories with trivially-acting subgroups (Pantev, ES '06)

Example: Consider a theory of electromagnetism (a $U(1)$ gauge theory)
in which all of the matter fields (electrons, ...) have charges that are multiples of k
so that $\mathbb{Z}_k \subset U(1)$ acts trivially.

This theory has a generalized symmetry,
that interchanges the bundles / instantons of the $U(1)$ gauge theory:

$$(U(1) \text{ bundle}) \mapsto (U(1) \text{ bundle}) \otimes (\mathbb{Z}_k \text{ bundle})$$

for *any* $\mathbb{Z}_k$ bundle

Because the subgroup $\mathbb{Z}_k \subset U(1)$ acts trivially on all matter,
the action S weighting these contributions is the same under the replacement above.

An action, *not* of an element of $\mathbb{Z}_k$, but rather a $\mathbb{Z}_k$ *bundle*

This is a generalized symmetry, denoted $B\mathbb{Z}_k$

So far, we've seen that a gauge theory with a trivially-acting subgroup has a generalized symmetry, that interchanges instanton sectors.

Let's try to characterize such symmetries more precisely....

One way to think about these symmetries is in terms of operators.

Noether's theorem:

Consider an ordinary global symmetry.

Under an infinitesimal symmetry transformation parametrized by α,

$$S \mapsto S + \int (d\alpha) \wedge j$$

(Hodge dual of typical description)

where j is a $(d-1)$-form (Hodge dual of Noether current),

which obeys $dj = 0$ (conservation law).

We can associate an operator

$$U_\alpha = \exp\left(\int_{M_{d-1}} j \right)$$

supported along a submfld M_{d-1} of dim $d-1$.

It's invariant under deformations of M_{d-1}, b/c of conservation law $dj = 0$.

"topological operator"

That picture can be generalized. Consider a symmetry parametrized by a p-form α.

$$S \;\mapsto\; S + \int_M (d\alpha)\wedge j$$

where j is a $(d-p-1)$-form, obeying $dj = 0$ (conservation law).

We can associate an operator

$$U_\alpha \;=\; \exp\left(\int_{M_{d-p-1}} j\right)$$

supported along a submanifold M_{d-p-1} of dim $d-p-1$.

It's invariant under deformations of M_{d-p-1}, b/c of conservation law $dj = 0$.

We call this a *p-form symmetry*.

Ordinary symmetries are 0-form symmetries.

Gauge theory with trivially-acting subgroup has a 1-form symmetry (BK).

In $d > 1$ spacetime dimensions,
if a local quantum field theory has a global $(d-1)$-form symmetry,
it is equivalent to a disjoint union of other local QFT's,
known in this context as `universes.'

We call this **decomposition**.

(2d: Hellerman et al '06;
d>2: Tanizaki-Unsal '19, Cherman-Jacobson '20)

When this happens, we say the QFT `decomposes.'
Decomposition of the QFT can be applied to give insight
into its properties, which I will explore in this talk.

In d spacetime dimensions,
if a local quantum field theory has a global $(d-1)$-form symmetry,
it is equivalent to a disjoint union of other local QFT's,
known in this context as `universes.'

Prototypical example:

A two-dimensional G-gauge theory
with trivially-acting central subgroup $K \subset G$
is equivalent to
a disjoint union of $|K|$ copies of (G/K) gauge theories,
each with a possibly different (discrete) theta angle.

$$G\text{-gauge theory} \;=\; \coprod_{|K|} \left(G/K\text{-gauge theory}\right)_{\theta}$$

(the *universes* of the decomposition)

Why is the existence of decomposition surprising?

To explain, let me distinguish a *sum* of QFTs from a *product* of QFTs.

Consider two QFTs with path integrals: $Z(T_1) = \int [D\phi_1] \exp(-S_1)$ $\quad Z(T_2) = \int [D\phi_2] \exp(-S_2)$

In a product of QFTs, we *multiply* partition functions:

$$Z(T_1 \otimes T_2) = Z(T_1)\, Z(T_2) = \int [D\phi_1][D\phi_2] \exp(-S_1 - S_1)$$

There always exists a local action for a product. Here, it's $S_1 + S_2$

In a sum of QFTs, we *add* partition functions: (connected spacetime)

$$Z(T_1 \coprod T_2) = Z(T_1) + Z(T_2) = \int [D\phi_1] \exp(-S_1) + \int [D\phi_2] \exp(-S_2)$$

Ordinarily, no way to write this in the form $\int [D\phi_1][D\phi_2] \exp(-S)$ for some S: $\log(x+y) \neq (\log x) + (\log y)$

But that's exactly what happens in decomposition!

What does it mean for one local QFT to be a sum of other local QFTs?

(Hellerman et al ’06)

1) Existence of projection operators

The theory contains topological operators Π_i such that

$$\Pi_i \Pi_j = \delta_{i,j} \Pi_j \qquad \sum_i \Pi_i = 1 \qquad [\Pi_i, \mathcal{O}] = 0$$

Operators Π_i simultaneously diagonalizable; state space = $\mathcal{H} = \oplus_i \mathcal{H}_i$

In the language of extended objects / defects from earlier,
a $p = (d-1)$-form symmetry in d dimensions has operators supported along
submanifolds of dimension $d - p - 1$, which here $= d - (d-1) - 1 = 0$.
These are the projectors Π_i above.

In the case of gauge theories w/ triv’ acting subgroups, because the action is trivial,
the operators commute with everything — hence diagonalize the state space.

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Operators Π_i simultaneously diagonalizable; state space = $\mathcal{H} = \oplus_i \mathcal{H}_i$

Correlation functions:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \Pi_i \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle (\Pi_i \mathcal{O}_1) \cdots (\Pi_i \mathcal{O}_m) \rangle = \sum_i \langle \tilde{\mathcal{O}}_1 \cdots \tilde{\mathcal{O}}_m \rangle_i$$

2) Partition functions decompose

$$Z = \sum_{\text{states}} \exp(-\beta H) = \sum_i \sum \exp(-\beta H_i) = \sum_i Z_i$$

(on a connected spacetime)

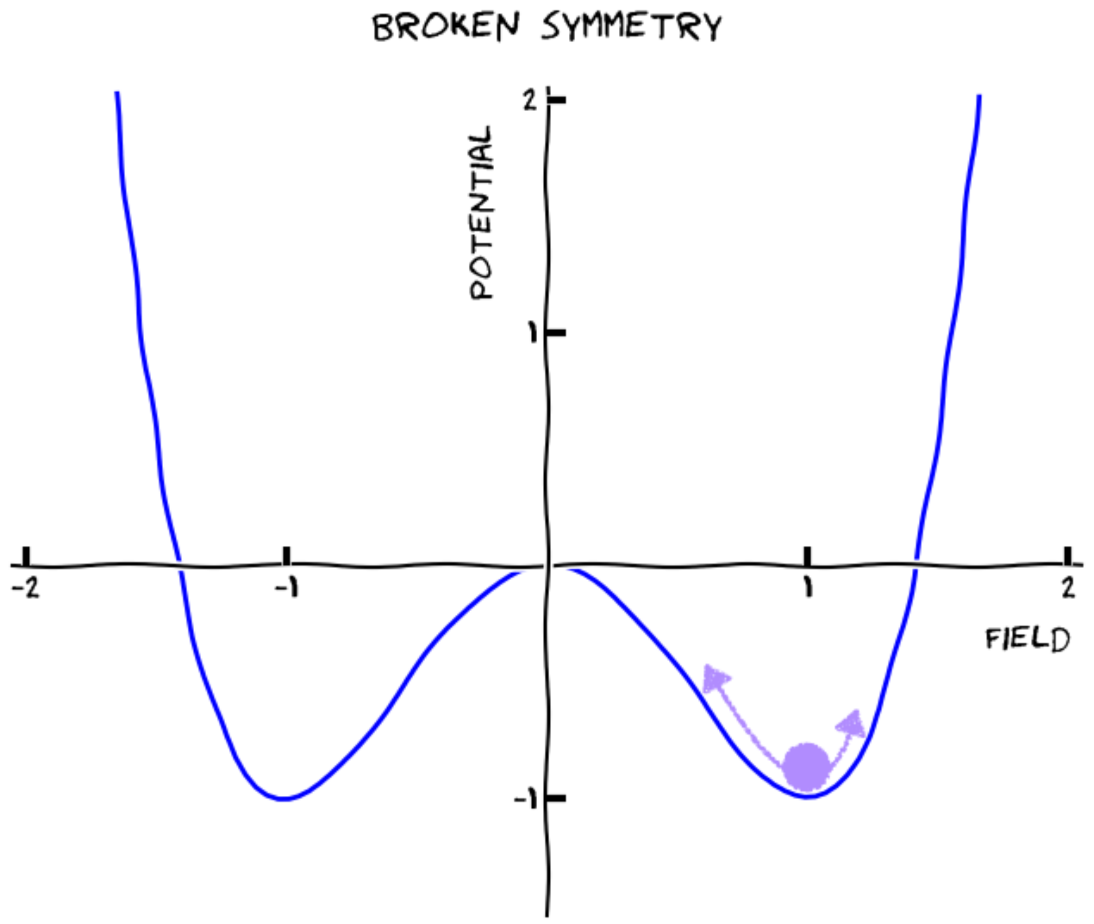
Decomposition $\neq$ spontaneous symmetry breaking

SSB:

Superselection sectors:

- separated by dynamical domain walls
- only genuinely disjoint in IR
- only one overall QFT

Prototype:

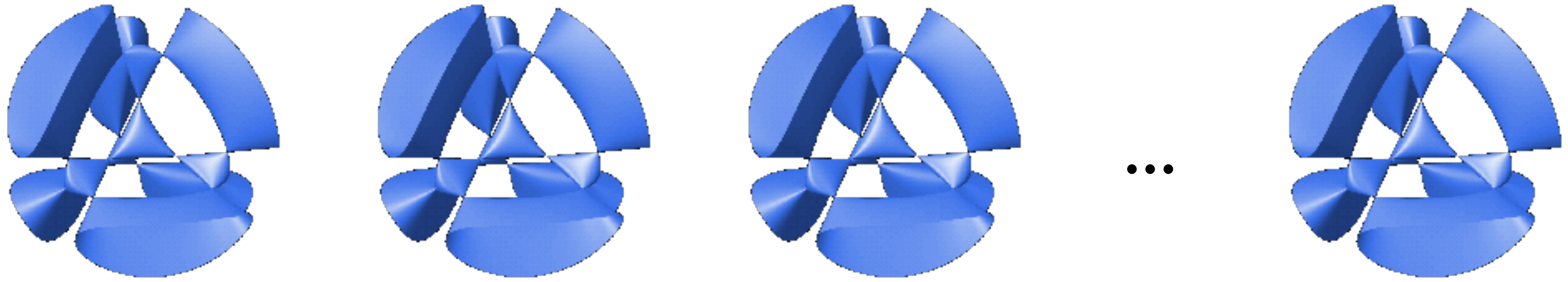


Decomposition:

Universes:

- separated by *non*dynamical domain walls
- disjoint at *all* energy scales
- *multiple* different QFTs present

Prototype:



(see e.g. Tanizaki-Unsal 1912.01033)

There are many examples of decomposition !

Finite gauge theories in 2d (orbifolds): (T Pantev, ES '05; D Robbins, ES, T Vandermeulen '21)

Common thread: a subgroup of the gauge group acts trivially.

Example: If $K \subset \text{center}(\Gamma) \subset \Gamma$ acts trivially, then $[X/\Gamma] = \coprod_{\text{irreps } K} [X/(\Gamma/K)]_{\hat{\omega}}$

Gauge theories:

- 2d $U(1)$ gauge theory with nonmin' charges = sum of $U(1)$ theories w/ min charges (Hellerman et al '06)

  Ex: charge p Schwinger model

- 2d G gauge theory w/ center-invt matter = sum of $G/Z(G)$ theories w/ discrete theta (ES '14)

  Ex: $SU(2)$ theory (w/ center-invt matter) = $SO(3)_+ \coprod SO(3)_-$ (w/ same matter)

- 2d pure G Yang-Mills = sum of trivial QFTs indexed by irreps of G (Nguyen, Tanizaki, Unsal '21) (U(1): Cherman, Jacobson '20)

  Ex: pure $SU(2) = \coprod_{\text{irreps } SU(2)}$ (sigma model on pt)

There are also higher-dimensional examples....

There are many examples of decomposition !

More examples :

- 3d Chern-Simons theory with gauged noneffectively-acting 1-form symmetry (Pantev, ES '22)

  = disjoint union of ordinary Chern-Simons theories

  (On the boundary, this reduces to 2d decomposition.)

- 3d orbifold by finite noneffectively-acting 2-group (Pantev, Robbins, ES, Vandermeulen '22)

  = disjoint union of ordinary 3d orbifolds

  (Example: Yetter model vs union of Dijkgraaf-Witten theories)

- 4d Yang-Mills w/ restriction to instantons of deg' divisible by k (Tanizaki, Unsal '19)

  = disjoint union of ordinary 4d Yang-Mills w/ different θ angles

More examples ....

There are many examples of decomposition !

More examples :

TFTs: 2d unitary TFTs w/ semisimple local operator algebras decompose to invertibles

(Implicit in Durhuus, Jonsson '93; Moore, Segal '06)

(Also: Komargodski et al '20, Huang et al 2110.02958)

Examples:

- 2d abelian BF theory at level k = disjoint union of k invertibles (sigma models on pts)

  (Hellerman, ES, 1012.5999)

- 2d G/G model at level k = disjoint union of invertible theories as many as integrable reps of the Kac-Moody algebra

  (Komargodski et al 2008.07567)

- 2d Dijkgraaf-Witten = sum of invertible theories, as many as irreps (In fact, is a special case of finite gauge theories already mentioned.)

Sigma models on gerbes = disjoint union of sigma models on spaces w/ B fields

Solves tech issue w/ cluster decomposition.

(T Pantev, ES '05)

So, there are lots of examples of decomposition in practice.

Next, I'll focus on one particular family of examples:
2d gauge theories with trivially-acting subgroups

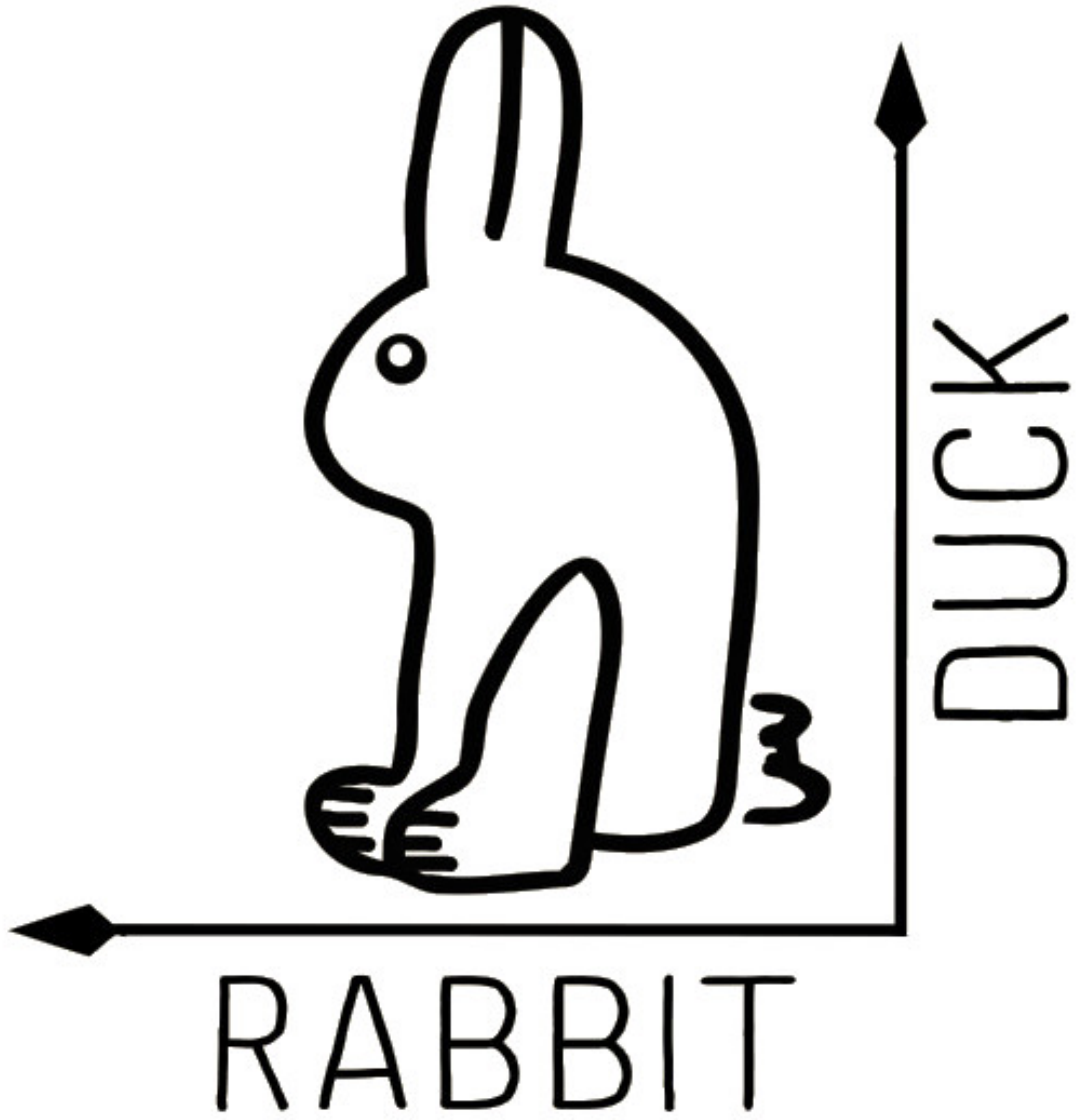
Decomposition in 2d gauge theories

(Hellerman et al '06)

S'pose have G-gauge theory, G semisimple, with finite central $K \subset G$ acting trivially.

As discussed previously, has 1-form symmetry (specifically, BK).

So far, this sounds like just one QFT.



However, I'll outline how, from another perspective,
QFTs of this form are also each
a disjoint union of other QFTs;
they "decompose."

Decomposition in 2d gauge theories

(Hellerman et al '06)

S'pose have G-gauge theory, G semisimple, with finite central $K \subset G$ acting trivially.

As discussed previously, has 1-form symmetry (specifically, BK).

Claim this theory decomposes.

Where are the projection operators?

Math understanding:

Briefly, the projection operators (twist fields, Gukov-Witten) correspond to elements of the center of the group algebra $\mathbb{C}[K]$.

Existence of those projectors (idempotents), forming a basis for the center, is ultimately a consequence of Wedderburn's theorem.

Universes $\longleftrightarrow$ Irreducible representations of K

Partition functions & relation of decomp' to restrictions on instantons....

Decomposition in 2d gauge theories

(Hellerman et al '06)

S'pose have G-gauge theory, G semisimple, with finite central $K \subset G$ acting trivially.
As discussed previously, has 1-form symmetry (specifically, BK).

Statement of decomposition (in this example):

$$\mathrm{QFT}(G\text{−gauge theory}) \;=\; \coprod_{\mathrm{char's}\ \hat{K}} \mathrm{QFT}\left(G/K\text{−gauge theory w/ discrete theta angles}\right)$$

Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where ± denote discrete theta angles (w_2)

Perturbatively, the $SU(2)$, $SO(3)_\pm$ theories are identical
— differences are all nonperturbative.

Decomposition in 2d gauge theories

(Hellerman et al '06)

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where $\pm$ denote discrete theta angles (w_2)

$SU(2)$ instantons (bundles) $\subset SO(3)$ instantons (bundles)

The discrete theta angles weight the non-$SU(2)$ $SO(3)$ instantons so as to cancel out of the partition function of the disjoint union.

Summing over the $SO(3)$ theories projects out some instantons, giving the $SU(2)$ theory.

Decomposition in 2d gauge theories

(Hellerman et al '06)

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Formally, the partition function of the disjoint union can be written

$$Z = \underbrace{\sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \overbrace{\left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)}^{\text{projection operator}}$$

where we have moved the summation inside the integral.

This is an interference effect between universes: **multiverse interference**

Decomposition in 2d gauge theories

(Hellerman et al '06)

projection operator

$$Z = \sum_{\theta \in \hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right] = \int [DA] \exp(-S) \left(\sum_{\theta \in \hat{K}} \exp\left[\theta \int \omega_2(A)\right] \right)$$

Disjoint union

Decomposition in 2d gauge theories

(Hellerman et al ’06)

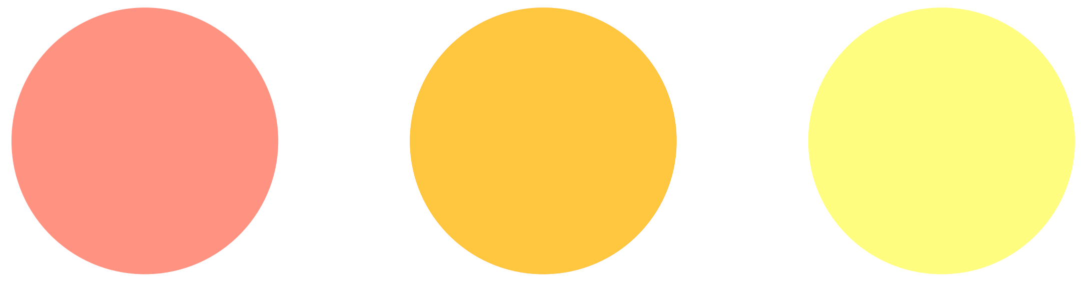
One effect is a projection on nonperturbative sectors:

$$\underbrace{\sum_{\theta\in\hat{K}} \int [DA] \exp(-S) \exp\left[\theta \int \omega_2(A)\right]}_{\text{Disjoint union}} = \int [DA] \exp(-S) \left(\overbrace{\sum_{\theta\in\hat{K}} \exp\left[\theta \int \omega_2(A)\right]}^{\text{projection operator}} \right)$$

Disjoint union of several QFTs / universes = `One’ QFT with a restriction on nonperturbative sectors = `multiverse interference’

Schematically,
two theories combine to form a distinct third:

universe ($SO(3)_+$)



universe ($SO(3)_-$)

multiverse interference effect ($SU(2)$)

Before going on, let's quickly check these claims for pure $SU(2)$ Yang-Mills in 2d.

The partition function Z, on a Riemann surface of genus g, is

(Migdal, Rusakov)

$$Z(SU(2)) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SU(2) reps

$$Z(SO(3)_+) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SO(3) reps

(Tachikawa '13)

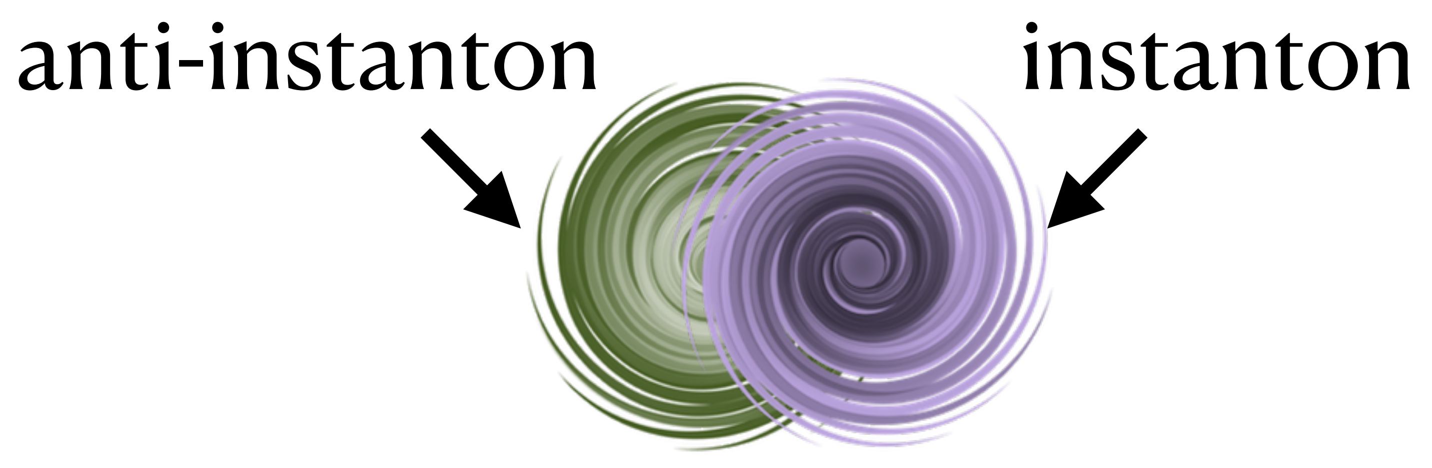
$$Z(SO(3)_-) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SU(2) reps that are not SO(3) reps

Result: $Z(SU(2)) = Z(SO(3)_+) + Z(SO(3)_-)$ as expected.

Suppose we try to require that the total instanton number always vanish in our QFT.

Start with a field configuration with no net instantons.

Now, move them far away from one another:



Nonzero instanton number here!

Total instanton number : 0

Nonzero instanton number here!

If physics is local ("cluster decomposition"),
then in those widely-separated regions, the theories have instantons.
So, even if we start with no net instantons,
cluster decomposition implies we get instantons!

Cluster decomposition:

For this reason, Steven Weinberg taught us:

All local quantum field theories must sum over all instantons, so as to preserve cluster decomposition.

Loophole: Disjoint unions of QFTs also violate cluster decomposition (ex: multiple dimension zero operators), but in principle are straightforward to deal with.

So, if a theory with a restriction on instantons is also a disjoint union, of theories which are well-behaved, then all is OK.

Recap:

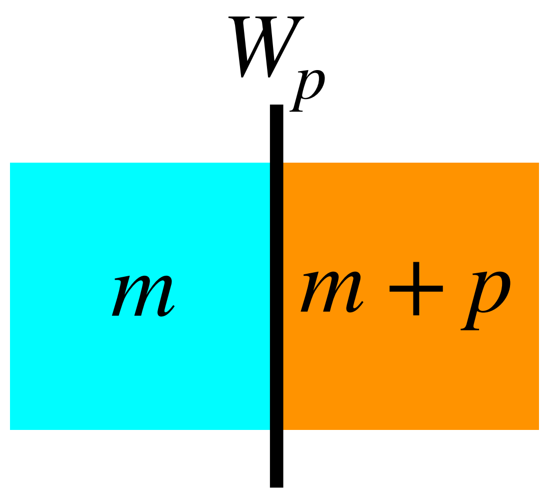
So far we have discussed, in a simple set of examples,
the form of decomposition,
and how it explains restrictions on instantons —
— as a multiverse interference effect.

What if one has a Wilson line that is charged under the trivially-acting $K \subset G$?

Such Wilson lines are defects linking different universes.

Here's an easy example in a different context:

Ex: 2d abelian BF theory at level k



Projectors: $\Pi_m = \frac{1}{k} \sum_{n=0}^{k-1} \xi^{nm} \mathcal{O}_n \qquad \xi = \exp(2\pi i/k)$

Clock-shift commutation relations: $\mathcal{O}_p W_q = \xi^{pq} W_q \mathcal{O}_p \quad \Leftrightarrow \quad \Pi_m W_p = W_p \Pi_{m+p \mod k}$

We'll also see wormholes between universes in another example later....

Decomposition has been checked in many ways,
including, for example, gauge duals & mirrors.

In such a dual, the nonperturbative physics of the original theory becomes perturbative in the dual theory, and so one can see decomposition perturbatively.

Example: susy $\mathbb{CP}^N$ model

The susy $\mathbb{CP}^N$ model is a 2d susy $U(1)$ gauge theory,
with $N+1$ (chiral super)fields each of charge $+1$.

Semiclassically, the Higgs moduli space is $\mathbb{CP}^N$, thus the name.

The mirror to this theory is a susy Landau-Ginzburg model with superpotential

$$W \;=\; \exp(-Y_1) + \exp(-Y_2) + \cdots + \exp(-Y_{N-1}) \;+\; q\exp(+Y_1 + Y_2 + \cdots + Y_{N-1})$$

The mirror encodes the nonperturbative physics of the original theory (eg instantons) as classical / perturbative physics in the mirror.

Decomposition?

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Example: gerby susy $\mathbb{CP}^N$ model

Consider a 2d susy $U(1)$ gauge theory with $N+1$ chiral superfields of charge $k > 1$.

Has $B\mathbb{Z}_k$ one-form symmetry, decomposes into k copies of $\mathbb{CP}^N$ model.

Mirror was computed using methods of (Hori, Vafa '00) in (Pantev, ES, '06); result:

$$W \; = \; \exp(-Y_1) + \exp(-Y_2) + \cdots + \exp(-Y_{N-1}) \; + \; q\Upsilon \exp(+Y_1 + Y_2 + \cdots + Y_{N-1})$$

where Υ is a $\mathbb{Z}_k$-valued field.

Path integral sum over values of Υ = disjoint union, perturbatively.

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$$W = \exp(-Y_1) + \exp(-Y_2) + \cdots + \exp(-Y_{N-1}) + q\Upsilon \exp(+Y_1 + Y_2 + \cdots + Y_{N-1})$$

where Υ is a $\mathbb{Z}_k$-valued field.

Path integral sum over values of Υ = disjoint union, perturbatively.

In passing:

Ordinarily I describe decomposition in terms of universes with variable θ angles or B fields — complex Kahler parameters.

In the mirror, these become complex structure parameters.

Mathematical interpretation:

So far I've just talked abstractly about 2d QFTs & 1-form symmetries.

This has a mathematical interpretation: "gerbes" $\neq$

A G-gerbe is a fiber bundle

whose fibers are 1-form symmetry `groups', specifically, BG.

A sigma model whose target is a G-gerbe has a global BG symmetry,

just as a sigma model whose target is a G-bundle has a global G-symmetry,

from translations on the fibers.

Furthermore, $BG = [\text{point}/G]$

so whenever a group acts trivially,

you should expect a gerbe structure (1-form symmetry) somewhere.

Mathematical interpretation:

Twenty years ago, I was interested in studying
`sigma models on gerbes' as possible sources of new string compactifications.

Potential issues, since solved:

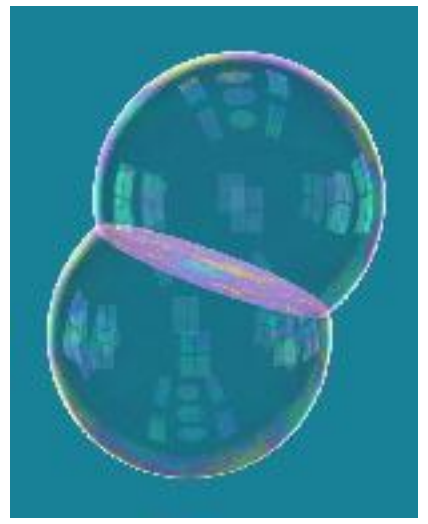
construction of QFT; cluster decomposition; moduli;
mod' invariance & unitarity in orbifolds; potential presentation-dependence.

What we eventually learned was that these theories are well-defined,
but,
are disjoint unions of ordinary theories, at least in (2,2) susy cases,
because of decomposition.

Not really new compactifications, but instead other applications.
I'll list 4 of my favorites next....

Application: GW invariants

The Gromov-Witten (GW) invariants count minimal-area surfaces in a given space.



There exists a def'n of GW invariants of gerbes.

(Chen, Ruan; Abramovitch, Graber, Vistoli ~2000)

Decomposition predicts,

GW invariants of a gerbe = sum of GW invariants of universes

✓ Checked by (H-H Tseng, Y Jiang, et al '08 on)

Application: GLSMs

(Caldararu et al '07)

Consider the GLSM for e.g. $\mathbb{P}^3[2,2] = T^2$.

This is a $U(1)$ gauge theory, with ϕ_i charge $+1$, p_a charge -2.

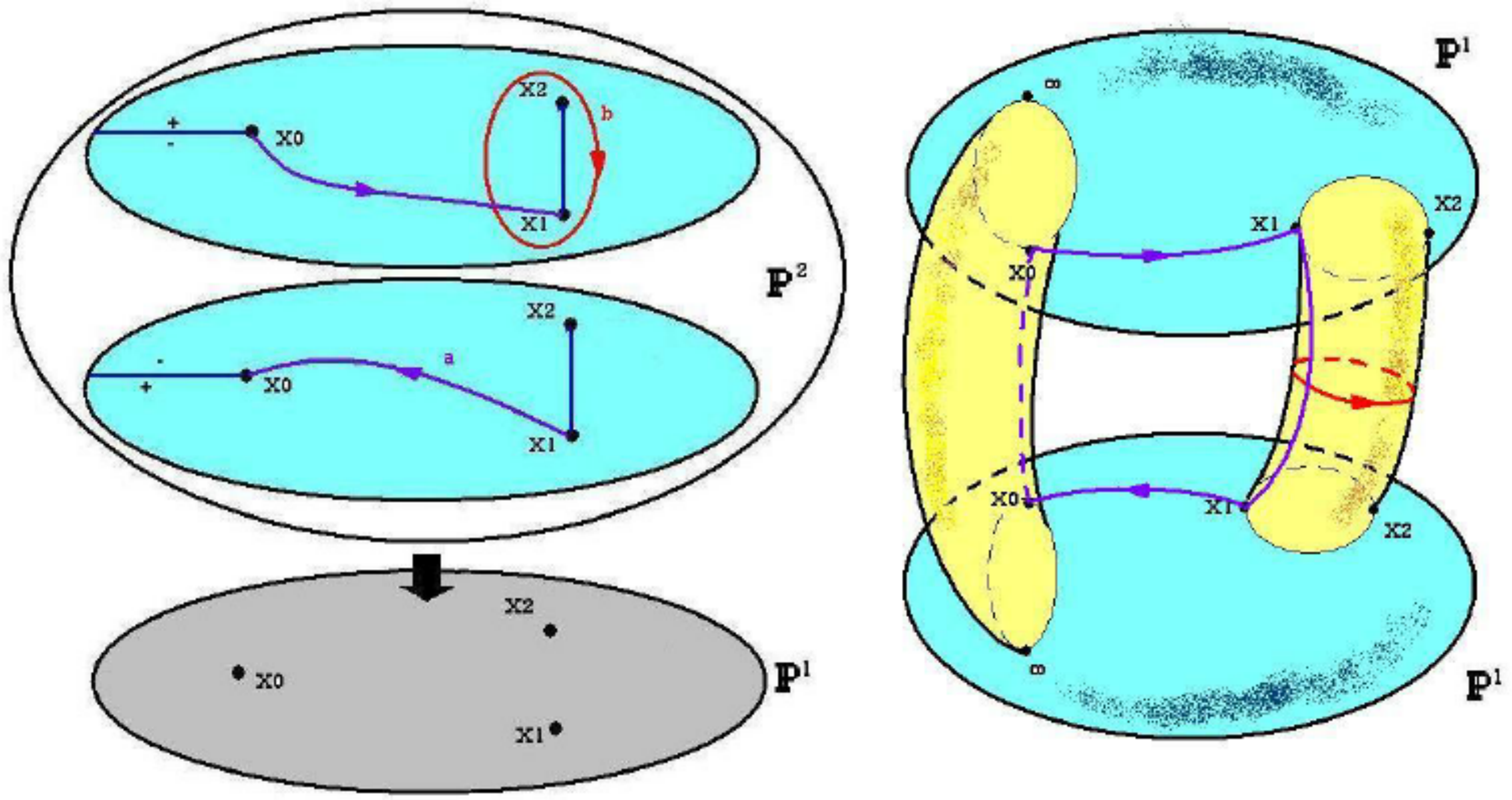
The LG point has superpotential

$$W = \sum_{ij} A^{ij}(p)\, \phi_i \phi_j \quad \text{— mass matrix for } \phi \text{ fields.}$$

Away from zeroes of eigenvalues of A^{ij},

looks like sigma model on $\mathbb{P}^1 = \mathrm{Proj}\, \mathbb{C}[p_1, p_2]$, with $B\mathbb{Z}_2$ symmetry.

Decomposition $\Rightarrow$ Double cover of $\mathbb{P}^1$, branched over $\{\det A = 0\} = \{4 \text{ points}\}$



Another T^2 !
geometry
realized
nonperturbatively
via decomposition

Application: elliptic genera of pure susy gauge theories

(R Eager, ES '20)

We can use decomposition to predict elliptic genera of pure (2,2) susy gauge theories, using knowledge of IR susy breaking for various discrete theta angles.

Example: for $SU(k)/\mathbb{Z}_k$, susy unbroken only for discrete theta $\theta = -(1/2)k(k-1) \mod k$

(as derived from 2d nonabelian mirrors)

$$\text{EG}(G/K, \theta) = 0 \quad \text{if susy broken in IR}$$

$$\textbf{Decomposition} \Rightarrow \text{EG}(G) = \sum_{\theta} \text{EG}(G/K, \theta)$$

Can then algebraically recover elliptic genera.

Example: $$\text{EG}(SU(k)/\mathbb{Z}_k, \theta) = (1/k)\text{EG}(SU(k)) \sum_{m=0}^{k-1} (-)^{m(k+1)} \exp(im\theta)$$

For $k = 2$, matches (Kim, Kim, Park '17).

Numerous other low-rank exs checked with susy localization.

Application: anomalies

Consider a finite G-gauge theory, $[X/G]$, with a gauge anomaly (so that the theory does not actually exist).

Two methods to resolve the anomaly:

1) Make G bigger. (Wang-Wen-Witten '17, Tachikawa '17)

Replace G by Γ, $1 \longrightarrow K \longrightarrow \Gamma \stackrel{\pi}{\longrightarrow} G \longrightarrow 1$

where $\pi^* \alpha$ trivial for $\alpha \in H^3(G, U(1))$ the anomaly,

and replace original orbifold with $[X/\Gamma]_B$ for suitable phases $B \in H^1(G, H^1(K, U(1)))$.

2) Make G smaller.

Replace original orbifold with $[X/\mathrm{ker} f]$ for some hom' $f: G \to H$ s.t. $\alpha|_{\mathrm{ker} f} = 0$

Decomposition: $[X/\Gamma]_B \; = \;$ (copies of) $[X/\mathrm{ker}\, B]$ (Robbins, ES, Vandermeulen '21)

So the two possibilities are equivalent.

Application: moduli spaces

Gerbe structures are common on moduli spaces of SCFTs.

Moduli stack of susy sigma models = $\mathbb{Z}_2$ gerbe over moduli stack of CYs

Bagger-Witten line bundle = `fractional' bundle over that gerbe

(a bundle on the gerbe that is not a pullback from the underlying moduli space)

(Donagi et al '17, '19)

Example: moduli space of elliptic curves

$$\mathcal{M} = [\mathfrak{h}/SL(2,\mathbb{Z})]$$ for $\mathfrak{h}$ the upper half plane

However, the Bagger-Witten line bundle lives on $\mathcal{N} = [\mathfrak{h}/Mp(2,\mathbb{Z})]$

where $$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Mp(2,\mathbb{Z}) \longrightarrow SL(2,\mathbb{Z}) \longrightarrow 1$$ (Gu, ES '16)

which reflects a subtle $\mathbb{Z}_2$ extending T-duality in susy theories.

(Pantev, ES '16)

Summary

Decomposition: sometimes one QFT secretly $= \sum$ QFTs $= \coprod$ universes

Restrictions on instantons arise from such sums as
interference effect between universes

Examples include gauge theories w/ trivially-acting subgroups

Applications include Gromov-Witten theory, GLSMs, elliptic genera, anomalies.

Thank you for your time!

Details of another 2d example, involving orbifolds

Let's first construct a family of examples in $d = 2$ spacetime dimensions.

We'll gauge a noneffectively-acting $(d-2) = 0$-form symmetry,
to get a global 1-form symmetry (& hence a decomposition).

Specifically, consider the orbifold $[X/\Gamma]$, where

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad \sim \ \omega \in H^2(G,K)$$

is a central extension, and K, Γ, G are finite, K abelian, and K acts trivially.
(Decomposition exists more generally, but today I'll stick w/ easy cases.)

The orbifold $[X/\Gamma]$ has a global $BK \ = \ K^{(1)}$ symmetry, & should decompose.

I'm going to outline one way to see that

$$\mathrm{QFT}\left([X/\Gamma]\right) \ = \ \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)} \right)$$

where
$$H^2(G,K) \ \longrightarrow \ H^2(G,U(1))$$
$$\omega \ \mapsto \ \rho(\omega)$$
gives the discrete torsion on universe ρ

Claim: $\mathrm{QFT}([X/\Gamma]) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)} \right)$

Let's establish this in partition functions on T^2.

Universally, for any Γ orbifold on T^2,

$$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|} \sum_{\gamma_1 \gamma_2 = \gamma_2 \gamma_1} Z_{\gamma_1, \gamma_2}(X) \quad \text{where} \quad Z_{g,h} = \Big(g$$

$$h \longrightarrow X \Big)$$

("twisted sectors")

(Think of $Z_{g,h}$ as sigma model to X with branch cuts g, h.)

We need to count commuting pairs of elements in Γ ....

Claim: $\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$

Let's establish this in partition functions on T^2.

Universally, for any Γ orbifold on T^2, $Z_{T^2}\left([X/\Gamma]\right) = \frac{1}{|\Gamma|} \sum_{\gamma_1\gamma_2 = \gamma_2\gamma_1} Z_{\gamma_1,\gamma_2}(X)$

We need to count commuting pairs of elements in Γ ....

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad \sim \ \omega \in H^2(G, K)$$

Write $\gamma \in \Gamma$ as $\gamma = (g \in G, k \in K)$ where $\gamma_1\gamma_2 = (g_1g_2, k_1k_2\omega(g_1, g_2))$

Then, $\gamma_1\gamma_2 = \gamma_2\gamma_1 \Leftrightarrow g_1g_2 = g_2g_2$ and $\omega(g_1, g_2) = \omega(g_2, g_1)$

commuting pairs in G such that $\omega(g_1, g_2) = \omega(g_2, g_1)$

Restriction on nonperturbative sectors

(In an orbifold, nonperturbative sectors = twisted sectors)

Claim: $\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)} \right)$

Let's establish this in partition functions on T^2.

Universally, for any Γ orbifold on T^2, $Z_{T^2}\left([X/\Gamma]\right) = \frac{1}{|\Gamma|} \sum_{\gamma_1\gamma_2=\gamma_2\gamma_1} Z_{\gamma_1,\gamma_2}(X)$

We need to count commuting pairs of elements in Γ .... $1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$

These are commuting pairs in G such that $\omega(g_1, g_2) = \omega(g_2, g_1)$

So:

$$Z_{T^2}\left([X/\Gamma]\right) = \frac{1}{|\Gamma|} \sum_{\gamma_1\gamma_2=\gamma_2\gamma_1} Z_{\gamma_1,\gamma_2}(X) = \frac{|K|^2}{|\Gamma|} \sum_{g_1g_2=g_2g_1} \delta\left(\frac{\omega(g_1, g_2)}{\omega(g_2, g_1)} - 1 \right) Z_{g_1,g_2}$$

where we have used $Z_{\gamma_1,\gamma_2} = Z_{g_1,g_2}$ since K acts trivially.

Claim: $\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$

Let's establish this in partition functions on T^2.

So far:

$$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|} \sum_{\gamma_1\gamma_2=\gamma_2\gamma_1} Z_{\gamma_1,\gamma_2}(X) = \frac{|K|^2}{|\Gamma|} \sum_{g_1 g_2 = g_2 g_1} \delta\left(\frac{\omega(g_1,g_2)}{\omega(g_2,g_1)} - 1\right) Z_{g_1,g_2}$$

Next, write

$$\delta\left(\frac{\omega(g_1,g_2)}{\omega(g_2,g_1)} - 1\right) = \frac{1}{|\hat{K}|} \sum_{\rho\in\hat{K}} \frac{\rho\circ\omega(g_1,g_2)}{\rho\circ\omega(g_2,g_1)}$$

where $\rho \circ \omega \in H^2(G, U(1))$

(discrete torsion!)

so that, after rearrangement,

$$Z_{T^2}([X/\Gamma]) = \frac{|G||K|^2}{|\Gamma||\hat{K}|} \sum_{\rho\in\hat{K}} Z_{T^2}\left([X/G]_{\rho\circ\omega}\right) = \sum_{\rho\in\hat{K}} Z_{T^2}\left([X/G]_{\rho\circ\omega}\right)$$

consistent with decomposition !

Adding the universes projects out some sectors — interference effect.

So far we have demonstrated that for T^2 partition functions,

$$\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$$

which is the statement of decomposition in this case ($K \subset \Gamma$ central).

Similar computations can be done at any genus,
and for local operators, etc.

Next, we'll walk through details in a simple example....

To make this more concrete, let's walk through an example, where everything can be made completely explicit.

Example: Orbifold $[X/D_4]$ in which the $\mathbb{Z}_2$ center acts trivially.

(T Pantev, ES '05)

— has $B\mathbb{Z}_2$ (1-form) symmetry

$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ so this is closely related to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Decomposition predicts

$$\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)} \right)$$

which here means

$$\mathrm{QFT}\left([X/D_4]\right) = \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Let's check this explicitly....

Example, cont'd

$$\mathrm{QFT}\left([X/D_4]\right) = \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

At the level of operators, one reason for this is that the theory admits projection operators:

Let $\hat{z}$ denote the (dim 0) twist field associated to the trivially-acting $\mathbb{Z}_2$:

$\hat{z}$ obeys $\hat{z}^2 = 1.$

Using that relation, we form projection operators:

$$\Pi_{\pm} = \frac{1}{2}(1 \pm \hat{z}) \qquad \text{(= specialization of general formula)}$$

$$\Pi_{\pm}^2 = \Pi_{\pm} \qquad \Pi_{\pm}\Pi_{\mp} = 0 \qquad \Pi_+ + \Pi_- = 1$$

Note: untwisted sector lies in both universes; universes = lin' comb's of twisted & untwisted.

Next: compare partition functions....

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 \; = \; \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

Take the (1+1)-dim'l spacetime to be T^2.

The partition function of any orbifold $[X/\Gamma]$ on T^2 is

$$Z_{T^2}([X/\Gamma]) \; = \; \frac{1}{|\Gamma|} \sum_{gh=hg} Z_{g,h}$$

where $Z_{g,h} = \Big(g$  $\longrightarrow X \Big)$, h

("twisted sectors")

(Think of $Z_{g,h}$ as sigma model to X with branch cuts g, h.)

We're going to see that

$$Z_{T^2}\left([X/D_4]\right) \; = \; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) \; + \; Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\rm d.t.}\right)$$

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

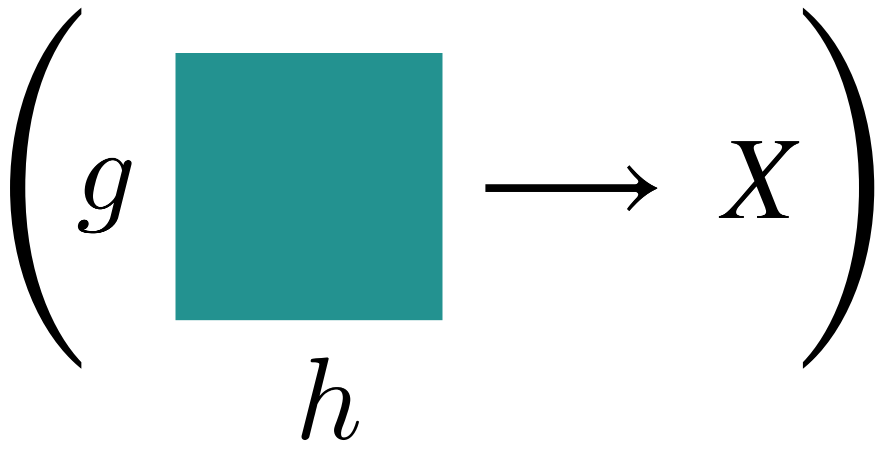
$$D_4 \;=\; \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \;=\; \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

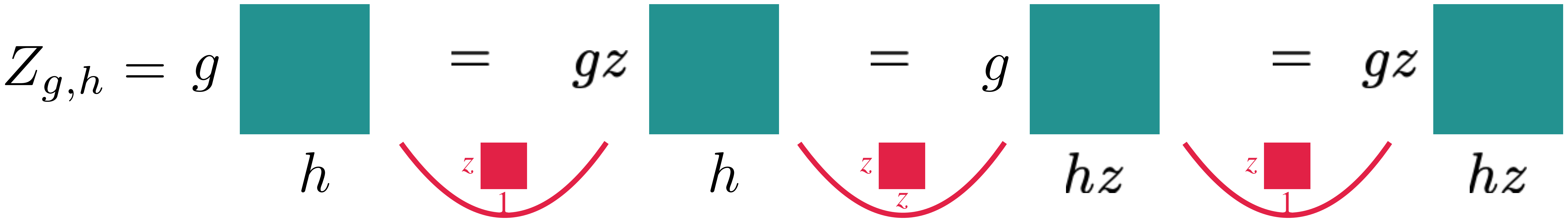
$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{1}{|D_4|} \sum_{g,h \in D_4,\; gh=hg} Z_{g,h}$$

where $Z_{g,h}$ =



Since z acts trivially,

$Z_{g,h}$ is symmetric under multiplication by z



This is the $B\mathbb{Z}_2$ 1-form symmetry.

Example, cont'd

Compute the partition function of $[X/D_4]$ (T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

$$Z_{T^2}([X/D_4]) = \frac{1}{|D_4|} \sum_{g,h \in D_4,\ gh=hg} Z_{g,h}$$

where $Z_{g,h} = \left(g \right.$  $\left. \longrightarrow X \right)$, h below the square

Each D_4 twisted sector ($Z_{g,h}$) that appears is the same as a $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sector,

appearing with multiplicity $|\mathbb{Z}_2|^2 = 4$,

except for the sectors $\overline{a}$  $\overline{b}$, $\overline{a}$  $\overline{ab}$, $\overline{b}$  $\overline{ab}$ which do **not** appear.

Restriction on nonperturbative sectors

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}([X/D_4]) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors}) \right)$$

$$= 2 \left(Z_{T^2}([X/\mathbb{Z}_2 \times \mathbb{Z}_2]) - (\text{some twisted sectors}) \right)$$

Different theory than $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Physics knows when we gauge even a trivially-acting group!

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

Fact: given any one partition function $Z_{T^2}([X/G]) = \frac{1}{|G|} \sum_{gh=hg} Z_{g,h}$

we can multiply in $SL(2,\mathbb{Z})$-invariant phases $\epsilon(g,h)$

to get another consistent partition function (for a different theory)

$$Z' = \frac{1}{|G|} \sum_{gh=hg} \epsilon(g,h)\, Z_{g,h}$$

There is a universal choice of such phases, determined by elements of $H^2(G, U(1))$

This is called "discrete torsion."

Example, cont'd

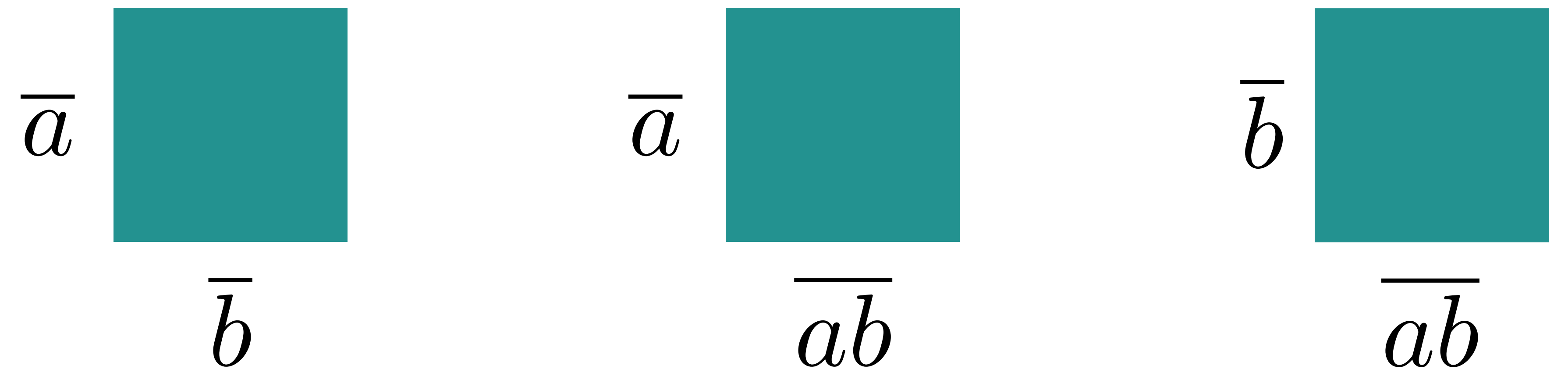
Compute the partition function of $[X/D_4]$ (T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - (\text{some twisted sectors}) \right)$$

$$= 2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - (\text{some twisted sectors}) \right)$$

In a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, discrete torsion $\in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and the nontrivial element acts as a sign on the twisted sectors



the same sectors which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) + Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Adding the universes projects out some sectors — interference effect.

Example, cont'd

Compute the partition function of $[X/D_4]$

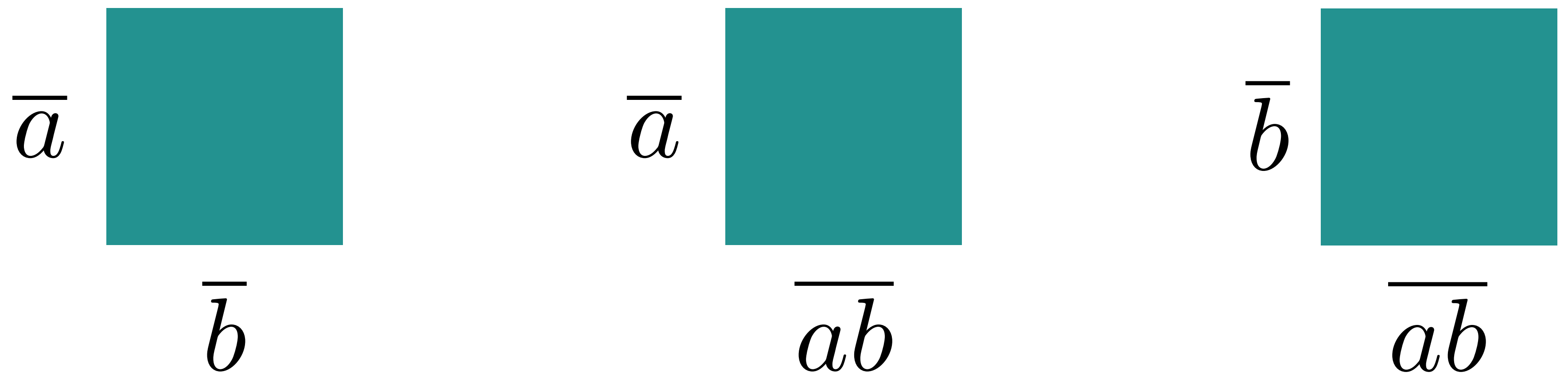
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) \;=\; \frac{|\mathbb{Z}_2\times\mathbb{Z}_2|}{|D_4|}\,|\mathbb{Z}_2|^2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors})\right)$$

$$= \; 2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) \;-\; (\text{some twisted sectors})\right)$$

Discrete torsion is $H^2(\mathbb{Z}_2\times\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and acts as a sign on the twisted sectors



which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) \;=\; \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) \;=\; \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

The computation above demonstrated that the partition function on T^2 has the form predicted by decomposition.
The same is also true of partition functions at higher genus
— just more combinatorics.
(see hep-th/0606034, section 5.2 for details)

Only slightly novel aspect: in gen'l, one finds dilaton shifts, which mostly I'll suppress in this talk.

Example, cont'd

Massless states of $[X/D_4]$ for $X = T^6$ (T Pantev, ES '05)

Massless states of $[T^6/D_4]$

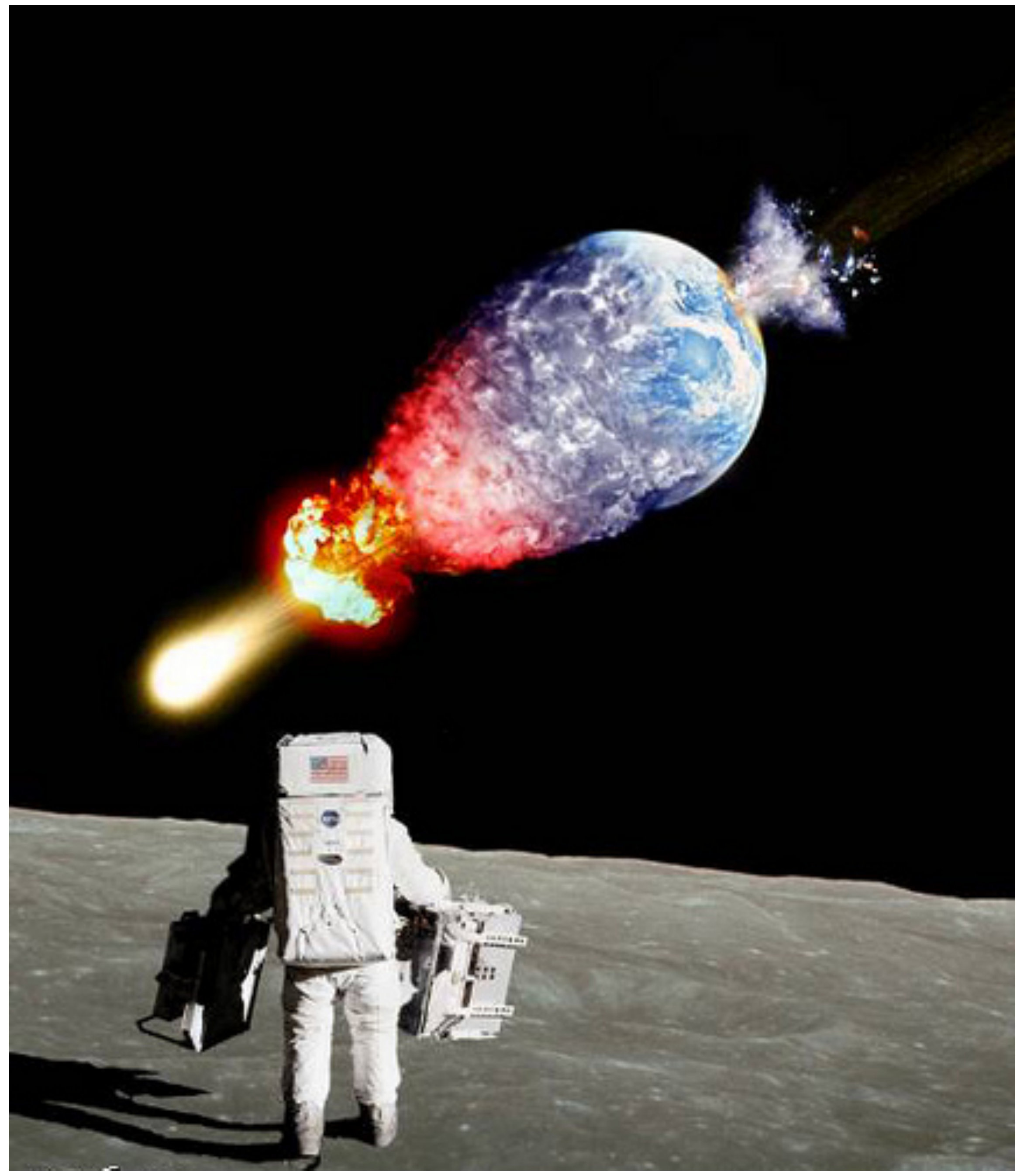
$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}
$$

Signals mult' components / cluster decomp' violation

If we didn't know about decomposition, the 2's in the corners would be a problem...

A big problem!

They signal a violation of
cluster decomposition,
the same axiom that's violated
by restricting instantons.



Ordinarily, I'd assume that the computation was wrong.

However, decomposition saves the day....

Example, cont'd

Massless states of $[X/D_4]$ for $X = T^6$ (T Pantev, ES '05)

Massless states of $[T^6/D_4]$

$$
\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}
\;=\;
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 51 & & 0 & \\
1 & & 3 & & 3 & & 1 \\
 & 0 & & 51 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\;+\;
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 51 & & 51 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Signals mult' components / cluster decomp' violation

spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb' w/o d.t.

spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb' w/ d.t.

matching the prediction of decomposition

$$\mathrm{CFT}\left([X/D_4]\right) \;=\; \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{w/o\,d.t.}}\right) \coprod \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{d.t.}}\right)$$